Comput Manag Sci (2017) 14:135–160
DOI 10.1007/s10287-016-0263-4

ORIGINAL PAPER

Optimal pension fund composition for an Italian private pension plan sponsor

Sebastiano Vitali[1] · **Vittorio Moriggia**[1] · **Miloš Kopa**[2]

Received: 30 October 2015 / Accepted: 18 August 2016 / Published online: 27 August 2016

Abstract We address the problem of a private pension plan sponsor who has to find the best pension funds for its members. Starting from a descriptive analysis of the pension plan members we identify a set of representative subscribers. Then, the optimal allocation for each representative will become a pension fund of the pension plan. For each representative, we propose a multistage stochastic program (MSP) which includes a multi-criteria objective function. The optimal choice is the portfolio allocation that minimizes the average value at risk deviation of the final wealth and satisfies a wealth target in the final stage and other constraints regarding pension plan regulations. Stochasticity arises from the investor's salary process and from asset returns. Numerical results show the optimal dynamic portfolios with respect to the investor's preferences and then the best pension funds the sponsor might offer.

Keywords Pension fund · Optimal policy · Multistage stochastic programming · Cluster analysis

---

The research was partially supported by MIUR-ex60% 2014 and MIUR-ex60% 2016 sci.resp. Vittorio Moriggia. The research of Miloš Kopa was supported by Czech Science Foundation (Grant 402/12/G097).

✉ Sebastiano Vitali
sebastiano.vitali@unibg.it

Vittorio Moriggia
vittorio.moriggia@unibg.it

Miloš Kopa
kopa@karlin.mff.cuni.cz

1 Department of Management, Economics and Quantitative Methods, University of Bergamo, Via dei Caniana 2, 24127 Bergamo, Italy

2 Department of Probability and Mathematical Statistics, Faculty of Mathematics and Physics, Charles University in Prague, Sokolovska 83, 18675 Prague, Czech Republic

Mathematics Subject Classification 90C15 · 91B30

1 Introduction

The pension system has become more and more complex and structured all over Europe in the last decades. Because of the financial and social crisis, several countries implemented strong reforms in the state welfare system in order to reduce the pension costs on the state budget balance. Furthermore, they encouraged the establishment of private pension facilities (see OECD 2013, pp. 18–25). In general, a private pension fund is an investment fund which periodically receives contributions from a private investor and then provides an annuity during the retirement. In order to manage a pension fund several asset and liability management (ALM) structures have been deeply explored in the last years. Milestone models, among others, are: the Russell-Yasuda Kasai Model (see Cariño et al. 1994, 1998b; Cariño and Ziemba 1998a), the InnoALM model for multistage managing of a pension fund (see Geyer and Ziemba 2008, Ziemba 2006), and the CALM model for long-term pension fund planning (see Consigli and Dempster 1998b, a). More recently, Mulvey et al. 2006 suggests a multiperiod model to increase the understanding of the risks and rewards in a long-term horizon framework for pension plans and other long-term investors (see also Mulvey et al. 2007, 2008). An innovative formulation of the ALM problem is proposed in Consigli et al. (2011), Consigli and di Tria (2012) and in Consigli and Moriggia (2014).

Clearly, each country required an adjustment of a general model in order to consider the specificity of the country's regulations. Some models have been built starting from the country's pension system. Høyland and Wallace (2001) analyse the Norwegian regulations; Dupačová and Polívka (2009) focus on the Czech Republic public's scheme; Hilli et al. (2007) study the case of a Finnish pension company; Kouwenberg (2001), Bovenberg and Knaap (2005), Streutker et al. (2007), Haneveld et al. 2010b and Haneveld et al. (2010a) explore the Dutch system; Dondi et al. (2007) analyse the Swiss setting; in Fabozzi et al. (2005) there is a comparison between 28 defined benefit pension funds of the Netherlands, Switzerland, the United Kingdom, and the United States.

The main function of a pension plan is to provide a reasonable and sure annuity to the subscribers, i.e. to guarantee an integration of the public retirement pension so that the total income before and after retirement does not differ substantially. Typically, a pension plan is composed of pension funds which are sufficiently different from each other that the investors can choose the optimal pension investment from well diversified strategies. Often, such pension funds are issued following some standard investment allocations: guaranteed capital, low risk profile, high yield investment, etc. The competition between private pension plan providers is becoming stronger and stronger. They would all like to offer suitable and reliable pension plans for their contributors. This aim cannot be pursued by huge providers whose offer is somehow standard, while for small and medium pension plan sponsors it is simpler to proceed with a rigorous analysis of the subscribers and then to customize the pension plan investment strategies according to the actual contributors' needs.

We deal with a real-life problem of a medium-size Italian bank that wants to issue a pension plan specifically for its employees. For this reason, the bank wishes to identify the optimal allocation for its pension funds in order to match its employees' features and, of course, comply with the Italian regulations and laws. Indeed, some methodological adjustments adopted in this paper have been requested by the bank and several constraints have been designed according to specific requirements included in the already existing pension fund rules (see OECD 2013, pp. 284–288). For instance, in Italy second-pillar pension funds are strongly encouraged and the employer contributes proportionally to the subscriber's payment. Further, the second-pillar pension funds start paying the annuities when the employee receives the public pension. To define the optimal pension plan strategies, the bank has to alter its usual point of view: instead of optimizing the asset allocation in an ALM perspective, the sponsor wants to find a solution to the individual asset allocation problem of its employees. Given a strategic optimal composition designed for the individual, the pension fund manager will define an ALM model accordingly. The individual asset allocation problem was first investigated in Merton (1969, 1971), introducing the concept of consumption and optimal investment through a dynamic programming approach in order to maximize the utility of a private investor over a fixed time horizon. Richard (1975) introduces the concept of lifetime uncertainty, labor and insured wealth as further elements to take into account. Geyer et al. (2009) extends Richard's model by using a multistage stochastic programming approach. Berger and Mulvey (1998) proposes a tool named Home Account Advisor, a multistage model which optimizes the investor's financial objective considering investments, savings and borrowing simultaneously.

We propose a two-step model. The first step is to make a precise statistical analysis of the data concerning the plan's subscribers to identify a set of representative members. The second step is to formulate and implement a multistage stochastic program (MSP) in order to define the optimal investment allocation for each representative member. Lastly, the optimal investment portfolio for each representative will define the strategic allocation of each pension fund. We must remark that, among the actual pension funds composing the whole pension plan of the bank, there is no life-cycle pension fund, see Gomes et al. (2008). Moreover, the bank does not want to issue such pension fund. However, getting closer to retirement, each subscriber can personally switch from one pension fund to another. Thus, the optimal investment portfolios have to consider this hidden need too.

This paper is structured as follows. The statistical analysis of the first step is presented in Sect. 2. The second step, analysed in Sect. 3, explores the formulation of the MSP. In Sect. 4, numerical results provide optimal dynamic portfolios with respect to investors' preferences. The paper's conclusions are in Sect. 6.

2 Population analysis

2.1 Statistical description of the population

The population analysis consists in a statistical description of 5577 employees of the bank. They belong to an homogeneous population and are the active population of the pension plan. The focus of the study is twofold: to give the pension plan provider a complete and rigorous view of the actual participants and to investigate their main characteristics in order to have a reliable starting point for the later clusterization. The considered members' features are:

– age and remaining working life
– accumulated wealth
– average annual contribution
– choice of percentage of the salary to contribute
– attitude to diversification
– switching behavior.

The age analysis uses as input data the year in which each member started to contribute to the plan. For the considered population the result shows a uniform distribution in the last decades. The analysis highlights a huge variety in the accumulated wealth, from the younger employees with almost no wealth to the top managerial positions, which create a heavy right tail. The mean value is EUR 70,000, the standard deviation is EUR 46,000. To better analyse the accumulation process, we introduce a contribution ratio given by the accumulated wealth divided by the number of years spent in the plan. The contribution ratio distribution is highly concentrated between EUR 3000 and EUR 6000 per year.

A particular focus is placed on the diversification choice. Up to now, the plan has been composed of seven pension funds, and we analyse the number of positions opened for each strategy and for each member. The first analysis investigates the preferred pension funds, the second shows the individual inclination to invest simultaneously in more than one fund, i.e. to adopt for the pension savings perspective the same diversification strategies that are usually performed for investment portfolios.

The option of switching between the strategies is not widely used. Only 4 % of the pension plan population moved their accumulated wealth at least once. In those cases, the switch occurs typically from risky pension funds to low risk ones. This switching behavior is consonant with the fact that there is no life-cycle pension fund, so the contributors try to implement a life-cycle investment on their own. We would expect a higher occurrence of switching events. They are limited probably because the actual pension fund investment strategies are not suitable for satisfying this need.

We observe a strong correlation between features of the switch and the attitude to diversification. The participants who require a switch experience a diversified portfolio. Generally, we can distinguish between members with static and concentrated portfolios and members implementing dynamic and diversified strategies.

The study of the choice of strategy also displays the attitudes towards risk of the members of the pension plan. The strategies with the lowest risk constitute the main investment. A few contributors switch to riskier position for two reasons: the perspective of a long investment window if they are young members, or a natural aptitude for risk which leads the contributors to invest their savings so as to seek an extra gain during periods of high volatility in the markets.

2.2 Cluster analysis

The aim of the cluster analysis is to extract a set of representative participants from the whole population. For each of them, we propose an optimal portfolio allocation considering the investor's features and the stochastic environment. The pension plan sponsor wants to offer the best pension funds to the active population. Once the optimal portfolios are obtained (one for each representative), they will be proposed to the pension plan provider (the bank) in order to create similar pension funds. Clearly, the cardinality of the set of representative member will coincide with the number of pension funds. Therefore, the provider must decide on the number of clusters, taking into account its suitability for the members and its manageability for the pension fund manager.

Thanks to the results of the previous section, we start the cluster analysis having three elements as the main characterizing features: the accumulated wealth, the portfolio risk level, and the remaining working years. All three variables are normalized for better comparability. The portfolio risk level is measured by the bank with a value on a scale from 1 (very low risk) to 10 (very high risk).

In order to create the cluster sets, we adopt the k-means Lloyd's algorithm using the cityblock distance, which measures the distance between two elements x_i, x_j having p attributes by $d(x_i, x_j) = \sum_{k=1}^{p} |x_i^k - x_j^k|$, i.e. each centroid is the component-wise median of the points in that cluster, see Lloyd (1982) and Kaufman and Rousseeuw (2009). As already mentioned, the number of clusters is the provider's decision. This choice should consider the representativeness of the clusters. In our case, due to the costs of management, the bank wants to reduce the number of pension funds (from the current 7) to at most 5. This request is quantitatively justified by the scree-plot shown in Fig. 1, which captures the gain in term of reliability for an increasing number of clusters.

The scree-plot only shows that a minimal reasonable number of clusters is three. Table 1 shows the results assuming, in turn, three, four, and five clusters.

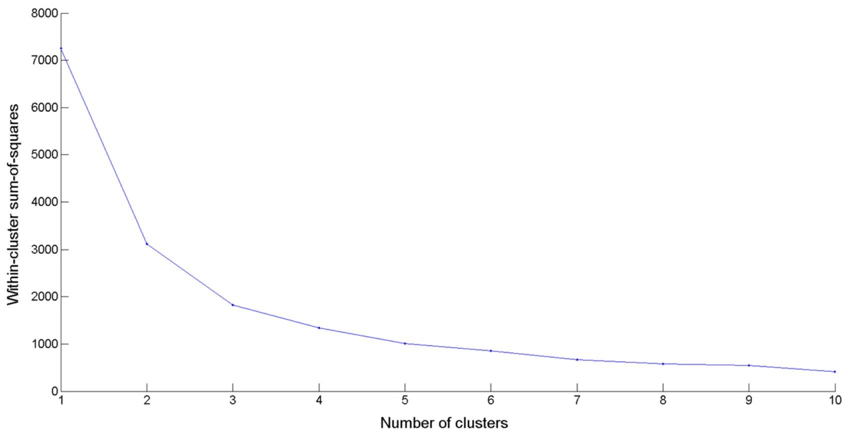


Fig. 1 Cluster analysis scree-plot

Table 1 Centroid features for choice of the number of clusters

	Wealth (EUR)	Risk profile	Years to retirement
Three clusters			
Rep. 1	105,000	1	13
Rep. 2	43,000	1	28
Rep. 3	41,000	7	32
Four clusters			
Rep. 1	104,000	1	13
Rep. 2	43,000	1	27
Rep. 3	66,000	3	28
Rep. 4	38,000	10	33
Five clusters			
Rep. 1	132,000	1	9
Rep. 2	78,500	1	17
Rep. 3	66,800	3	28
Rep. 4	35,000	1	31
Rep. 5	38,000	10	33

To compare different choices of the number of clusters, Rousseeuw (1987) proposes the silhouette analysis. The silhouette value $s(x_i)$ describes how each point i is similar to the points in its own cluster. It is defined as

$$s(x_i) = \frac{m(x_i) - a(x_i)}{\max[m(x_i), a(x_i)]}$$

where $a(x_i)$ is the average distance from the ith point to the points in the same cluster and $m(x_i)$ is the minimum average distance from the ith point to the points in different clusters, see Kaufman and Rousseeuw (2009). As distance measure we adopt the cityblock distance, to be consonant with the used k-mean algorithm employed. A negative silhouette value for some elements of the population suggests an inefficient choice of the number of clusters. The silhouette analysis yields Figs. 2, 3 and 4 for three, four and five clusters, respectively.

According to the silhouette approach, to precisely quantify the quality of a choice of the number of clusters, de Amorim and Hennig (2015) suggests using the silhouette index defined as $1/N \sum_{i=1}^{N} s(x_i)$. A higher silhouette index implies a better choice. In all the provided computations, the clustering input values were first normalized by dividing each series by its average. We tested two other normalization techniques. The first divides each series by its median, the second subtracts from each series its minimum and then divides by the difference between its maximum and its minimum. The silhouette indexes associated to each clustering choice and to each normalization technique are presented in Table 2. Both the mean-normalization and the median-normalization produce comparable results, identifying four clusters as the best choice. The minmax-normalization identifies three clusters as the best choice. In order to stay

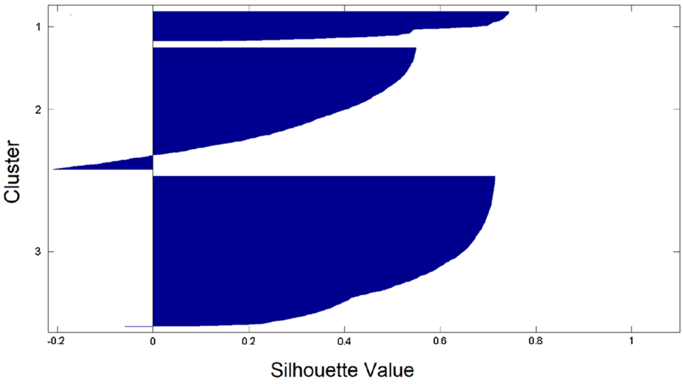


Fig. 2 Silhouette of the three-cluster case

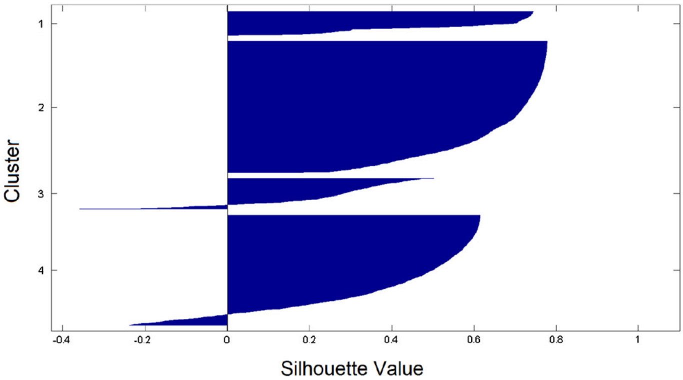


Fig. 3 Silhouette of the four-cluster case

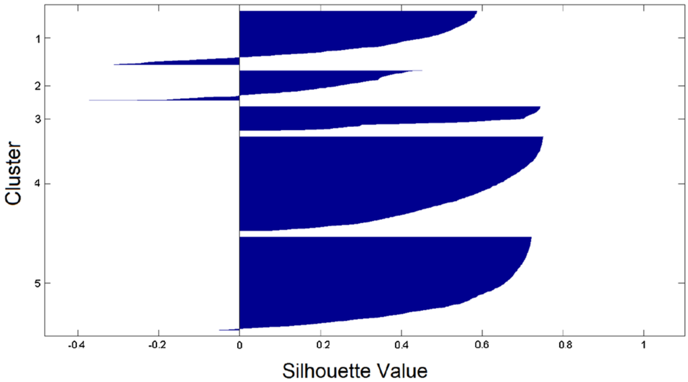


Fig. 4 Silhouette of the five-cluster case

closer to the pension fund provider's expectations and to produce a richer analysis, we adopt the results for the four clusters obtained with the mean-normalization. Therefore, all the provided figures, as well as the table for the centroids, were produced after the

Table 2 Silhouette index for each choice of number of clusters and each normalization type

Number of clusters	Silhouette index		
	Mean normalization	Median nomalization	MinMax nomalization
Three clusters	0.4818	0.4622	0.4744
Four clusters	0.5145	0.5234	0.3901
Five clusters	0.4946	0.5030	0.3572

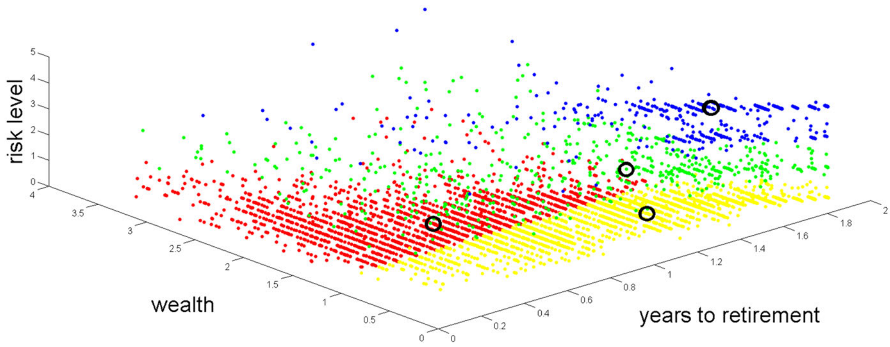


Fig. 5 Cluster analysis. Four-cluster case

mean-normalization. Figure 5 represents the data normalized by the mean and clustered into four clusters. The x-axis represents the wealth, the y-axis the remaining working years, and the z-axis the risk profile.

3 The optimal policies for the pension plans

The individual investment problem has been thoroughly studied in the last decades. The main feature of this class of models is to consider jointly all variables which characterize the investor's investment, i.e. the salary process, consumption, borrowings, etc. See, for example, Consiglio et al. (2004, 2007), Consigli (2007) and Medova et al. (2008). Moreover, much research has focused on the individual investment in a pension perspective framework.

Given the representative employees defined in Sect. 2.2, the optimal strategy for each pension fund will coincide with the optimal portfolio allocation of each representative investor. Therefore, a model to describe the pension problem for a private investor is needed. The aim of this procedure is to define the optimal asset allocation for an employee in a retirement perspective. We deal with two main features: a long term horizon with a fixed and given sequence of portfolio rebalancing stages, and an uncertain environment regarding the asset returns and the evolution of the salary. These elements lead naturally to a Multistage Stochastic approach (see Dupačová et al. 2002). The considered framework is a defined contribution pension fund. The distinction between defined contribution and defined benefit, especially in terms of the securities included, is described and analysed in Consiglio et al. (2015). Several asset classes are involved in a pension fund portfolio. Nevertheless, the most suitable in terms of a risk/reward profile are government and corporate bonds, see Lozza et al. (2013) and Abaffy et al. (2007). The robustness of a model can be measured by analysing its sensitivity as proposed in Bertocchi et al. (2000a, b).

We assume that the decision times correspond to all the stages except the last one, in which we just compute the accumulated final wealth. The stochasticity arises from two sources: the asset returns and the salary process. The investment universe is composed of n assets, which are the benchmarks that the fund manager is able to replicate. The first asset is a guaranteed capital investment with a risk free return. The risk free rate, r_t, is modeled in (1) assuming it follows the Vasicek process as proposed in Vašiček (1977). There are far more complex models of the interest rate, but Vasicek's allows for negative interest rates, which are nowadays a common circumstance. The asset return processes, $\mathbf{R}_t^R = (R_{t,1}, \ldots, R_{t,n})'$, are modeled in (2) as Geometric Brownian motions. The stochasticity of the salary is crucial in the definition of a consistent model for a private investor's optimal allocation. Based on the approach proposed in Cairns et al. (2006), in (3) we assume that the salary process, Y_t, is correlated with the riskiest assets and with the risk free dynamics.

$$dr_t = \alpha(\beta - r_t)dt + \nu dW_t^r, \quad \forall t \tag{1}$$

$$d\mathbf{R}_t = \boldsymbol{\mu}\mathbf{R}_t dt + \boldsymbol{\Sigma} d\mathbf{W}_t^R, \quad \forall t \tag{2}$$

$$dY_t = \omega Y_t dt + \sigma_r dW_t^r + \boldsymbol{\sigma}_R d\mathbf{W}_t^R, \quad \forall t \tag{3}$$

where W_t^r is the Wiener process for the Vasicek model and α, β and ν are its parameters, $\mathbf{W}_t^R = (W_{t,1}, \ldots, W_{t,n})'$ is the vector of the Wiener processes for the risky returns, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)'$ is the mean return vector for the risky assets, $\boldsymbol{\Sigma}$ is the variance-covariance matrix of the risky assets, ω is the growth rate of the salary, σ_r expresses the relation between the salary and the asset dynamics, and $\boldsymbol{\sigma}_R$ that between express the relation between the salary and the risk free process and the asset dynamics.

The stochasticity is represented by a discrete scenario tree composed of S paths and characterized by a regular branching.

We define the non-negative decision variables $c_{i,t,s}$, $r_{i,t,s}^+$ and $r_{i,t,s}^-$, where $i = 1, \ldots, n$ represents the assets, $t = t_0, \ldots, T$ the stages and $s = 1, \ldots, S$ the scenarios. Thus, $c_{i,t,s}$ expresses the level of contribution we want to invest in the asset i at the stage t in scenario s; the rebalancing variables $r_{i,t,s}^+$ and $r_{i,t,s}^-$ allow redistributing the accumulated wealth between the chosen assets, quantifying how much we buy and how much we sell of each asset at the beginning of each stage, i.e. before adding the contribution. We write $\rho_{i,t,s}$ for the asset return processes found by solving and applying (1) and (2), and $\rho_{t,s}^{sal}$ for the salary growth rate process given by (3).

Lastly, we can list the set of constraints in order to express the regulatory bounds and the cash balance conditions.

Salary process Fixing the initial level $sal_{t_0,s}$ equal to the actual salary of the employee, we can easily describe the salary process.

$$sal_{t,s} = sal_{t-1,s} \cdot (1 + \rho_{t,s}^{sal}), \quad \forall t > t_0, \forall s \tag{4}$$

Maximum contribution level At each stage, the employee does not want to invest more than a certain maximum percentage of the employee's salary. Therefore, we introduce the parameter propensity-to-save denoted by λ and a coefficient e which represents a percentage supplementary contribution added by the employer. Moreover, the time structure of the problem defines the stages every Δt years, but in real life the contribution is added yearly (sometimes also monthly) to the pension plan. Therefore, assuming that the growth rate of the salary is constant over each period and equal to the discount rate, and assuming that the contribution is paid at the beginning of each year, we compute the actual value of a growing annuity paying one euro for Δt years by simply multiplying by Δt. Thus, the constraint describing the maximum contribution level assumes the form

$$\sum_{i=1}^{n} c_{i,t,s} \leq sal_{t,s} \cdot \lambda \cdot (1+e) \cdot \Delta t, \quad \forall t, \forall s \tag{5}$$

Portfolio balance We define the set of constraints that describes the portfolio allocation, the rebalancing decisions and the wealth account. For this purpose, we introduce the holding variable $h_{i,t,s}$ which represents the amount we hold in each asset, and the total wealth variable $w_{t,s}$. Moreover, we define the *initial portfolio* vector $h_{i,0}$ in case the investor already has a position in the pension plan and the *initial cash* parameter w_0 if the investor wants to add an amount of money, i.e. shift something from another pension plan and/or make an initial extra contribution.

$$h_{i,t_0,s} = h_{i,0} + r^{+}_{i,t_0,s} - r^{-}_{i,t_0,s} + c_{i,t_0,s}, \quad \forall i, \forall s \tag{6}$$

$$\sum_{i=1}^{n} r^{+}_{i,t_0,s} = \sum_{i=1}^{n} r^{-}_{i,t_0,s} + w_0, \quad \forall s \tag{7}$$

$$r^{-}_{i,t_0,s} \leq h_{i,0}, \quad \forall i, \forall s \tag{8}$$

$$\sum_{i=1}^{n} r^{-}_{i,t_0,s} \leq \theta \sum_{i=1}^{n} h_{i,0}, \quad \forall s \tag{9}$$

$$h_{i,t,s} = h_{i,t-1,s} \cdot (1+\rho_{i,t,s}) + r^{+}_{i,t,s} - r^{-}_{i,t,s} + c_{i,t,s}, \quad \forall i, t_0 < t < T, \forall s \tag{10}$$

$$\sum_{i=1}^{n} r^{+}_{i,t,s} = \sum_{i=1}^{n} r^{-}_{i,t,s}, \quad t_0 < t < T, \forall s \tag{11}$$

$$r^{-}_{i,t,s} \leq h_{i,t-1,s} \cdot (1+\rho_{i,t,s}), \quad \forall i, t_0 < t < T, \forall s \tag{12}$$

$$\sum_{i=1}^{n} r^{-}_{i,t,s} \leq \theta \cdot w_{t,s}, \quad t_0 < t < T, \forall s \tag{13}$$

$$w_{t,s} = \sum_{i=1}^{n} \left(h_{i,t-1,s} \cdot (1+\rho_{i,t,s})\right), \quad t > t_0, \forall s \tag{14}$$

Equation (6) defines the holding in the first stage for each asset to be equal to the initial portfolio allocation $h_{i,0}$ plus $r^{+}_{i,t_0,s}$ and $r^{-}_{i,t_0,s}$, which are the buying and selling of the initial portfolio and the buying and selling of the initial wealth, plus the first period contribution $c_{i,t_0,s}$. The initial portfolio re-allocation is defined using Eqs. (7)–(9). In particular, Eq. (7) defines the buying as the re-allocation of the initial portfolio plus the allocation of the initial wealth. For the next stages, Eq. (10) defines the holding as the capitalization of the previous holding for each asset plus the re-allocation of the accumulated wealth and plus the contribution. The portfolio re-allocation follows Eqs. (11), (12) and (13). Equations (9) and (13) express the turnover constraints through the parameter θ which states that it is not possible to sell more than a fixed percentage θ of the portfolio. Lastly, Eq. (14) computes the accumulated wealth in each stage and for each scenario. We build the target constraints and the objective function in terms of this wealth variable. Moreover, we include a risk exposure constraint. This constraint, and specifically its linear formulation, has been explicitly required by the bank and assigns to each asset a risk coefficient ϕ_i and then sets a risk level Φ that the portfolio cannot exceed on average. This threshold level is extracted from the risk/reward profile of the subscriber.

$$\sum_{i=1}^{n} h_{i,t,s} \cdot \phi_i \leq \Phi \cdot \sum_{i=1}^{n} h_{i,t,s}, \quad \forall t, \forall s \tag{15}$$

Since we use a stochastic tree structure, see Dupačová et al. (2009), we include the set of all the non-anticipativity constraints on the decision variables to ensure that the decision variables depend only on the past values of the stochastic processes. Therefore, we include the set of non-anticipativity constraints

$$c_{i,t,\dot{s}} = c_{i,t,\tilde{s}}, \quad \forall i, \forall t \tag{16}$$

$$r^{+}_{i,t,\dot{s}} = r^{+}_{i,t,\tilde{s}}, \quad \forall i, \forall t \tag{17}$$

$$r^{-}_{i,t,\dot{s}} = r^{-}_{i,t,\tilde{s}}, \quad \forall i, \forall t \tag{18}$$

for each pair of scenarios $\dot{s}$ and $\tilde{s}$ which have shared the same history up to stage t.

As suggested in Kilianová and Pflug (2009), we define the multicriteria objective function including two wealth targets and the Average Value at Risk Deviation ($AV@RD$) as the risk measure, where $AV@RD(x) = \mathbb{E}(x) - AV@R(x)$. We adopt the ϵ-constrained approach

$$\min \quad \sum_{s=1}^{S} \left(w_{T,s} \cdot p_s\right) - a + \frac{1}{\alpha} \sum_{s=1}^{S} (z_s \cdot p_s) \tag{19}$$

$$\text{s.t.} \quad -a + w_{T,s} + z_s \geq 0, \qquad z_s \geq 0, \forall s \tag{20}$$

$$\sum_{s=1}^{S} w_{T,s} \cdot p_s \geq \Pi_T \tag{21}$$

$$(5)\text{—}(15) \tag{22}$$

In (19) we minimize the $AV@RD$ at last stage, i.e. the final wealth, for the given confidence level α. According to Rockafellar and Uryasev (2000, 2002), the discrete definition of the $AV@RD$ needs the inequality (20) in order to define jointly the variables a and z_s. The wealth target (21) requires that the average accumulated wealth at the final stage be greater than or equal to a fixed amount Π_T. In order to compute this value, the definition of a benchmark wealth $w^b_{t,s}$ is needed. We suppose that we have a benchmark portfolio with returns $\rho^b_{t,s}$ which invests uniformly (see DeMiguel et al. 2009) only in those assets which singly satisfy the risk exposure constraint, i.e. $I = \{i|\phi_i \leq \Phi\}$, then $\rho^b_{t,s} = 1/|I| \sum_{i \in I} \rho_{i,t,s}$, $\forall t$, $\forall s$. Then, assuming that the contribution touches the bound in (5), i.e.

$$C^b_{t,s} = sal_{t,s} \cdot \lambda \cdot (1 + e) \cdot \Delta t, \quad \forall t, \forall s \tag{23}$$

and starting from the initial wealth, i.e. $w^b_{t_0,s} = \sum_{i=1}^{n} h_{i,0} + w_0$, $\forall s$, the evolution of the benchmark wealth is

$$h^b_{t_0,s} = w^b_{t_0,s} + C^b_{t_0,s}, \quad \forall s \tag{24}$$

$$h^b_{t,s} = h^b_{t-1,s} \cdot (1 + \rho^b_{t,s}) + C^b_{t_0,s}, \quad t > t_0, \forall s \tag{25}$$

$$w^b_{t,s} = h^b_{t-1,s} \cdot (1 + \rho^b_{t,s}), \quad t > t_0, \forall s. \tag{26}$$

Then, the value of the target becomes

$$\Pi_T = \mathbb{E}[w^b_{T,s}] \tag{27}$$

The $AV@RD$ minimization is a general formulation and has been implemented in order to have a general model that the bank could use to test the sensitivity to different settings. However, since (21) already pushes the average wealth to a relatively high target for our particular setting, such a formulation could be replaced with the maximization of the $AV@R$. Moreover, since we have asset return scenarios generated from a multi-variate normal distribution, the typical variance minimization must also produce almost the same results. We propose the minimization of the $AV@RD$ because it is more general, in case the bank were to decide to test the sensitivity with respect to different types of target wealth or in case the bank were to adopt scenarios generated from a non-symmetric distribution. With the current settings, we ran tests adopting alternatively one of the three formulations, producing no significant difference in the solutions. We also highlight that the variance minimization would lead to a quadratic problem, while the $AV@RD$ minimization gives a linear programming problem.

Using an Intel(R) Xeon(R) 2.40 GHz with 8.00 GB RAM virtual machine running Windows 8.1, we analysed the computational complexity for 6-stage problems with various branchings and numbers of scenarios. The results are summarized in Table 3.

Table 3 Computation statistics of the model with $AV@RD$ objective

Branching	10–10–5–5–2–2	10–10–10–5–2	10–10–10–10–5	10–10–10–10–10	50–20–10–5–2
Scenarios	1000	10,000	50,000	100,000	100,000
Nodes	1811	16,111	61,111	111,111	161,051
Equations	30,789	273,889	1,038,889	1,888,889	2,737,868
Variables	40,276	352,776	1,277,776	2,277,776	3,526,276
Execution time	<1 m	<2 h	<10 h	<55 h	<60 h
Iteration count	47,941	580,140	1,863,079	1,900,583	4,784,633

Table 4 Vasicek model estimation on the Euribor series

	Whole period from 1 Jan 1999 to 13 Mar 2015	Pre-crisis period from 1 Jan 1999 to 31 Dec 2008	Post-crisis period from 1 Jan 2009 to 13 Mar 2015
α	0.066	0.065	1.634
β	0	0.025	0.003
ν	0.003	0.004	0.002

4 Settings and results

The proposed model has been applied to the four representative members defined by the cluster analysis in order to identify the four optimal pension funds that should be issued by the sponsor of the pension plan. Let us assume that the pension fund manager is able to replicate six different securities (assets) which compose the investment universe we deal with. Each pension fund is a combination of these assets, which are: a guaranteed capital security, two low risk, one medium risk and two high risk assets. For the risk free asset, we analysed the historical series of the Euribor 3-months from 1 Jan 1999 to 1 Sep 2015. We performed a maximum likelihood estimation for all the parameters and we obtained the results reported in Table 4 according to three different estimation periods. In particular, we split the whole Euribor historical data into a pre-crisis period and a post-crisis period.

The estimations made for the post-crisis period and for the whole period produce a very low value of the parameter β. Since an equilibrium interest rate equal to 2.5 % seems more reasonable for a long-term horizon problem, in (1) we adopt the parameters of the Vasicek model estimated for the pre-crisis period.

We artificially create a universe of risky assets to cover a larger number of profiles. For that reason, we assume that their dynamics, as described in (2), follow a multivariate normal distribution characterized by the following statistics.

$$\boldsymbol{\mu} = \begin{bmatrix} 1.5\,\% \\ 2.0\,\% \\ 4.5\,\% \\ 5.0\,\% \\ 5.5\,\% \end{bmatrix} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} 1.5\,\% \\ 2.0\,\% \\ 9.5\,\% \\ 10.0\,\% \\ 10.5\,\% \end{bmatrix}^{\top} \begin{bmatrix} 1 & 0.9 & -0.1 & -0.1 & -0.1 \\ 0.9 & 1 & 0 & 0 & 0 \\ -0.1 & 0 & 1 & 0.9 & 0.8 \\ -0.1 & 0 & 0.9 & 1 & 0.9 \\ -0.1 & 0 & 0.8 & 0.9 & 1 \end{bmatrix} \begin{bmatrix} 1.5\,\% \\ 2.0\,\% \\ 9.5\,\% \\ 10.0\,\% \\ 10.5\,\% \end{bmatrix}$$

Here, $\boldsymbol{\mu}$ is the vector of mean returns of the risky asset in (2) and $\boldsymbol{\Sigma}$ is the variance-covariance matrix given by the product of the vector of the volatilities with the correlation matrix. Moreover, the risk coefficients associated to each asset are

$$\phi_i = [0 \;\; 1 \;\; 2 \;\; 3 \;\; 7 \;\; 8]$$

The salary process in (3) is characterized by the parameters

$$\omega = 1.0\,\% \quad \sigma^r = 0.5 \quad \boldsymbol{\sigma}^R = \begin{bmatrix} 0 \\ 0 \\ 0.9 \\ 0.9 \\ 0.9 \end{bmatrix}$$

In (5) the propensity-to-save parameter λ is 7 %, while the employer contribution e is 50 %. In (13) the turnover coefficient θ is 20 %. Moreover, we left the solver free to find the best here-and-now solution by setting to zero the initial portfolio, i.e. $h_{i,0} = 0 \ \ \forall i$, and accumulating the whole wealth as an extra initial contribution w_0. The initial salary is fixed for each representative participant to EUR 15,000, i.e. $sal_{t_0,s} = 15{,}000$. This choice is driven by the evidence for a highly dishomogeneous salary level among the cluster elements, thus, we adopt as a fixed initial salary the average net salary of the whole population. In the multicriteria objective function (19), the average value at risk deviation ($AV@RD$) is computed considering the confidence level $\alpha = 5\,\%$.

We propose different time lengths between stages depending on the representative member we are going to consider, and with the same tree branching 50–20–10–5–2, i.e. 100,000 scenarios. In all the figures, the white asset represents the guaranteed capital security. Then, going from the dark green to the dark red, each color represents a specific asset in a rising scale of risk/reward profile.

The first one is characterized by an initial wealth of EUR 104,000, $w_0 = 104{,}000$, by a very low risk profile, $\Phi = 1$, and by 13 remaining working years. We assume the time length between the 6 stages as follows: 1, 2, 2, 3 and 5 years. The dynamic optimal allocation evolution is depicted in Fig. 6. In particular, for each stage we show the average allocation over all the nodes of that stage. The evolution of the wealth through the stages (in terms of the average over the nodes belonging to each stage) is presented in Fig. 7. Figure 8 describes the distribution of the final wealth and its basic statistics for the first representative member.

The here-and-now allocation is predominantly focused on the less risky assets. The riskiest asset is always present in a very small portion. The strategy moves to a safer portfolio, increasing the investment in the guaranteed capital asset, which is less than 15 % in the here-and-now solution but almost 50 % in the last decisional stage. The evolution of the wealth is consonant with the allocation. We observe a slight growth, mainly due to the contribution and residually to the financial gains. A relatively short time horizon does not allow for a remarkable rebalance through the stages.

Thank to the choice of a safe allocation, the optimal solution achieves a great reduction in the risk profile of the solution. The standard deviation is 30 % lower than for the benchmark, and the distribution is highly concentrated on the target value. The mean and the median are very close to the value of $AV@R$, highlighting the quality of the solution with respect to the benchmark portfolio.

The second representative member is characterized by an initial wealth of EUR 43,000, $w_0 = 43{,}000$, by a very low risk profile, $\Phi = 1$, and by 27 remaining working years. We put the time lengths between the 6 stages as 1, 2, 5, 8 and 11 years. The dynamic optimal allocation evolution is depicted in Fig. 9. The evolution of the

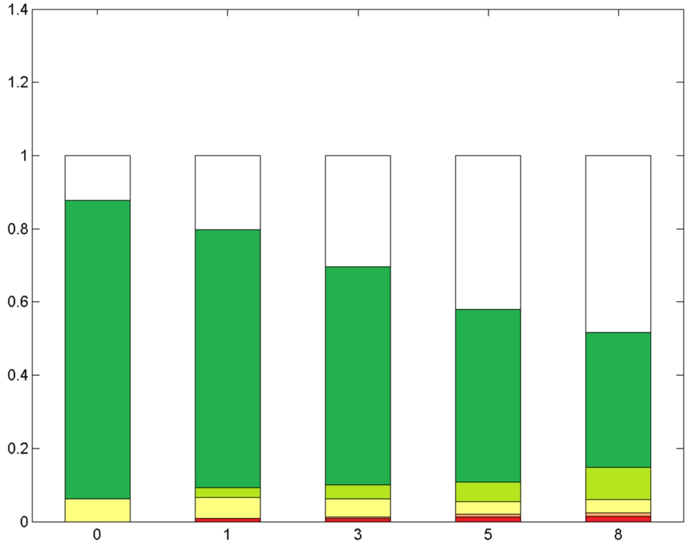


Fig. 6 First representative member: evolution of the allocation

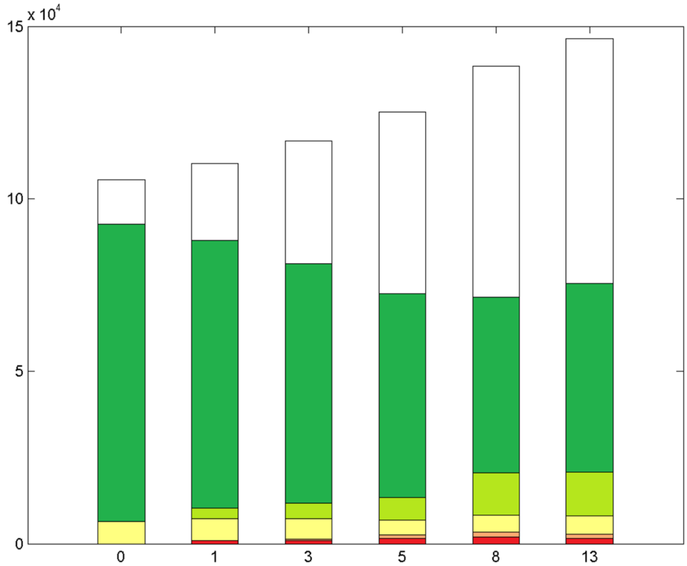
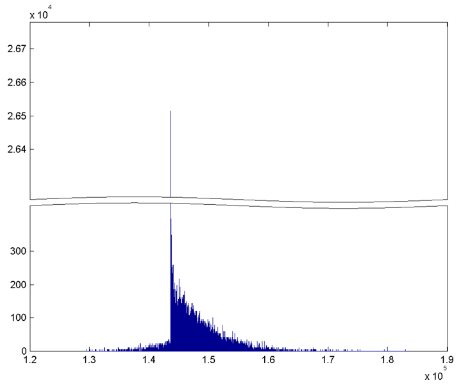


Fig. 7 First representative member: evolution of the wealth

wealth is presented in Fig. 10. Figure 11 describes the distribution of the final wealth and its basic statistics for the second representative member.

The allocation is slightly more aggressive than the first representative here-and-now allocation with only 4 % invested in the risk free asset. The strategy moves to an even safer position in the last stage, with more than 40 % in the risk free asset. The wealth process has great growth, induced mainly by the long-term horizon and then



	Opt Ptflio	Bnchmk Ptflio
mean	146,397	146,397
median	145,510	146,258
st. dev.	4097	5946
$V@R_{0.05}$	142,920	136,857
$AV@R_{0.05}$	140,630	134,575
kurtosis	4.84	3.01
skewness	0.96	0.13

Fig. 8 Final wealth distribution and related statistics for the first representative member

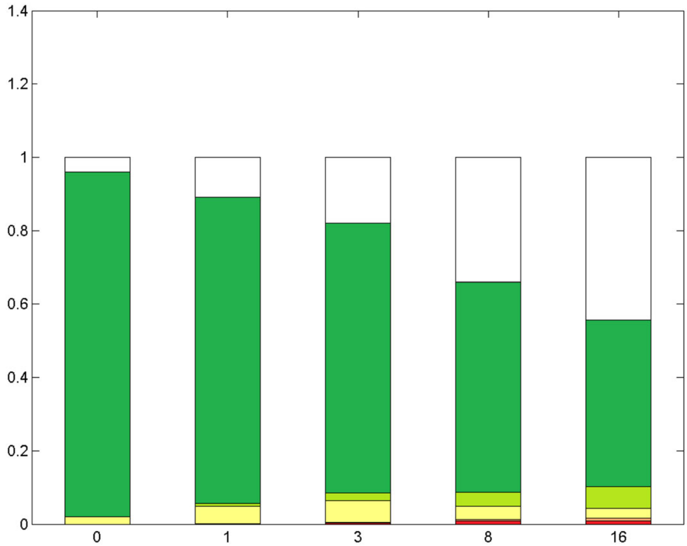


Fig. 9 Second representative member: evolution of the allocation

by the contribution. The financial gains are quite low because of the very low risk allocation.

The final wealth is again very concentrated around the mean value. The reduction of the risk in terms of both standard deviation and $AV@R$ is remarkable. As for the first representative, the whole dynamic is in line the very low risk/reward profile of the investor and this allows the optimal solution to achieve the target and to reduce effectively the objective function, i.e. the $AV@RD$. The $V@R$ and $AV@R$ are 7 % higher than for the benchmark.

The third representative member is characterized by an initial wealth of EUR 66,000, $w_0 = 66{,}000$, by a medium risk profile, $\Phi = 3$, and by 28 remaining working years. We define the lengths between the 6 stages as follows: 1, 2, 5, 8 and 12 years.

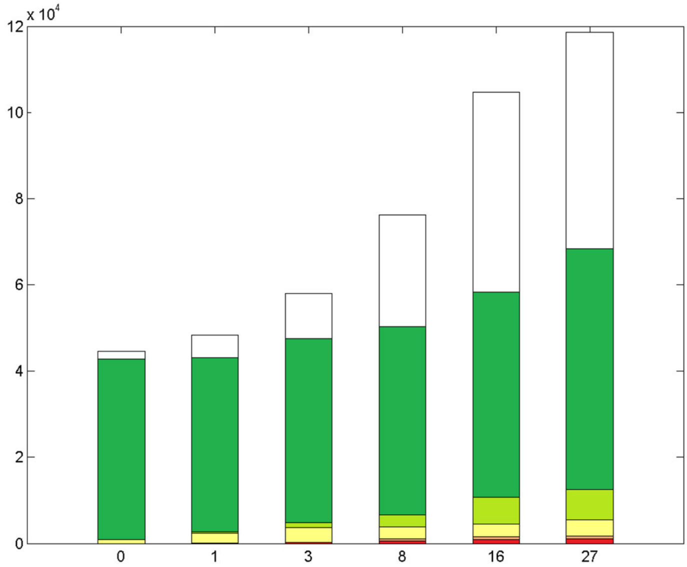
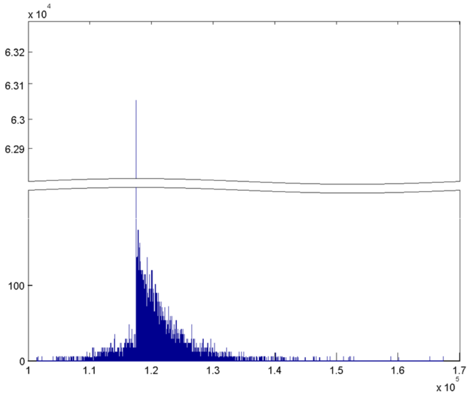


Fig. 10 Second representative member: evolution of the wealth



	Opt Ptflio	Bnchmk Ptflio
mean	118,576	118,576
median	116,710	118,383
st. dev.	3800	6452
$V@R_{0.05}$	116,440	108,329
$AV@R_{0.05}$	114,090	105,966
kurtosis	14.85	3.05
skewness	2.34	0.19

Fig. 11 Final wealth distribution and related statistics for the second representative member

The dynamic optimal allocation evolution is depicted in Fig. 12. The evolution of the wealth is presented in Fig. 13. Figure 14 describes the distribution of the final wealth and its basic statistics for the third representative member. In the here-and-now solution, only the four riskiest assets are included. The most used asset is still a low risk one, but the allocation includes the riskiest assets at almost 20 %. Then, getting close to the end horizon, the portfolio moves to the two less risky assets and reduces the portion invested in the two more risky ones. The evolution of the wealth profits from both the long-term horizon and the quite aggressive allocation. The contributions are substantial and also the financial gains.

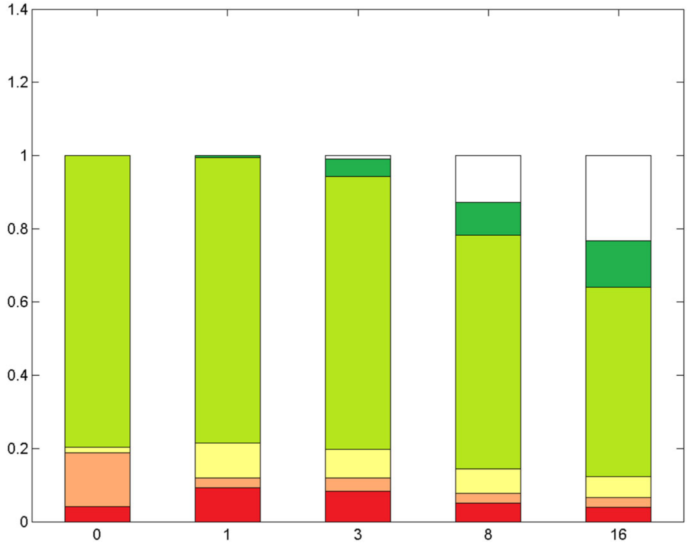


Fig. 12 Third representative member: evolution of the allocation

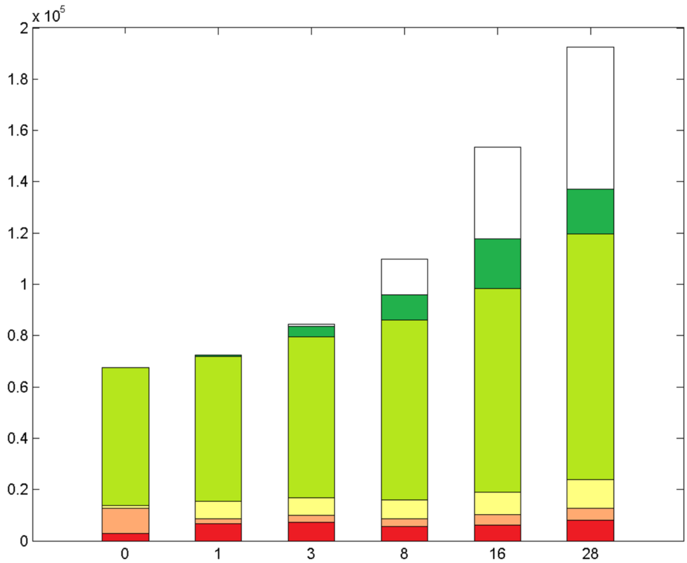
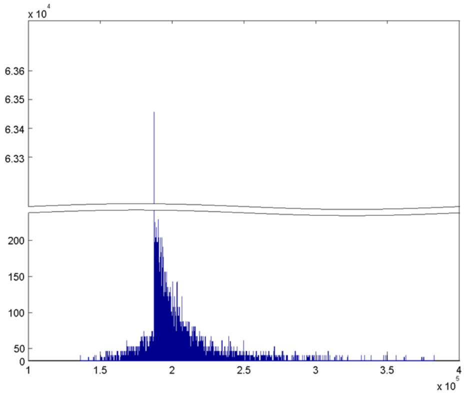


Fig. 13 Third representative member: evolution of the wealth

The final wealth of the third representative highlights a higher riskiness than the first two representatives. The standard deviation is almost 4 times that of the previous two representatives and the $AV@RD$ is larger too. Nevertheless, the solution is aligned with the risk/reward profile of the investor, who reaches the wealth targets over the stages and still obtains a low risk final performance. The standard deviation is almost



	Opt Ptflio	Bnchmk Ptflio
mean	192,566	192,566
median	185,010	188,974
st. dev.	15965	28994
$V@R_{0.05}$	184,420	152,102
$AV@R_{0.05}$	177,260	144,769
kurtosis	26.27	4.33
skewness	3.52	0.82

Fig. 14 Final wealth distribution and related statistics for the third representative member

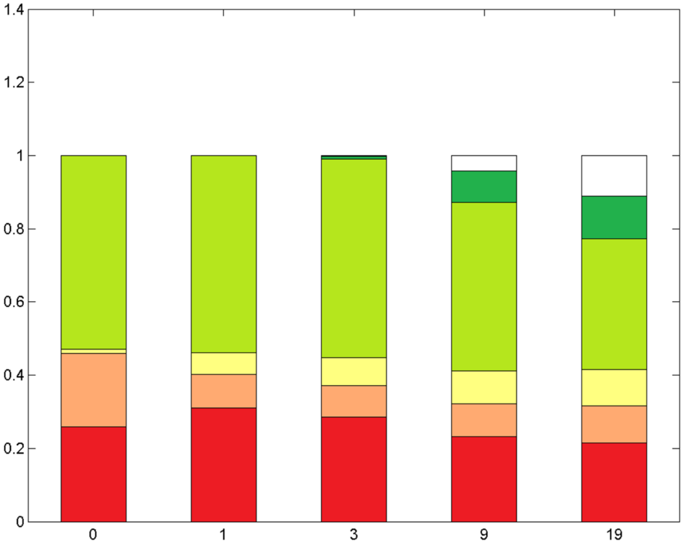


Fig. 15 Fourth representative member: evolution of the allocation

half that of the benchmark and the $V@R$ and the $AV@R$ are the 20 % higher than for the benchmark.

The fourth representative member is characterized by an initial wealth of EUR 38,000, $w_0 = 38{,}000$, by a high risk profile, $\Phi = 10$, and by 33 remaining working years. We assume that the time lengths between the six stages are 1, 2, 6, 10 and 14 years. The dynamic optimal allocation evolution is depicted in Fig. 15. The evolution of the wealth is presented in Fig. 16. Figure 17 describes the distribution of the final wealth and its basic statistics for the fourth representative member. The here-and-now allocation of the fourth representative is extremely risky. Almost 50 % is invested in the riskiest assets. The guaranteed capital asset is introduced from the fourth stage and its portion increases in the last decisional stage, reaching only 10 %. On the other hand,

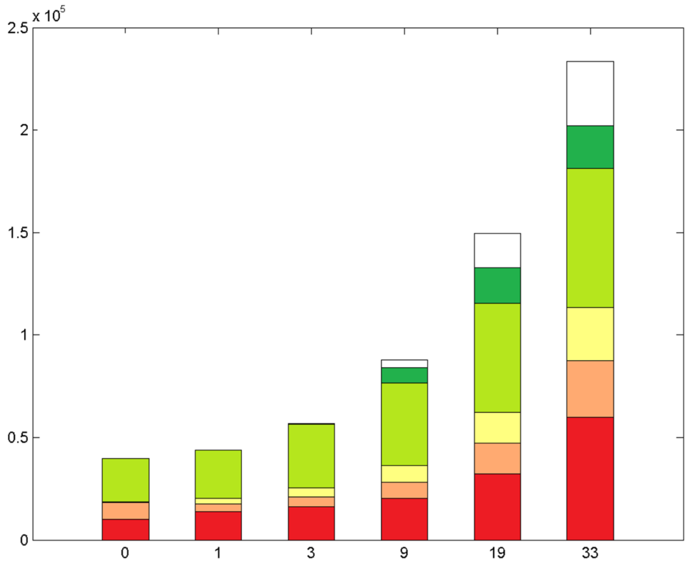
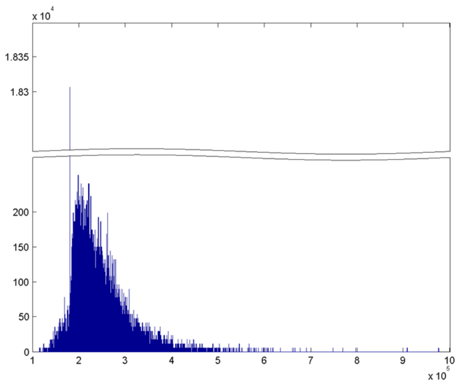


Fig. 16 Fourth representative member: evolution of the wealth



	Opt Ptflio	Bnchmk Ptflio
mean	233,531	233,531
median	219,280	221,217
st. dev.	59877	70839
$V@R_{0.05}$	177,300	143,513
$AV@R_{0.05}$	164,330	130,578
kurtosis	13.85	6.03
skewness	2.24	1.27

Fig. 17 Final wealth distribution and related statistics for the fourth representative member

as for the other representatives, the allocation reduces the risk profile by increasing the investment in the two low risk assets. The wealth process has a huge financial return accompanied by a high contribution. The investment in the riskiest asset is constant as a percentage allocation, but increases in monetary terms.

The final wealth distribution of the fourth representative, like the previous ones, is very concentrated on the mean value. The standard deviation is 18 % lower than that for the benchmark, while the $V@R$ and the $AV@R$ are more than 20 % higher. Therefore, the optimal solution guarantees achieving a less risky result than the benchmark portfolio and produces a high final wealth satisfying the targets and in accordance with the investor's risk/reward profile.

Table 5 Sensitivity analysis with respect to the turnover parameter

	$\theta = 0.10$	$\theta = 0.15$	$\theta = 0.20$	$\theta = 0.25$	$\theta = 0.30$
Wealth statistics					
Mean	233,531	233,531	233,531	233,531	233,531
Median	218,154	216,612	219,280	217,672	217,408
St. dev.	62,166	62,426	59,877	58,226	56,284
$V@R_{0.05}$	174,191	176,882	177,300	183,853	186,785
$AV@R_{0.05}$	162,054	163,497	164,330	168,786	170,652
Kurtosis	9.93	10.31	13.85	13.88	14.58
Skewness	1.81	1.96	2.24	2.34	2.53
H&N allocation					
Guaranteed capital	0 %	0 %	0 %	0 %	0 %
Low risk 1	0 %	0 %	0 %	0 %	0 %
Low risk 2	69.5 %	63.5 %	52.9 %	51.7 %	48.4 %
Medium risk	2.6 %	4.3 %	1.2 %	2.6 %	2.7 %
High risk 1	7.5 %	10.1 %	20.1 %	20.4 %	19.8 %
High risk 2	20.5 %	22.0 %	25.8 %	25.4 %	29.0 %

5 Parameter sensitivity analysis

Since the propensity-to-save λ and the employer contribution e affect both the optimal portfolio and the benchmark portfolio in the same way, see Eqs. (5) and (23), the optimal dynamic solution is not sensitive to these parameters. Further research could consider the adoption of an alternative and unlinked benchmark to test such sensitivity.

Given the setting of the fourth representative, whose risk profile allows a wide diversification, we tested the sensitivity of the optimal solution to the turnover parameter θ and to the years-to-retirement. Tables 5 and 6 show the here-and-now solutions and final wealth statistics for a set of turnover values and stage lengths. A higher turnover coefficient implies a more flexible model, i.e. a portfolio which can more easily change its composition through the stages. Then, such flexibility results in a riskier here-and-now allocation, which is consonant with a strategy which seeks wealth in the first stages and moves to a safer position to reduce the risk at the end horizon. As remarked by the final wealth statistics, despite the riskier here-and-now allocation, a more flexible portfolio can more efficiently reduce the risk and squeeze the left tail. Conversely, a higher turnover induces a jumping strategy along the stages. For all the proposed turnover coefficients the average final wealth corresponds to the benchmark portfolio average final wealth.

A change in the end horizon, i.e. in the structure of the lengths of the stages, shows a reasonably corresponding behavior of the solution. A short time horizon does not allow a dynamic strategy to effectively reduce the risk. Thus, the here-and-now solution must already be less risky than in the case of a longer horizon, that is, a longer horizon permits a more aggressive initial strategy. The final wealth statistics reflect both the

Table 6 Sensitivity analysis with respect to the years-to-retirement structure

Years-to-retirement	23	28	33	38	43
Stage lengths	1–2–4–6–10	1–2–5–8–12	1–2–6–10–14	1–2–7–12–16	1–2–8–14–18
Wealth statistics					
Mean	143,420	183,313	233,531	293,801	369,613
Median	137,631	175,880	219,280	270,854	328,110
St. dev.	26,067	36,558	59,877	85,225	13,0511
$V@R_{0.05}$	116,900	145,651	177,300	224,530	281,594
$AV@R_{0.05}$	110,692	136,174	164,330	203,611	248,560
Kurtosis	5.53	9.80	13.85	14.61	20.36
Skewness	1.32	1.64	2.24	2.45	3.14
H&N allocation					
Guaranteed capital	0 %	0 %	0 %	0 %	0 %
Low risk 1	0 %	0 %	0 %	0 %	0 %
Low risk 2	76.4 %	66.3 %	52.9 %	48.4 %	34.3 %
Medium risk	0 %	4.1 %	1.2 %	6.0 %	5.5 %
High risk 1	13.0 %	11.9 %	20.1 %	13.2 %	14.4 %
High risk 2	10.6 %	17.8 %	25.8 %	32.3 %	45.8 %

riskiness of the strategy and the time horizon, highlighting higher *AV@RD*, higher wealth, and larger dispersion for longer term strategies.

6 Conclusion

In conclusion, nowadays, a quantitative approach is strongly recommended not only to manage a single pension fund in an ALM perspective, but also to define the optimal pension funds offered by pension plan sponsors. We propose a two-step approach to address and solve this problem, considering the case in which the pension fund is issued for a homogeneous group of people. In the first step, we use a cluster analysis in order to analyse the population and identify a set of representative members. Then, for each representative, the optimal here-and-now allocation (the first stage decision of a multistage stochastic problem) in a dynamic pension perspective strategy is found. Each of these optimal here-and-now allocations describes a pension fund that the pension provider should issue.

For the analysed population, having 4 representatives was found to be optimal. The corresponding 4 here-and-now solutions, one for each representative, are reported in Table 7.

The percentage allocation is similar for funds A and B. Both invest a high percentage in the two lower risk assets (93.9 and 98.0 % respectively) and only a residual percentage in the medium risk asset. Fund C moves to a more balanced allocation by investing 79.7 % in a lower risk asset and 20.3 % in the three most risky assets. The most aggressive pension fund is D which allocates more than 45 % in the two riskiest assets. Analysing the wealth evolutions, it is clear that young representative members

Table 7 Summary of the allocations of the pension funds

	Fund A (%)	Fund B (%)	Fund C (%)	Fund D (%)
Guaranteed capital	12.3	4.0	0	0
Low risk 1	81.6	94.0	0	0
Low risk 2	0	0	79.7	52.9
Medium risk	6.1	2.0	1.4	1.2
High risk 1	0	0	14.8	20.1
High risk 2	0	0	4.1	25.8

can afford more risky positions and achieve higher returns than older investors. The main driver of the allocation is still the risk/reward profile of the investor no matter if he/she is young or old. Consequently, the final wealth distribution reflects the portfolio risk attitude.

The number of pension funds equals the number of clusters. Clearly, the pension plan provider has to decide whether the proposed pension funds are sufficiently different from each other, in order to justify the implementation of all of them. The correct balance between the pension plan sponsor's effort and the members' satisfaction is hard to attain, but the proposed method may provide many hints. In the case of our data, a simple cluster analysis suggests considering four representatives. However, the optimal allocations show that maybe three funds are enough. The final decision about the number of funds is up to the sponsor. Anyway, we can conclude that either three or four pension funds should be issued.

References

Abaffy J, Bertocchi M, Dupačová J, Moriggia V, Consigli G (2007) Pricing nondiversifiable credit risk in the corporate Eurobond market. J Bank Financ 31(8):2233–2263

Berger AJ, Mulvey JM (1998) The home account advisor: asset and liability management for individual investors

Bertocchi M, Dupačová J, Moriggia V (2000a) Sensitivity analysis of a bond portfolio model for the Italian market. Control Cybern 29:595–615

Bertocchi M, Moriggia V, Dupačová J (2000b) Sensitivity of bond portfolio's behavior with respect to random movements in yield curve: a simulation study. Ann Oper Res 99(1–4):267–286

Bovenberg L, Knaap T (2005) Ageing, funded pensions and the Dutch economy

Cairns AJG, Blake D, Dowd K (2006) Stochastic lifestyling: optimal dynamic asset allocation for defined contribution pension plans. J Econ Dyn Control 30(5):843–877

Cariño DR, Ziemba WT (1998a) Formulation of the Russell–Yasuda Kasai financial planning model. Oper Res 46(4):433–449

Cariño DR, Kent T, Myers DH, Stacy C, Sylvanus M, Turner AL, Watanabe K, Ziemba WT (1994) The Russell–Yasuda Kasai model: an asset/liability model for a Japanese insurance company using multistage stochastic programming. Interfaces 24(1):29–49

Cariño DR, Myers DH, Ziemba WT (1998b) Concepts, technical issues, and uses of the Russell–Yasuda Kasai financial planning model. Oper Res 46(4):450–462

Consigli G (2007) Individual asset liability management for individual investors. Handbook of asset and liability management, vol B, pp 752–827

Consigli G, Dempster MAH (1998a) The CALM stochastic programming model for dynamic asset-liability management. Worldw Asset liabil Model 10:464

Consigli G, Dempster MAH (1998b) Dynamic stochastic programming for asset-liability management. Ann Oper Res 81:131–162
Consigli G, Moriggia V (2014) Applying stochastic programming to insurance portfolios stress-testing. Quant Financ Lett 2(1):7–13
Consigli G, di Tria M (2012) Optimal long-term property and casualty ALM with risk capital control. Asset Liabil Manage Financ Inst 1:137–150
Consigli G, di Tria M, Gaffo M, Iaquinta G, Moriggia V, Uristani A (2011) Dynamic portfolio management for property and casualty insurance. In: Stochastic optimization methods in finance and energy. Springer, pp 99–124
Consiglio A, Cocco F, Zenios SA (2004) http://www.personal_asset_allocation. Interfaces 34(4):287–302
Consiglio A, Cocco F, Zenios SA (2007) Scenario optimization asset and liability modelling for individual investors. Ann Oper Res 152(1):167–191
Consiglio A, Tumminello M, Zenios SA (2015) Designing guarantee options in defined contributions pension plans. The Wharton Financial Institutions Center
de Amorim RC, Hennig C (2015) Recovering the number of clusters in data sets with noise features using feature rescaling factors. Inf Sci 324:126–145
DeMiguel V, Garlappi L, Uppal R (2009) Optimal versus naive diversification: how inefficient is the 1/n portfolio strategy? Rev Finan Stud 22(5):1915–1953
Dondi G, Herzog F, Schumann LM, Geering HP (2007) Dynamic asset and liability management for Swiss pension funds. In: Zenios SA, Ziemba WT (eds) Handbook of asset and liability management: applications and case studies, North-Holland Finance Handbook Series. Elsevier, USA, pp 963–1028
Dupačová J, Polívka J (2009) Asset-liability management for Czech pension funds using stochastic programming. Ann Oper Res 165(1):5–28
Dupačová J, Hurt J, Štěpán J (2002) Stochastic modeling in economics and finance. Applied optimization. Springer, Berlin
Dupačová J, Bertocchi M, Moriggia V (2009) Testing the structure of multistage stochastic programs. Comput Manag Sci 6(2):161–185
Fabozzi FJ, Focardi SM, Jonas CL (2005) Market experience with modeling for defined-benefit pension funds: evidence from four countries. J Pension Econ Financ 4(3):313–327
Geyer A, Ziemba WT (2008) The Innovest Austrian pension fund financial planning model InnoALM. Oper Res 56(4):797–810
Geyer A, Hanke M, Weissensteiner A (2009) Life-cycle asset allocation and consumption using stochastic linear programming. J Comput Financ 12(4):29–50
Gomes FJ, Kotlikoff LJ, Viceira LM (2008) Optimal life-cycle investing with flexible labor supply: a welfare analysis of life-cycle funds. Tech. rep, National Bureau of Economic Research
Haneveld WKK, Streutker MH, van der Vlerk MH (2010a) An ALM model for pension funds using integrated chance constraints. Ann Oper Res 177(1):47–62
Haneveld WKK, Streutker MH, van der Vlerk MH (2010b) Indexation of Dutch pension rights in multistage recourse ALM models. IMA J Manage Math 21(2):131–148
Hilli P, Koivu M, Pennanen T, Ranne A (2007) A stochastic programming model for asset liability management of a Finnish pension company. Ann Oper Res 152(1):115–139
Høyland K, Wallace SW (2001) Analyzing legal regulations in the Norwegian life insurance business using a multistage asset-liability management model. Eur J Oper Res 134(2):293–308
Kaufman L, Rousseeuw PJ (2009) Finding groups in data: an introduction to cluster analysis. Wiley, USA
Kilianová S, Pflug GC (2009) Optimal pension fund management under multi-period risk minimization. Ann Oper Res 166(1):261–270
Kouwenberg R (2001) Scenario generation and stochastic programming models for asset liability management. Eur J Oper Res 134(2):279–292
Lloyd S (1982) Least squares quantization in PCM. IEEE Trans Inf Theor 28(2):129–137
Lozza SO, Vitali S, Cassader M (2013) Reward and risk in the fixed income markets. In: Financial regulation and supervision in the after-crisis period, Proceedings of 14th international conference on finance and banking, pp 329–340
Medova EA, Murphy JK, Owen AP, Rehman K (2008) Individual asset liability management. Quant Financ 8(6):547–560
Merton RC (1969) Lifetime portfolio selection under uncertainty: the continuous-time case. The review of economics and statistics, pp 247–257

Merton RC (1971) Optimum consumption and portfolio rules in a continuous-time model. J Econ Theor 3(4):373–413
Mulvey JM, Simsek KD, Zhang Z (2006) Improving investment performance for pension plans. J Asset Manage 7(2):93–108
Mulvey JM, Ural C, Zhang Z (2007) Improving performance for long-term investors: wide diversification, leverage, and overlay strategies. Quant Financ 7(2):175–187
Mulvey JM, Simsek KD, Zhang Z, Fabozzi FJ, Pauling WR (2008) OR PRACTICE-assisting defined-benefit pension plans. Oper Res 56(5):1066–1078
OECD (2013) Pensions at a glance 2013: OECD and G20 Indicators. OECD Publishing
Richard SF (1975) Optimal consumption, portfolio and life insurance rules for an uncertain lived individual in a continuous time model. J Financ Econ 2(2):187–203
Rockafellar TR, Uryasev S (2000) Optimization of conditional value-at-risk. J Risk 2:21–42
Rockafellar TR, Uryasev S (2002) Conditional value-at-risk for general loss distributions. J Bank Financ 26(7):1443–1471
Rousseeuw PJ (1987) Silhouettes: a graphical aid to the interpretation and validation of cluster analysis. J Comput Appl Math 20:53–65
Streutker M, van der Vlerk MH, Haneveld WKK (2007) Implementation of new regulatory rules in a multistage ALM model for Dutch pension funds. Tech. rep., University of Groningen, Research Institute SOM (Systems, Organisations and Management)
Vašiček O (1977) An equilibrium characterization of the term structure. J Financ Econ 5(2):177–188
Ziemba WT (2006) The Russell Yasuda, InnoALM and related models for pensions, insurance companies and high net worth individuals. Handbook of asset and liability management