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ORIGINAL PAPER

Optimal maintenance and scrapping versus the value of back ups

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Abstract This paper investigates optimal maintenance of equipment under uncertainty and the options of scrapping versus keeping the equipment as a back up (at a cost). This set up is used to analyze three points. The first observation is that the continuous, deterministic and even the unconstrained stochastic problem allow for closed form analytical solutions, realistic constraints require numerical means to solve the corresponding stochastic managerial problem. Second, the possibility to switch at negligible costs between different modes (here running or mothballing the equipment) depending on current states requires a condition in addition to the familiar value matching and smooth pasting conditions, namely continuity of the second derivative of the value function (or super contact). Equipped with these findings the analysis turns to the third point of quantifying numerically the value of keeping equipments as a back-up instead of scrapping.

Keywords Itô-process · Stopping · Switching · Super contact · Real option

1 Introduction

This paper addresses the following points within managerial decisions requiring dynamic and stochastic optimizations:

1. Theoretical analysis and computation are often complementary. More precisely, in the context of stochastic dynamic optimization brute force numerics (i.e., trying to solve numerically directly the partial differential equations resulting from the Hamilton–Jacobi–Bellman equations) fails. On the other hand, analytical solutions

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are often of less use than it appears at the first sight due to the complexity of the solution. This point is demonstrated indirectly in Lyasoff (2004) for one of the most famous and celebrated analytical results of economics, business and finance, the Black-Scholes formula. Lyasoff (2004) shows that the explicit analytical expression of the value of a call option offers little compared with the description of this value as a path integral (in the sense of Feynman). The latter approach can be readily implemented e.g. in Mathematica (even symbolically) and is more general because it applies to arbitrary nonlinear payoffs.

2. Many management decisions are not constrained to using or scrapping an equipment but have often the possibility to keep (at a cost) without using some equipment before mustering it out (e.g., old power plants that are mothballed for possible future supply shortages); or in partnerships, investments terminate before ultimately breaking up (or not, if prospects improve due to unforeseen events). The two transitions between these three modes—operation, on hold, and scrapping—differ. While the last action is irreversible (a scrapped, or sold machine cannot be used in the future) the others are reversible. Now if the costs of re-installing and also in the other direction, moving a machine from operation to back up, are negligible, then the familiar conditions of value matching and smooth pasting are insufficient and super contact provides the remaining condition.
3. An investigation of the managerial problem of maintenance, mothballing and scrapping of an equipment or another profit generating unit (the activity is labelled 'maintenance' throughout the paper although it can cover a variety of efforts such as advertising). Numerical examples are used to determine the value and sensitivity of this option.

The basic framework is introduced in Sect. 2. This paper focuses inter alia on reversible stopping—an action (here 'maintenance') can be suspended for some time without precluding future operation—while most of the related literature considers stopping decisions that are irreversible (*killing the option*). In fact, such an irreversible stopping (i.e. *scrapping*, in the application of this paper) is sometimes assumed even if a later re-entry is actually optimal (almost for sure e.g. in the global warming model in Pindyck 2000). Re-entry decisions associated with costs have been studied in Dixit (1989) emphasizing hystereses effects (associated with real option decisions) and drawing attention to the two different kinds of options: to enter and to exit. Section 3 analyzes the optimal use of an equipment differentiating between operation, maintenance, back up and scrapping for discrete maintenance (e.g., you can change the oil in your car or not). This section discusses the limits of theoretical analysis (documented in detail for an example in the Appendix), numerical difficulties and gives examples including a sensitivity analysis. Section 4 sketches the extension to continuous maintenance.

2 Model

The firm chooses a maintenance strategy, $\{m(t) \in M\}$, M is the admissible set of maintenance strategies, at each point in time (t) that maximizes the expected net present value of profits over costs generated by this equipment,

$$F(\pi_0) \equiv \max_{\{m(t)\in M\}} E \int_0^\infty e^{-rt}\Pi(t)\,dt. \tag{1}$$

The current period profit $\Pi(t)$ consists of the net revenues ($\pi(t)$, i.e. the operating costs are already subtracted) if the unit is operating, the costs for maintenance and mothballing if the equipment serves only as a backup, but is not scrapped; the further possibility of upgrading an existing equipment at a cost is a further possibility but is left out of the analysis in order to reduce the number of cases. In order to simplify and to facilitate analytical solutions where possible, the following assumptions are made: costs of mothballing are constant, $h > 0$, per period, maintenance costs are assumed to be quadratic, which is not crucial except for a short discussion in Sect. 5. In addition it is assumed that maintenance is restricted to operating units. This gives in total four cases (allowing for maintenance of back up units adds a fifth case and a further stopping decision yet without additional insights) with the following instantaneous payoffs:

$$\Pi(t) = \left\{\begin{array}{ccc} \pi(t) - \frac{c}{2}m^2(t) & & \text{operating and maintained} \\ \pi(t) & \text{if equipment is} & \text{only operating but } m = 0 \\ -h & & \text{on hold, not operating} \\ 0 & & \text{scrapped} \end{array}\right\}. \tag{2}$$

Assuming that expected depreciation is linear (a) and that the noise is Brownian (dz is the increment of a standardized Wiener process z), the net revenues π are stochastic processes evolving in the above four different regimes according to,

$$d\pi = \left\{\begin{array}{ccc} [m(t) - a]\,dt + \sigma dz & & \text{operating and maintained} \\ -adt + \sigma dz & \text{if equipment is} & \text{only operating but } m = 0 \\ \sigma dz & & \text{on hold, not operating} \\ 0 & & \text{scrapped} \end{array}\right\}, \tag{3}$$

with the initial condition, $\pi(0) = \pi_0$. That is, the net revenues (π) from an equipment/outlet/factory/etc. are expected to depreciate by the amount a which can be reduced by maintenance m.

This framework can be interpreted easily despite its mathematical set up with the help of practical examples, even personal ones, like owning a car. First of all the benefit from a car (π) is random, because it depends on the urgency, importance and benefit (e.g. weather) of trips and on the availability of the car, which will diminish over time due to repairs and other reasons for outages (unexpected damages either mechanical or due to accidents). For concreteness assume that the annual benefit of a brand new car is \$10,000 (above insurance and other related costs) and that the car would last 5 years without any maintenance, which implies for linear depreciation that the expected benefit declines by, $a =$ \$2,000; this depreciation may include beyond the mentioned outages, less fun and less comfort as the car ages. Maintenance, say a regular service at the cost $C(m)$ reduces this annual expected depreciation of benefits by $m =$ \$ 1,000 (and increases expected lifetime to 10 years). Mothballing means in this case that the car is moved into a garage (at the cost h per year covering the expenses for the garage and the insurance). The car is only used if needed (either as a back up, as a second car, or not used at all, if individual service demand and thus the benefit from a car has drastically shrunk), i.e., if the associated benefit π justifies this. The stochastic process, $d\pi = \sigma dz$, describes the hypothetical profit off-line (while the car is in the garage, potential benefits from driving change), but clearly, the car in the garage delivers no instantaneous benefit (and does not age from a benefit perspective). The uncertainty results from conditions outside the control of the individual. Finally, the additional and simplifying assumption that operation and maintenance go only hand in hand means that a car in the garage will receive no service, which seems to be plausible.

Within this set up, the firm has in addition to operation and maintenance, the first line in (3), three different stopping options not all of them always economical:

1. To suspend maintenance but keep operating, $m = 0$.
2. To suspend operations and maintenance but to keep the machine as a back up and for potential reuse if profit opportunities improve sufficiently. This case, labeled mothballing, reduces revenues and depreciation to zero and causes costs h per period. In this case, π describes the hypothetical profit off-line (and $d\pi = \sigma dz$ its evolution) that can be only realized after bringing the unit back into the production.
3. To scrap the machine/plant at no cost except that all future uses and benefits are sacrificed. This stopping is irreversible, i.e., $\Pi(t) = 0$ for all $t > T =$ scrapping date, once this option is executed in contrast to the above two stopping decisions that only suspend actions.

Continuous dynamic programming implies that the value function F must satisfy the Hamilton–Jacobi–Bellman equation across the four different cases (in the same sequence as in (2) and (3) and omitting the function arguments t as well as π)

$$rF = \left\{ \begin{array}{c} \max\limits_{m \in M} \left\{ \pi - \frac{c}{2}m^2 + F'(m - a) + \frac{1}{2}\sigma^2 F'' \right\} \\ \pi - aF' + \frac{1}{2}\sigma^2 F'' \\ -h + \frac{1}{2}\sigma^2 F'' \\ 0 \end{array} \right\}. \tag{4}$$

Since the above framework involves switching (at points 1 and 2) and only one conventional stopping condition (to scrap in 3), additional boundary conditions are necessary to solve the problem. The derivation of such boundary conditions is the content and objective of the next section.

3 Constant maintenance ($\overline{m}$)

Assuming a binary control, $M = \{0, \overline{m}\}$, i.e., maintenance can be applied or not (e.g. a car can get the oil changed or not) allows to apply the apparatus of real options documented in Dixit and Pindyck (1994); the extension for continuous maintenance is briefly addressed in the following section.

3.1 Operation and maintenance

Substituting the interior policy, $m = \overline{m}$, into the Hamilton–Jacobi–Bellman equation (4) implies the following linear differential equation,

$$rF^i = \pi - \frac{c}{2}\overline{m}^2 + F^{i\prime}(\overline{m} - a) + \frac{\sigma^2}{2}F^{i\prime\prime}. \tag{5}$$

The general solution of (5) consists of a particular solution $\left(F_p^i\right)$ and of the general solution $\left(F_h^i\right)$ of the homogeneous part of the differential equation (5) according to the principle of superposition. The particular solution is

$$F_p^i = \frac{\pi - \frac{c}{2}\overline{m}^2}{r} + \frac{\overline{m} - a}{r^2} = E\int_0^\infty e^{-rt}\left(\pi(t) - \frac{c}{2}\overline{m}^2\right)dt, \tag{6}$$

which is easily verified. This particular solution is the fundamental term, because it determines the expected net present value of profits of running and maintaining the equipment forever [see the right hand side in (6)]. The general solution of the homogenous part of (5) is

$$F_h^i = c_1 e^{\beta_1 \pi} + c_2 e^{\beta_2 \pi}, \quad \beta_{12} = \frac{a - \overline{m} \pm \sqrt{(a - \overline{m})^2 + 2r\sigma^2}}{\sigma^2} \tag{7}$$

in which β_{12} are the roots of the characteristic equation of the homogeneous part of (5). The general solution, $F_p^i + F_h^i$, accounts for the fundamental term and the value of the option to stop using the equipment. This possibility of stopping is clearly valuable if operation yields a negative cash flow. Yet as the profits are large, the probability of near future stopping is unlikely, which reduces the value of this option. This is only possible if the coefficient of the positive exponential vanishes, $c_1 = 0$, so that the solution of the interior part of the value function becomes

$$F^i = \frac{\pi - \frac{c}{2}\overline{m}^2}{r} + \frac{\overline{m} - a}{r^2} + c_2 e^{\frac{a - m - \sqrt{(a-m)^2 + 2r\sigma^2}}{\sigma^2}\pi}. \tag{8}$$

3.2 Scrapping

This part of the solution is trivial. With scrapping at date T all future profits terminate, i.e., the stochastic process evaporates ($d\pi = 0$) and the corresponding payoff (F^s) is simply

$$F^s = 0 \;\forall t > T. \tag{9}$$

Suppose that scrapping is the only option, i.e., operation and maintenance are Siamese twins (either for technical or economic reasons such that maintenance is not sufficiently profitable) and mothballing is not possible or too costly such that scrapping is optimal. Let π^s denote the scrapping threshold (i.e., the profit level at which scrapping is optimal), then applying value matching, $F^{i}(\pi^{s}) = F^{s}(\pi^{s}) = 0$ and smooth pasting, $F^{i\prime}(\pi^{s}) = F^{s\prime}(\pi^{s}) = 0$, yields two equations

$$\frac{\pi^{s} - \frac{c}{2}\overline{m}^{2}}{r} + \frac{\overline{m} - a}{r^{2}} + c_{2}e^{\beta_{2}\pi^{s}} = 0, \tag{10}$$

$$\frac{1}{r} + \beta_{2}c_{2}e^{\beta_{2}\pi^{s}} = 0, \tag{11}$$

that determine the two unknowns, explicitly,

$$\pi^{s} = \frac{1}{\beta_{2}} - \frac{m - a}{r} + \frac{c}{2}\overline{m}^{2}, \quad c_{2} = -\frac{e^{-\beta_{2}\pi^{s}}}{r\beta_{2}}, \tag{12}$$

and thus the value function analytically.

This explicit analytical solution allows to document that direct attempts of numerically solving the Hamilton–Jacobi–Bellman equation must fail. The optimal decision between using and scrapping requires to solve

$$rF = \pi - \frac{c}{2}\overline{m}^{2} + (\overline{m} - a)F' + \frac{\sigma^{2}}{2}F'', \quad F\left(\pi^{s}\right) = 0, \quad F'\left(\pi^{s}\right) = 0, \tag{13}$$

i.e., a second order, in this case even an ordinary and linear, differential equation, subject to two boundary conditions. The only difficulty seems that the stopping level π^{s} is unknown, which could be found by iterating until the maximum value is found. Yet as the above derived general solution highlights, this is numerically difficult, because any deviation from the true π^{s} (and any numerical guess is off of this value) is equivalent to $c_1 \neq 0$. This introduces an exponentially growing term which sooner or later dominates leading to a huge deviation from the true value function. Figure 1 documents this for a numerical example using excellent approximations ($\pm 0.01\%$ from the true π^{s}).

Replacing this direct (Runge–Kutta) procedure by finite difference algorithms improves the approximation at some computational cost (see Dangl and Wirl 2004).

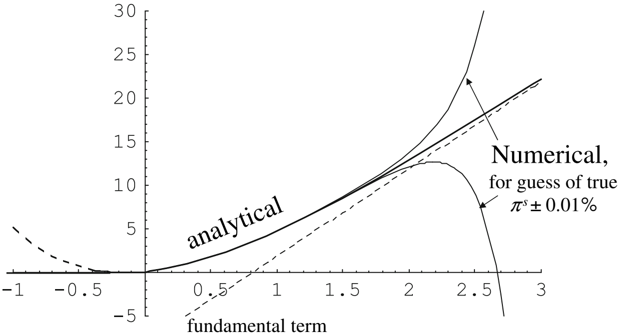

Fig. 1 Comparing direct numerical solutions of (13) using the Runge–Kutta algorithm and excellent guesses of the scrapping level; $a = 0.1$, $r = 0.1$, $\bar{m} = 0.025$, $c = 100$, $\sigma = 0.2$

However, finite difference algorithms require additional boundary conditions, e.g., that the solution of the Hamilton–Jacobi–Bellman equation can be approximated at large values of π by the fundamental term, the particular solution. Tying any solution between the boundary solution and the particular solution and applying the method of finite differences avoids the impact of the exploding solution paths displayed in Fig. 1. However, this approach requires the knowledge of the particular solution, which is not the case in general. For example, a geometric representation of the stochastic process (in the interior),

$$d\pi = [m - a]\,\pi dt + \sigma \pi dz, \quad \pi\,(0) = \pi_0, \tag{14}$$

has no closed form analytical solution for the corresponding expected net present value of profits (simultaneously a particular solution) for operating and maintaining forever.

3.3 Operating, no maintenance

This case of operation without maintenance saves the corresponding costs and thus implies the Hamilton–Jacobi–Bellman equation (this part of the overall value function is identified by the subscript o):

$$rF^o = \pi + F^{o\prime}\,(\overline{m} - a) + \frac{\sigma^2}{2} F^{o\prime\prime}. \tag{15}$$

Applying the procedure outlined in Sect. 3.1 yields the solution

$$F^o = \frac{\pi}{r} - \frac{a}{r^2} + d_1 e^{\gamma_1 \pi} + d_2 e^{\gamma_2 \pi}, \quad \gamma_{12} = \frac{a \pm \sqrt{a^2 + 2r\sigma^2}}{\sigma^2} \tag{16}$$

In contrast to the above interior solution (8), both exponentials can be part of the value function because they capture the two different options: to re-start maintenance and to stop operation (leading to mothballing or scrapping, depending on parameters).

3.4 Mothballing

Suspending the equipment (no maintenance and no operation thus also no depreciation) reduces the stochastic process to $d\pi = \sigma dz$ and eliminates the revenues π in the objective (presumably negative otherwise) but adds instead the per unit costs h. This implies the following functional equation for the value function in this domain, identified by the subscript m,

$$rF^m = -h + \frac{\sigma^2}{2} F^{m\prime\prime}. \tag{17}$$

This differential equation has the following solution (again applying the principle of superposition)

$$F^m\,(\pi) = -\frac{h}{r} + k_1 e^{\alpha_1 \pi} + k_2 e^{\alpha_2 \pi}, \quad \alpha_{12} = \pm \frac{\sqrt{2r}}{\sigma} \tag{18}$$

where the particular solution $(-h/r)$ determines the fundamental term (the expected net present value of holding the equipment forever as a backup without ever activating it) while the exponential terms account for the potential benefits from the two different and actually opposite options: future operation and scrapping. The exponents are the roots of the characteristic equation of the homogeneous part of the differential equation (17), the options—to re-start operations and scrapping—are potentially valuable, thus $k_1 > 0$ and also $k_2 > 0$.

3.5 Boundary conditions

Assuming that all options are economical in the state space (which need not be the case, as is demonstrated with examples), the familiar conditions of value matching $F(\pi^s) = F^m(\pi^s) = F^s(\pi^s) = 0$ and smooth pasting $F'(\pi^s) = F^{m\prime}(\pi^s) = F^{s\prime}(\pi^s) = 0$ at the scrapping level (π^s) yield:

$$-\frac{h}{r} + k_1 e^{\alpha_1 \pi^s} + k_2 e^{\alpha_2 \pi^s} = 0, \tag{19}$$

$$\alpha_1 k_1 e^{\alpha_1 \pi^s} + \alpha_2 k_2 e^{\alpha_2 \pi^s} = 0. \tag{20}$$

Analogously, value matching $F^i(\pi^m) = F^o(\pi^o)$ and smooth pasting, $F^{i\prime}(\pi^m) = F^{o\prime}(\pi^m)$, when stopping maintenance yet continuing operation, imply:

$$-\frac{c\overline{m}^2}{2r} + \frac{\overline{m}}{r^2} + c_2 e^{\beta_2 \pi^o} = d_1 e^{\gamma_1 \pi^o} + d_2 e^{\gamma_2 \pi^o}, \tag{21}$$

$$\beta_2 c_2 e^{\beta_2 \pi^o} = \gamma_1 d_1 e^{\gamma_1 \pi^o} + \gamma_2 d_2 e^{\gamma_2 \pi^o}. \tag{22}$$

Finally, at the level of mothballing, $F^i(\pi^m) = F^m(\pi^m)$ and $F^{i\prime}(\pi^m) = F^{m\prime}(\pi^m)$ imply

$$\frac{\pi^m}{r} - \frac{a}{r^2} + d_1 e^{\gamma_1 \pi^m} + d_2 e^{\gamma_2 \pi^m} = -\frac{h}{r} + k_1 e^{\alpha_1 \pi^m} + k_2 e^{\alpha_2 \pi^m}, \tag{23}$$

$$\frac{1}{r} + \gamma_1 d_1 e^{\gamma_1 \pi^m} + \gamma_2 d_2 e^{\gamma_2 \pi^m} = \alpha_1 k_1 e^{\alpha_1 \pi^m} + \alpha_2 k_2 e^{\alpha_2 \pi^m}. \tag{24}$$

Summarizing, the familiar conditions of value matching and smooth pasting yield six equations that contain yet eight unknowns, the coefficients $(k_1, k_2, d_1, d_2, c_2)$ and the threshold levels (π^s, π^m, π^o).

Therefore additional conditions are needed. The approach in Dixit (1989) cannot be extended to the limiting case of zero exit and entry costs (this reversible case of stopping and entering at no charge is called *switching* in the following) because it also ends up in the lack of a boundary condition. Dumas (1991) considers transaction between actions and shows that the usual boundary conditions move up one notch: value matching → smooth pasting, and smooth pasting → super contact, i.e., continuity of the second derivative. This does not solve our problem of switching at no cost. However, all three conditions, value matching, smooth pasting and super contact hold for the limit of zero transaction costs:

Lemma *Consider the stochastic dynamic optimization problem,*

$$G(y_0) \equiv \max_{x \in X \subseteq \Re_+} \left\{ E \left(\int_0^\infty e^{-rt} u(x(t), y(t)) dt \right) \right\},$$

$$dy(t) = a(x(t), y(t))\, dt + b(y(t))\, dz(t), \quad y(0) = y_0,$$

in which the functions u, a and b are twice continuously differentiable and ensure concavity, $u_{xx} + a_{xx} G' < 0$*, and the boundary strategy,* $x = 0$*, is admissible,* $|u_x(0, y)| < \infty$*. Furthermore, assume without loss in generality that* $y < Y$ *is the interior domain and* $y > Y$ *the stopping domain and Y denotes the corresponding threshold level. Then,*

$$\lim_{y \nearrow Y} G''(y) = \lim_{y \searrow Y} G''(y).$$

This result applies to discrete (real option), $X = \{0, \overline{x}\}$*, as well as to continuous controls,* $X = \Re_+$*.*

Proof Wirl (2007, forthcoming).

Application of this lemma ensures that switching at no cost between interior and boundary strategies implies continuity of the second derivative of the value function. This property of super contact, i.e., continuity of F'' at the corresponding threshold levels π^o and π^m, renders the needed two additional equations,

$$\beta_2^2 c_2 e^{\beta_2 \pi^o} = \gamma_1 d_1 e^{\gamma_1 \pi^o} + \gamma_2 d_2 e^{\gamma_2 \pi^o}, \quad (25)$$

$$\gamma_1^2 d_1 e^{\gamma_1 \pi^m} + \gamma_2^2 d_2 e^{\gamma_2 \pi^m} = \alpha_1^2 k_1 e^{\alpha_1 \pi^m} + \alpha_2^2 k_2 e^{\alpha_2 \pi^m}. \quad (26)$$

The resulting simple equations document the suitability and applicability of the super contact condition.

The eight Eqs. (19)–(26) are linear in the coefficients $(k_1, k_2, d_1, d_2, c_2)$ yet non-linear in an essential way in the thresholds (π^s, π^m, π^o) such that explicit analytical solutions are not possible. And although the implicit function theorem allows in principle to determine qualitative properties, its application is highly cumbersome even after reducing the system to three equations by substituting the expressions of coefficients (obtained from solving the linear system). This holds a fortiori for complex processes that render analytical solutions even if available, of often very limited use; a particular example that allows for a rather complex closed form solution is elaborated in the Appendix.

3.6 Examples

For the reasons expounded above a numerical example serves to trace the managerial consequences. The reference parameters of this example are

$$r = 0.10, \quad a = 0.10, \quad h = 0.1, \quad \sigma = 0.10, \quad m = 0.05, \quad c = 200. \quad (27)$$

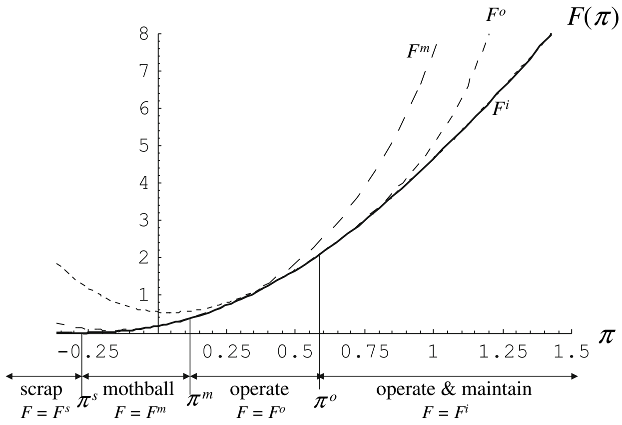

Fig. 2 Value function (s) for reference example: $a = 0.1$, $r = 0.1$, $m = 0.05$, $c = 200$, $h = 0.1$, $\sigma = 0.2$

Figure 2 shows the value function (bold), the thresholds (π^s, π^m, π^o) and the associated separation of the state space into the different actions; the dashed lines refer to the 'solutions' associated with particular sets of actions (i.e., to the functions labeled F^j, $j = i, o, m$ and how they make up the value function F). From a managerial point of view two results seem relevant: units that would produce a loss and are costly to mothball are nevertheless kept ($\pi^s < 0$) while units still delivering profits are retracted from operation ($\pi^m > 0$). At first sight it seems a little bit puzzling that maintenance is active even if profitability is very high. This is a consequence of risk-neutral objective and the linear and unbounded contribution of maintenance (or better efforts) to profits.

Figures 3 and 4 vary model parameters of which some parameter constellations eliminate cases shown in Fig. 2 for the reference example. Not surprising, high discounting enlarges the scrapping domain, reduces the domain of joint production and maintenance and reduces the value (and domain) of mothballing. At low discount rates, the option of stopping maintenance becomes worthless (i.e., maintenance and operation become Siamese twins) and this induces a non-monotonicity into the threshold π^o. The consequences of higher uncertainty are, as seems common, a mirror image of discounting: it enlarges the domain of joint production and maintenance (including a decline in π^o) and of mothballing (sufficient uncertainty is necessary to render this option valuable), but reduces the scrapping domain (with high uncertainty equipment yielding a considerably losses are nevertheless kept) and the domain of operation only.

Figure 4 plots the consequences of the costs of maintenance[1] and of mothballing. Higher maintenance costs enlarge the domain of operating only largely at the expense of shrinking the domain of joint operation and maintenance (i.e., a substantial increase of the threshold π^o), while the remaining thresholds are insensitive. Low maintenance costs eliminate the case of operation only. Increasing the costs of mothballing has two effects: it increases the domain where scrapping is optimal rendering the option

[1] Variations in m do not cover the efficiency in maintenance since any increase in m increases the corresponding costs to the squared. Thus variations in c are best suited to trace the consequences of efficiency of maintenance.

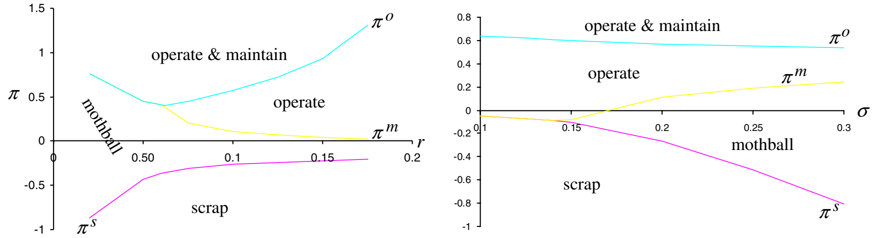

Fig. 3 Optimal thresholds for varying discount rate (r) and standard error (σ) for the reference parameters: $a = 0.1, r = 0.1, m = 0.05, c = 200, h = 0.1, \sigma = 0.2$

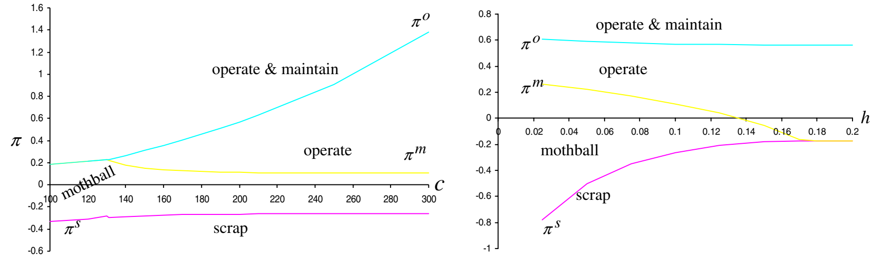

Fig. 4 Optimal thresholds for varying costs of maintenance (c) and of mothballing (h) for the reference parameters: $a = 0.1, r = 0.1, m = 0.05, c = 200, h = 0.1, \sigma = 0.2$

of mothballing worthless at sufficiently high costs (e.g. for $h > 0.18$); second the enlargement of the domain of operation cuts from above into the mothballing segment.

4 Continuous

The solution of the continuous problem is only sketched since the special case (scrapping versus operating and maintenance) is analyzed in Dangl and Wirl (2004) and the full blown case would require to append all the different options (i.e., the determination of the option value coefficients and the thresholds) to the continuous program which would enlarge the paper, complicate the computations without adding an equivalent in substance.

Since maximization of the right hand side of the first row in (4) with respect to a continuous choice of maintenance yields $m^* = F'/c$, the Hamilton–Jacobi–Bellman becomes in the interior

$$rF = \pi - aF' + \frac{1}{2c}F'^2 + \frac{\sigma^2}{2}F'', \tag{28}$$

but is unchanged in all the other cases. Hence, one has to solve the above second order differential equation accounting for all the boundary conditions. These conditions are structurally identical to those in Sect. 3.5 but yield different values and upper bars are introduced to indicate this difference. The thresholds $\overline{\pi}^s$ and $\overline{\pi}^m$ are contingent on stopping the interior program at $\overline{\pi}^o$ at which value matching, smooth pasting and super contact must hold. That is, conditional on a guess, $\widehat{\pi}^o$, it is necessary to locate the stable branch of the second order differential equation (28) which can be achieved by a projection algorithm (originally suggested in Judd 1992, applied in Dangl and Wirl 2004) because that follows the stable flow avoiding the exploding exponentials that render standard procedures inadequate as demonstrated in Fig. 1.

5 Summary

This paper investigated the managerial problem of maintenance, mothballing and scrapping of a profit generating unit (a machine, a product sold, an outlet, etc.). Aside from analyzing this particular managerial problem, the paper emphasized two issues, that of reversible versus irreversible stopping (scrapping) and how analytics and numerics are complements. The first point of reversible stopping or switching is related to the issue of stopping operation without immediately scrapping the unit and having the option of keeping it for potential future utilization. In this case, the familiar conditions of value matching and smooth pasting are insufficient to solve the problem and it is shown that super contact holds across such switching points and this condition proves useful in solving the problem (it provides the lacking boundary condition). Along the managerial application (maintenance) it was also demonstrated that analytical and numerical aspects are often complements and that sometimes analytical solutions (although much acclaimed in the theoretical literature) offer little insight; Judd (1998) and Miranda and Fackler (2002) are two particularly useful text books on numerical methods in business, finance and economics. Conversely, brute force numerics without any analytical look at the problem can prove faulty. Finally, a particular characteristic of this paper is that numerics become useful (if not necessary) for a handy small-scale model rather than the large-scale models (say in transportation, logistics and production) that must entirely rely on number crunching.

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Appendix: Example: analytical but complex solution

The objective of the following example is to demonstrate that an analytical solution may often be of less use than stressed in the literature. For this purpose, consider the following variant of a stochastic process along operation and maintenance of an equipment:

$$d\pi = [m - \delta\pi]\, dt + \sigma\sqrt{\pi} dz, \quad \pi\,(0) = \pi_0, \tag{29}$$

in which linear depreciation is replaced by geometric depreciation and the root accounts for the standard error; geometric noise ($\sigma\pi dz$) and/or relative improvement due to maintenance ($m\pi$) lead to more complicated expressions ruling even out, as already mentioned, an analytical expressions for the particular solution of the Hamilton–Jacobi–Bellman equation.

Let us restrict the analysis to the interior policy, $m = \overline{m}$. This implies the following linear differential equation (ignoring for the moment the superscript i),

$$rF = \pi - \frac{c}{2}\overline{m}^2 + F' \left(\overline{m} - a\pi\right) + \frac{1}{2}\sigma^2 \pi F'', \tag{30}$$

which has the particular solution

$$F_p = \frac{\pi}{r+a} - \frac{1}{r}\frac{c}{2}\overline{m}^2 + \frac{\overline{m}}{(r+a)\,r}.$$

The homogenous part of the Eq. (30) is (slightly re-arranged)

$$\pi F'' + \left(\frac{2\overline{m}}{\sigma^2} - \frac{2a}{\sigma^2}\pi\right) F' - \frac{2r}{\sigma^2} F = 0. \tag{31}$$

defining

$$z = \frac{2a}{\sigma^2}\pi \tag{32}$$

and demanding

$$h(z) = F(\pi), \quad F' = h' \frac{2a}{\sigma^2}, \quad F'' = h'' \left(\frac{2a}{\sigma^2}\right)^2 \tag{33}$$

yields after elementary simplifications, dividing by $\left(\frac{2a}{\sigma^2}\right)$, and defining

$$b = \frac{2\overline{m}}{\sigma^2}, \quad \theta = \frac{r}{a}, \tag{34}$$

the second-order ordinary differential equation

$$zh'' + (b - z)\,h' - \theta h = 0, \tag{35}$$

which is sometimes called Kummer's differential equation (Abramowitz and Stegun 1964). It has two linear independent solutions (Jeffrey 2000; MacDonald 1948) for $b \notin \aleph$:

$$h_1(z) = H(\theta, b, z) \equiv 1 + \frac{\theta}{b}z + \frac{\theta\,(\theta+1)}{b\,(b+1)}\frac{z^2}{2!} + \ldots, \tag{36}$$

$$h_2(z) = z^{1-b} H(\theta - b + 1, 2 - b, z). \tag{37}$$

Since the second solution $h_2(z)$ behaves for small values of z like z^{1-b} and is thus not well defined for small values of z if $b > 1$ (while $h_1(z)$ remains bounded at $z = 0$), the general solution of (30) is

$$F(\pi) = \frac{\pi}{r+a} - \frac{1}{r}\frac{c}{2}\overline{m}^2 + \frac{\overline{m}}{(r+a)\,r} + C_1 H\left(\frac{r}{a}, \frac{2\overline{m}}{\sigma^2}, \frac{2a}{\sigma^2}\pi\right) \tag{38}$$

Clearly, applying the implicit function theorem to exploit analytical properties implicit in the value matching, smooth pasting and super contact conditions is not a suitable strategy, even knowing (e.g. from MacDonald 1948) that $H' = \frac{\theta}{b} H$ $(\theta + 1, b + 1, z)$.

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