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Optimal portfolios: new variations of an old theme

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Abstract We survey some recent developments in the area of continuous-time portfolio optimization. These will include the use of options and of defaultable assets as investment classes and the presentation of a worst-case investment approach that takes the possibility of stock market crashes into account.

Keywords Continuous-time portfolio optimization · Derivatives · Worst-case control

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1 Introduction

The problem to find an optimal portfolio process in a continuous-time market setting is nowadays well understood from a theoretical point of view. Its main results (such as those derived in the pioneering work by Merton 1969, 1971; Cox and Huang 1989; Pliska 1986; or Karatzas et al. 1987) are covered in monographs such as Korn (1997) or Merton (1990). However, from a practical point of view there are various aspects that have until now prevented a breakthrough of the use of continuous-time portfolio methods in reality. Various obstacles for such a practical application have to be overcome. They range from a satisfying treatment of transaction costs via a demand for including alternative investment classes besides stocks and bonds to a reasonable protection against (stock) market crashes. In this survey, we will consider aspects of crash protection and of investment into derivatives. After introducing to the standard

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results obtained in a complete market setting of the standard Black-Scholes type, we will consider the portfolio problem when derivatives form an alternative investment class. As a direct application of this, optimal investment strategies with defaultable assets will be derived in an explicit form. Finally, we present the worst-case portfolio approach (and variants of it) by Korn and Wilmott (2002) that explicitly considers the possibility of market crashes in a non-standard way.

2 Optimal portfolios in the Black-Scholes-setting

We consider a Black-Scholes-type market where a bond (or, more precisely, a money market account) with a price evolution given by

$$dP_0(t) = P_0(t)r\,dt, \quad P_0(0) = 1 \tag{1}$$

and n risky securities (shares of stock) with price dynamics given by

$$dP_i(t) = P_i(t)\left(b_i\,dt + \sum_{j=1}^{n} \sigma_{ij}\,dW_j(t)\right), \quad P_i(0) = p_i,\ i = 1, \ldots, n \tag{2}$$

can be traded continuously in time until the time horizon T. Here,the market coefficients r, b and σ are assumed to be constants with σ being a regular matrix and $\{W(t), f_t\}_{t\in[0,T]}$ denotes an n-dimensional Brownian motion. To describe the actions of an investor, we introduce portfolio processes as n-dimensional, f_t-progressively measurable processes $\pi(t)$ where $\pi_i(t)$ denotes the fraction of the investor's total wealth invested in stock i at time t. Here, the total wealth is the sum of the current values of the investor's holdings. For technical reasons, we have to restrict to those portfolio processes such that the corresponding wealth equation

$$\begin{aligned} dX^{\pi}(t) &= X^{\pi}(t)\left[\left(\pi(t)'b + (1 - \pi(t)'\underline{1})r\right)dt + \pi(t)'\sigma\,dW(t)\right], \\ X^{\pi}(0) &= x \end{aligned} \tag{3}$$

admits a unique strong solution satisfying

$$\int_0^T \|\pi(t)X^{\pi}(t)\|^2 dt < \infty, \quad P\text{-a.s.} \tag{4}$$

Note that the fraction of wealth invested in the bond at time t is given by

$$1 - \sum_{i=1}^{n} \pi_i(t) = 1 - \pi(t)'\underline{1}. \tag{5}$$

The aim of the investor is to find an optimal portfolio process. More precisely, given a utility function U, i.e. a strictly concave, increasing and differentiable function U satisfying

$$U'(0) > 0, \quad U'(+\infty) = 0, \tag{6}$$

the problem

$$\max_{\pi(\cdot)\in A(x)} E(U(X^{\pi}(T)))$$

is called the **continuous-time portfolio problem**. Here, the set $A(x)$ consists of those portfolio processes that lead to a non-negative wealth process when the investor is endowed with a positive initial wealth of x and which satisfy in addition

$$E(U^{-}(X^{\pi}(T))) < \infty. \tag{7}$$

We call such portfolio processes admissible. Note that we face a dynamic optimization problem as we have to decide about the optimal portfolio process at each time instant. Thus, **stochastic control methods** are obvious candidates for solving the continuous-time portfolio problems. Indeed, this is the approach of Merton (1969). One can directly write down a Hamilton–Jacobi–Bellman equation (for short HJB-equation) corresponding to the portfolio problem

$$\max_{\pi\in R^n} \left\{ v_t(t,x) + \frac{1}{2}\pi'\sigma\sigma'\pi x^2 v_{xx}(t,x) + \left(r + \pi'\left(b - r\underline{1}\right)\right) x v_x(t,x) \right\} = 0 \tag{8}$$

$$v(T,x) = U(x) \tag{9}$$

By standard verification theorems (see Korn and Korn 2001) one can show that if the HJB-equation possesses a classical solution (satisfying suitable growth conditions), it has to coincide with

$$v(t,x) = \max_{\pi(\cdot)\in A(t,x)} E^{t,x}(U(X^{\pi}(T))), \tag{10}$$

the value function of our problem (i.e. the optimal utility viewed as a function of the parameters (t, x) when the wealth process starts at time t with a value of x. Therefore, one can state an algorithm for solving the portfolio problem with this method:

Algorithm for solving the portfolio problem via stochastic control

Step 1 Solve the optimisation problem inside the HJB-equation to obtain

$$\pi^*(t) = \pi^*(t,x)$$

still depending on the yet unknown value function v and its partial derivatives.

Step 2 Insert $\pi^*(t)$ into the HJB-equation, drop the supremum operator, and solve the resulting partial differential equation.

Step 3 Check all the assumptions made in the previous steps!

Solving the HJB-equation by this algorithm (compare, e.g., Korn 1997, Chap. 3) for the choice of $U(x) = \frac{1}{\gamma}x^{\gamma}$, $\gamma < 1$, leads to an optimal portfolio process of

$$\hat{\pi}(t) = \frac{1}{1-\gamma}(\sigma\sigma')^{-1}(b - r\underline{1}). \tag{11}$$

A second approach to solve the continuous-time portfolio problem and which is tailored to the properties of the Black-Scholes type market is the so called **martingale approach** (see Korn 1997, Chap. 3). It is based on the possibility of a decomposition of the continuous-time portfolio problem into a **static optimization problem**

$$\max_{B \in B(x)} E(U(B)) \tag{12}$$

with $B(x) = \{Bf_T - measurable\,|B \geq 0, E(H(T)B) \leq x\}$, and a **representation problem**

$$Find\ a\ portfolio\ process\ \hat{\pi}(\cdot)\ with\ X^{\hat{\pi}}(T) = \hat{B}$$

where $\hat{B}$ solves the static optimization problem. Here, the static optimization problem can be solved by Lagrangian methods while the representation problem can be really involved and often needs problem specific treatment (depending on the utility function and on the assumptions on the market coefficients). Further details can be found in Korn (1997, Chap. 3). As an application, one can explicitly solve the continuous-time portfolio problem for the choice of leading to an optimal portfolio process of

$$\hat{\pi}(t) = (\sigma\sigma')^{-1}(b - r\underline{1}). \tag{13}$$

For the choice of $U(x) = 1 - e^{-\lambda x}$, $\lambda > 0$, application of the martingale method results in an optimal vector of money (not fractions!) invested into the different stocks given by

$$\hat{\pi}(t)\hat{X}(t) = \frac{1}{\lambda}(\sigma\sigma')^{-1}(b - r\underline{1})e^{-r(T-t)}. \tag{14}$$

However, for general utility functions numerical methods have to be used. Another weakness of the whole approach is the solution of the problem itself. While a constant portfolio process has a very appealing form the mathematical point of view, it raises a lot of practical problems. To see this, note that in particular for the case of a market consisting only of a bond and a stock, the investor following—say—the log-optimal strategy $\hat{\pi}(t) = (b - r)/\sigma^2$ has to rebalance his holdings at each time instant due to the fact that the stock price and the bond price move in a different way. In the presence of transaction costs, strictly following such a strategy leads to ruin. On the other hand, if one uses a relaxed version (*Let the portfolio process evolve freely as long as it is not too far away from the optimal portfolio; rebalance to the optimal portfolio only if the portfolio process has moved too far away*) one could obtain nearly the same optimal utility as obtained by following the log-optimal portfolio in a strict way. However, the relaxed version leads to finite transaction costs. For details see Rogers (2001).

3 Optimal investment with derivatives

Due to their high liquidity, leverage effects and their non-linear payoff profile, options and other derivatives are nowadays a widely used investment opportunity. It is therefore a natural generalization of the standard portfolio problem to allow for investment into options. However, a straight forward generalization of the stochastic control approach leads to a much more complicated form of the HJB-equations as an option price even in the most simple examples (such as a call or put option) is a non-linear function $f(t, P_1(t))$ of the underlying stock price, if not a functional of the whole price paths of a set of stocks.

However, the martingale approach of portfolio optimization is well suited to deal with this problem when we restrict ourselves to so-called weakly path dependent options. To explain this in more detail, we first have to introduce the notion of a trading strategy and of a replicating strategy for an option.

Definition 1 (i) A trading strategy is an $R^{(n+1)}$-valued, f_t-measurable process $\varphi(t)$ satisfying

$$\int_0^T |\varphi_0(t)|dt < \infty, \quad \sum_{i=1}^n \sum_{j=1}^n \int_0^T \left(\varphi_i(t)P_i(t)\sigma_{ij}(t)\right)^2 dt < \infty \quad P\text{-a.s.} \tag{15}$$

where $T > 0$ is given (the *time horizon*). The process

$$X(t) = \sum_{i=0}^n \varphi_i(t)P_i(t) \tag{16}$$

is called the wealth process corresponding to $\varphi(t)$.

(ii) A trading strategy $\varphi(t)$ will be called self-financing if the corresponding wealth process $X(t)$ satisfies

$$X(t) = X(0) + \sum_{i=0}^n \int_0^t \varphi_i(s)dP_i(s) \quad \forall t \in [0, T]. \tag{17}$$

(iii) A non-negative, f_T-measurable random variable C with

$$E(C^\mu) < \infty \tag{18}$$

for some $\mu > 1$ is called a European option. Each self-financing trading strategy $\varphi(t)$ such that the corresponding wealth process $X(t)$ satisfies

$$X(T) = C \quad P\text{-a.s.} \tag{19}$$

is called a replication strategy for the option C.

The concept of replication strategies is the basic idea to derive the fair price of an option in the Black-Scholes setting. For more details we refer to Chap. 3 of Korn and Korn (2001). The following result from Korn and Trautmann (1999) will be quite useful for our considerations below. It gives an explicit formula for the replication strategy of options of a special kind:

Theorem 1 *Assume that the price of an option at time t can be written as an at most polynomially growing $C^{1,2}-$ function $f(t, P_1(t), \ldots, P_n(t))$ of time and the underlying stock prices. Then we have*:

(a) *The replication strategy $\psi(\cdot)$ in bond and stocks for the option is given by*

$$\psi_i(t) = f_{p_i}(t, P_1(t), \ldots, P_n(t)), \quad i = 1, \ldots, N, \tag{20}$$

$$\psi_0(t) = \frac{f(t, P_1(t), \ldots, P_n(t)) - \sum_{i=1}^{n} f_{p_i}(t, P_1(t), \ldots, P_n(t)) P_i(t)}{P_0(t)}, \tag{21}$$

and the price function $f(t, p_1, \ldots, p_n)$ is the unique polynomial solution of the partial differential equation

$$f_t + \frac{1}{2} \sum_{i,j,k=1}^{n} \sigma_{ik}\sigma_{kj} p_i p_j f_{p_i p_j} + \sum_{i=1}^{n} r p_i f_{p_i} - rf = 0. \tag{22}$$

(b) *The price process $f(t, P_1(t), \ldots, P_n(t))$ of the option satisfies*

$$\begin{aligned} &df(t, P_1(t), \ldots, P_n(t)) \\ &= \left(rf(t, P_1(t), \ldots, P_n(t)) + \sum_{i=1}^{n} (b_i - r) p_i f_{p_i}(t, P_1(t), \ldots, P_n(t)) \right) dt \\ &+ \sum_{i,j=1}^{n} \sigma_{ij} p_i f_{p_i}(t, P_1(t), \ldots, P_n(t)) dW_j(t) \end{aligned}$$

Remark (i) Note that the non-linear character of the above stochastic differential equation for the option price as given in part (b) of the theorem is the reason for the complicated form of the HJB-equation that would correspond to a suitable portfolio problem involving options.

(ii) Part (a) of the theorem implicitly states that the relation between the option price and the replication strategy for the option is given by

$$f(t, P_1(t), \ldots, P_n(t)) = \sum_{i=0}^{n} \psi_i(t) P_i(t). \tag{23}$$

With the help of all our terms introduced so far, we can set up the option portfolio problem:

$$\max_{\varphi} E(U(X(T))) \tag{24}$$

with

$$X(t) = X^{\varphi}(t) = \varphi_0(t)P_0(t) + \sum_{i=1}^{n} \varphi_i(t) f^{(i)}(t, P_1(t), \ldots, P_n(t)) \tag{25}$$

(we omit the technical details of the definition of a trading strategy in options and refer the reader to Korn and Trautmann 1999 for the required integrability constraints). The basic idea to solve the option portfolio problem now consists in a two step-procedure:

Two-step procedure to solve the option portfolio problem

Step 1 Solve the corresponding portfolio problem if the underlying stocks and the bond would be tradable.

Step 2 Express the optimal positions in the underlying stocks determined in Step 1 with the help of a suitable strategy in the options and the bond.

The main result ensuring that this procedure indeed is valid can be found in Korn and Trautmann (1999):

Theorem 2 *Let the* ***Delta matrix*** $\psi(t) = (\psi_{ij}(t)),\ \ i, j = 1, \ldots, n$ *with*

$$\psi_{ij}(t) := f^{(i)}_{p_j}(t, P_1(t), \ldots, P_n(t)), \quad t \in [0, T) \tag{26}$$

be regular. Then, the option portfolio problem has the following explicit solution:

(a) *The optimal terminal wealth B^* coincides with the optimal terminal wealth of the corresponding stock portfolio problem.*
(b) *Let $\xi(t)$ be the optimal trading strategy of the corresponding stock portfolio problem. Then, the optimal trading strategy $\varphi(t)$ for the option portfolio problem is given by*

$$\bar{\varphi}(t) = (\psi(t)')^{-1}\bar{\xi}(t) \tag{27}$$

$$\varphi_0(t) = \left(X(t) - \sum_{i=1}^{n} \varphi_i(t) f(t, P_1(t), \ldots, P_n(t))\right) / P_0(t) \tag{28}$$

where $\bar{\varphi}(t)$,$\bar{\xi}(t)$ are the last n components of $\varphi(t)$ and $\xi(t)$.

Example 1 As an example to illustrate the above theorem and also the two-step procedure we consider the case of HARA-utility, i.e. of the choice of

$$U(x) = \tfrac{1}{\gamma} x^{\gamma} \tag{29}$$

with $\gamma < 1, \gamma \neq 0$. For simplicity, we also assume $n = 1$. From the explicit example of Sect. 1 we obtain

$$\xi_1(t) = \frac{b - r}{(1-\gamma)\sigma^2} \frac{X(t)}{P_1(t)}. \tag{30}$$

By noting that the optimal wealth in both the option portfolio problem and the portfolio problem including the underlying stock coincide (this is a consequence of part a) of Theorem 3, relations (27) and (30) result in

$$\varphi_1(t) = \frac{b-r}{(1-\gamma)\sigma^2}\frac{X(t)}{\psi_1(t)P_1(t)} \tag{31}$$

and in

$$\pi_{opt}(t) := \frac{\varphi_1(t) f(t, P_1(t))}{X(t)} = \frac{b-r}{(1-\gamma)\sigma^2}\frac{f(t, P_1(t))}{f_p(t, P_1(t))P_1(t)}. \tag{32}$$

By denoting by $\pi^*(t) = \pi^* = (b-r)/((1-\gamma)\sigma^2)$ the optimal portfolio process for the portfolio problem including the underlying stock, we obtain

$$\pi_{opt}(t) < \pi^* \Leftrightarrow f(t, P_1(t)) < f_p(t, P_1(t))P_1(t). \tag{33}$$

Thus, whenever the value of an option is smaller than the amount of money invested in the stock in its replication strategy, we optimally invest less money in the option than in the underlying in both corresponding portfolio problems. This is in particular the case if the option we are investing in is a (European) call option (to see this remind yourself of the explicit form of the Black-Scholes formula).

4 Optimal investment with defaultable securities

As a direct application of the results obtained in the foregoing section we will solve an optimal portfolio problem with defaultable securities in the framework of Merton's firm value model (see Merton 1974). The model is based on the assumption that the (unobservable) value process of a firm follows a geometric Brownian motion

$$dV = bVdt + \sigma V dW(t). \tag{34}$$

Further, the company has issued one share of stock and a zero coupon bond with notional B and maturity T. It is assumed that only at maturity it can be decided if the full notional of the bond can be repaid to the bondholders. If this is the case than the firm value falls by an amount of B. If, however, the bond cannot be fully repaid by the company the bond holders take over the firm and the share value drops to zero. Consequently, at the maturity T of the bond, we have the following relations for the share value $S(T)$ and the bond value $B(T)$

$$S(T) = (V(T) - B)^+, \quad B(T) = \min(V(T), B) = B - (B - V(T))^+. \tag{35}$$

Hence, the value of the share and of the corporate bond can be interpreted as the prices of call options or as $(B-)$ put options on the firm's value with strike B, respectively. If in addition we assume that there is an opportunity for a riskless investment with an interest rate of r then we obtain—in total analogy with the Black-Scholes formulae—the representations for the (defaultable) bond and share price as:

$$S(t) = V(t)\Phi(d_1(t)) - Be^{-r(T-t)}\Phi(d_2(t)), \tag{36}$$
$$B(t) = Be^{-r(T-t)}\Phi(d_2(t)) + V(t)\Phi(-d_1(t)) \tag{37}$$

with

$$d_1(t) = \frac{\ln\left(\frac{V(t)}{B}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2(t) = d_1(t) - \sigma\sqrt{T-t}. \tag{38}$$

Consequently, we can use the solution technique for optimal investment with derivatives of the preceding section to solve the following portfolio problems:

Proposition 1 *In the above market consisting of a riskless money market account, shares of a company that has issued a bond with maturity $T > 0$ and notional B, we consider an investor who is maximizing his expected utility from final wealth at time $T_1 < T$.*

(a) *If the investor is allowed to invest into the money market account and in the (defaultable) bond issued by the company then his optimal portfolio process (to be precise, the fraction of his wealth invested in the defaultable bond) is given by*

$$\pi_B^*(t) = \begin{cases} \frac{b-r}{\sigma^2}\frac{B(t)}{\Phi(-d_1(t))V(t)}, & for \;\; U(x) = \ln(x) \\ \frac{b-r}{(1-\gamma)\sigma^2}\frac{B(t)}{\Phi(-d_1(t))V(t)}, & for \;\; U(x) = {}^1\!/_{\gamma} x^{\gamma} \end{cases} \tag{39}$$

(b) *If the investor is allowed to invest into the money market account and in the share of the company then his optimal portfolio process (to be precise, the fraction of his wealth invested in the share) is given*

$$\pi_S^*(t) = \begin{cases} \frac{b-r}{\sigma^2}\frac{S(t)}{\Phi(d_1(t))V(t)}, & for \;\; U(x) = \ln(x) \\ \frac{b-r}{(1-\gamma)\sigma^2}\frac{S(t)}{\Phi(d_1(t))V(t)}, & for \;\; U(x) = {}^1\!/_{\gamma} x^{\gamma} \end{cases} \tag{40}$$

(c) *If the investor is allowed to invest into both the money market account and in the share of the company then his optimal trading strategy $(\varphi_0, \varphi_1, \varphi_2)$ in the money market account, the defaultable bond and the share is explicitly given in the following way: for any given trading strategy $\varphi_2(\cdot)$ in the share of the company, $\varphi_0(\cdot)$ and $\varphi_1(\cdot)$ are*

$$\varphi_1(t) = \frac{\xi_1(t) - \varphi_2(t)\Phi(d_1(t))}{\Phi(-d_1(t))}, \tag{41}$$
$$\varphi_0(t) = \xi_0(t) - (-\varphi_1(t) + \varphi_2(t))e^{-rT}B\Phi(d_2(t)) \tag{42}$$

with (ξ_0, ξ_1) being the optimal trading strategy in the (artificial) portfolio problem where the money market account and the firm value can be traded.

Proof Parts (a) and (b) are already proved in Korn and Kraft (2003). Alternatively, they can be derived as a direct consequence of Theorem 3. For the proof of part (c) one has to note that the stock and defaultable bond parts of the trading strategy have to be chosen in such a way that the sum of their intrinsic firm value components have to equal the optimal position in the firm value in the artificial problem. More precisely, let (ξ_0, ξ_1) be the optimal trading strategy in the (artificial) portfolio problem where the money market account and the firm value can be traded. Then, the optimal wealth process satisfies

$$X^*(t) = \xi_0(t)e^{rt} + \xi_1(t)V(t). \tag{43}$$

Due to the main theorem of the preceding section, this is also the optimal wealth process of our portfolio problem in (iii). Hence, we have to generate the positions in the firm value and in the money market account by investing according to a suitable trading strategy $(\varphi_0, \varphi_1, \varphi_2)$ in the money market account, the defaultable bond and the share. If we note further that the defaultable bond and the share satisfy

$$S(t) = (-Be^{-rT}\Phi(d_2(t)))e^{rt} + \Phi(d_1(t))V(t),$$
$$B(t) = Be^{-rT}\Phi(d_2(t))e^{rt} + \Phi(-d_1(t))V(t)$$

and that in our portfolio problem we must have

$$\begin{aligned} X^*(t) &= \varphi_0(t)e^{rt} + \varphi_1(t)B(t) + \varphi_2 S(t) \\ &= (\varphi_0(t) + \varphi_1(t)Be^{-rT}\Phi(d_2(t)) + \varphi_2(t)(-Be^{-rT}\Phi(d_2(t))))e^{rt} \\ &\quad +(\varphi_1(t)\Phi(d_1(t)) + \varphi_2 S\Phi(-d_1(t)))V(t) \end{aligned}$$

equating the coefficients $e^{rt}$of and of $V(t)$ leads to the asserted form of the optimal trading strategy. Note that we can choose one of the two components corresponding to the risky assets freely. We have without loss of generality chosen $\varphi_2(\cdot)$. As in Chap. 5 of Korn and Korn (2001) one can also show that this so obtained trading strategy is a self-financing and admissible one.

Remark (i) Note in particular that if we can only invest in the money market account and in the share then the fraction of wealth invested in the risky security is less than in the corresponding artificial portfolio problem where we can invest in the money market account and in the firm value. If, however, we replace the share by the defaultable bond as investment possibility then the fraction of wealth invested in the risky security is higher than that in the corresponding artificial portfolio problem. To see this, compare the explicit forms of $\pi_S^*(\cdot)$ and of $\pi_B^*(\cdot)$ with the representations for the share and the bond price.

(ii) If we choose

$$\varphi_1(t) = \varphi_2(t) = \xi_1(t) \tag{44}$$

then due to $V(t) = B(t) + S(t)$ we will exactly invest as much money in the risky assets as we would do in the artificial portfolio problem and—as a by-product—obtain $\varphi_0(t) = \xi_0(t)$.

(iii) Further problems in this area can be found in Korn and Kraft (2003) and Kraft and Steffensen (2006). As credit products and credit derivatives in much more complicated form than the above ones are traded at the financial markets it would be a very interesting aspect of future research to include such products as CDO and CDS into a suitable portfolio problem.

5 Optimal investment with crashes and unhedgeable risks

It is a well-known fact that the standard Black-Scholes model—which has also served as the basis of the foregoing sections—has deficiencies in explaining large movements of the stock prices observed from time to time at the financial markets. This is in particular true for large down movements, often referred to as "crashes". There are various approaches in the financial that try to overcome this problem by introducing various stochastic processes with non-normal returns for the related stock price models. Popular examples are stochastic volatility models (see e.g. Heston 1993), pure jump processes with hyperbolic distributions (see e.g. Eberlein and Keller 1995), normal inverse Gaussian distributions (see e.g. Barndorff-Nielsen 1998), Lévy processes as a whole class or even general semi-martingales. While all of these models exhibit an excellent fit to observed data, their use alone will not overcome the problems of large losses when a standard method of portfolio optimization is used. Here, we will pick up an alternative approach to crash modelling pioneered by Hua and Wilmott (1997) who assume that there are normal times in the life of a stock and crash times. They assume that the number and size of crashes (i.e. the percentage of the relative drop of the stock prices) in a given time interval are bounded. In particular, they make no probabilistic assumptions on height, number and times of occurrence of crashes. We will combine this with a worst-case approach to portfolio optimization of Korn and Wilmott (2002) to derive a new optimality concept for investment. For simplicity, we consider a market consisting of only one bond and one stock, and assume that at most one crash can happen in $[0, T]$ with a maximal height of $k^* < 1$. The security prices are assumed to follow the prices given by

$$dP_0(t) = P_0(t)r dt, \quad P_0(0) = 1, \tag{45}$$

$$dP_1(t) = P_1(t)(b dt + \sigma dW(t)), \quad P_1(0) = p \tag{46}$$

in *normal* times. At the **crash time** (if it occurs in $[0, T]$), the stock price falls by a factor of $k \in [0, k^*]$. Noting that the wealth before the crash at time t if the investor follows a portfolio process of $\pi(t)$ at that time is given by

$$X^\pi(t-) = (1 - \pi(t))X^\pi(t-) + \pi(t)X^\pi(t-), \tag{47}$$

we obtain the following simple relation between the wealth before and after a crash:

$$(1 - \pi(t))X^\pi(t-) + \pi(t)X^\pi(t-)(1 - k) = X^\pi(t-)(1 - \pi(t)k) = X^\pi(t)$$

Moreover, following the portfolio process $\pi(\cdot)$ if a crash of size k happens at time t leads to a final wealth of $X^{\pi}(T) = (1-\pi(t)k)\tilde{X}^{\pi}(T)$ where $\tilde{X}^{\pi}(\cdot)$ denotes the wealth process in the standard model without any crash possibility. Hence, in tendency, *high* values of $\pi(\cdot)$ lead to a high final wealth if no crash occurs at all, but to a high loss at the crash time. On the other hand, in tendency, *low* values of $\pi(\cdot)$ lead to a low final wealth if no crash occurs at all, but to a small loss (or even no loss at all!) at the crash time. We have thus two competing aspects (*Return and insurance*) for two different scenarios (*Crash or not*) and are therefore faced with a balance problem between risk and return.

If we would now want to optimize our portfolio in the usual way, we will realize that we do not have the full probabilistic information to treat this problem. We can only follow the idea presented in Korn and Wilmott (2002) to determine worst-case bounds for the performance of optimal investment. Thus, we are searching for the investment strategy that yields the best uniform worst-case bound, i.e. we are going to solve the problem

$$\text{(WP)} \quad \sup_{\pi(\cdot)\in A(x)} \inf_{0\le t\le T, 0\le k\le k*} E(\ln(X^{\pi}(T)))$$

where the final wealth satisfies $X^{\pi}(T) = (1-\pi(t)k)\tilde{X}^{\pi}(T)$ in the case of a crash of size k at time t. To avoid bankruptcy we require

$$\pi(t) < {}^{1}\!/_{k*} \tag{48}$$

which is a reasonable assumption anyway, given $k^* < 1$. Another natural, but important assumption is

$$b > r \tag{49}$$

which results in the fact that we do not have to consider portfolio processes $\pi(t)$ that can attain negative values as the log-utility function is increasing in x (convince yourself that the worst case bound for portfolio processes with negative values is always dominated from the one corresponding to their positive part).

Remark (i) Of course, the optimal strategy after the only crash has happened in $[0, T]$ is to follow the optimal portfolio in the crash free world, i.e. to follow

$$\pi(t) \equiv \pi^* := \frac{b-r}{\sigma^2}. \tag{50}$$

Thus, the occurrence of the crash has the positive effect that the investor can be sure that no further one will happen until T.

(ii) To give the reader an intuition for our worst-case approach, we will calculate the worst-case bound for two extreme strategies. The first one is $\pi(t) \equiv 0$ (i.e. *playing safe*). Here, the worst-case scenario is that no crash appears at all (!), as then the investor can never switch to π^*. This leads to the following worst-case bound of

$$WCB_0 = E(\ln(X^0(T))) = \ln(x) + rT. \tag{51}$$

The other extreme strategy consists of ignoring the crash and to follow the optimal strategy as if no crash possibility would be present, i.e. to use

$$\pi(t) \equiv \pi^* := \frac{b-r}{\sigma^2}. \tag{52}$$

Then, the worst-case scenario is the occurrence of a crash of maximum size k^* (at an arbitrary time instant!), leading to the following worst-case bound of

$$WCB_{\pi^*} = E(\ln(X^{\pi^*}(T))) = \ln(x)+rT+\frac{1}{2}\left(\frac{b-r}{\sigma}\right)^2 T+\ln(1-\pi^* k^*) \tag{53}$$

Comparing the two results of Remark (ii) leads to the insights that

– it depends on time to maturity which one of the above strategies is better
– a constant portfolio process cannot be the optimal one
– strategy $\pi(t) \equiv 0$ takes too few risk to be good if no crash occurs while the strategy $\pi(t) \equiv \pi$ is too risky to perform well if a crash occurs, and the optimal strategy should balance this out!

We can show even stronger properties of the optimal strategy, and moreover, can derive it explicitly as a unique solution of an ordinary differential equation (see Korn and Wilmott 2002):

Theorem 3 *Optimal investment in the presence of a crash with log utility There exists a portfolio process $\hat{\pi}(\cdot)$ such that the corresponding expected log-utility after an immediate crash equals the expected log-utility given no crash occurs at all. It is given as the unique solution $\hat{\pi}(\cdot) \in [0, 1/k^*)$ of the differential equation*

$$\pi'(t) = \frac{1}{k^*}\left(1-\pi(t)k^*\right)\left(\pi(t)(b-r) - \frac{1}{2}\left(\pi(t)^2\sigma^2 + \left(\frac{b-r}{\sigma}\right)^2\right)\right), \tag{54}$$

$$\pi(T) = 0. \tag{55}$$

*Further, this strategy yields the **highest worst-case bound** for our problem (WP). In particular, this bound is active at each future time point (uniformly optimal balancing). After the crash has happened the optimal strategy is given by*

$$\pi(t) \equiv \pi^* := \frac{b-r}{\sigma^2}. \tag{56}$$

Proof Note first that it is clear that after the only crash has happened, it is optimal to follow the strategy of the form of (54) as then we are in the crash-free standard market of Sect. 1. In this proof, we only indicate how to obtain the above portfolio process $\hat{\pi}(\cdot)$. For the optimality proof, we refer to Korn and Wilmott (2002). Let

$$v_0(t,x) = \ln(x) + \left(r + \frac{1}{2}\left(\frac{b-r}{\sigma}\right)^2\right)(T-t) \tag{57}$$

be the value function corresponding to the portfolio problem in the usual, crash-free Black-Scholes setting (see also Sect. 1). Note also that any portfolio process yielding the highest worst-case bound obviously has to satisfy $\pi(T) = 0$. The above indifference property between the worst possible crash and no crash happening at all is satisfied by a portfolio process $\hat{\pi}(\cdot)$ with

$$\ln(x) + r(T-t) + E\int_t^T \left(\hat{\pi}(s)(b-r) - \frac{1}{2}\hat{\pi}^2(s)\sigma^2\right) ds$$

$$= v_0(t, x(1-\hat{\pi}(t)k*)) = \ln(x) + \ln(1-\hat{\pi}(t)k*) + \left(r + \frac{1}{2}\left(\frac{b-r}{\sigma}\right)^2\right)(T-t)$$

Here, the left hand side is the expected log-utility given no crash occurs at all. If we now assume that there exists a differentiable (thus meaning in particular a deterministic !) such process $\hat{\pi}(\cdot)$, via differentiating both sides of the above equation with respect to t, we obtain the differential equation

$$\pi'(t) = \frac{1}{k*}(1-\pi(t)k*)\left(\pi(t)(b-r) - \frac{1}{2}\left(\pi(t)^2\sigma^2 + \left(\frac{b-r}{\sigma}\right)^2\right)\right). \quad (58)$$

What remains to show is the existence of a classical solution to this equation also satisfying the final condition $\pi(T) = 0$ and $\hat{\pi}(\cdot) \in \left[0, {}^{1}/_{k*}\right)$. This however follows by standard existence and uniqueness results, and by the explicit form of the equation.

Remark The problem can also be treated in a more general setting including

- more general utility functions,
- changing market coefficients after a crash,
- insurance risk as a further risk component,
- a multi-asset market.

For those topics we refer to Korn (2005) and to Korn and Menkens (2005).

To illustrate the qualitative behaviour of the optimal strategy we consider the following example where we have chosen $b = 0.2, r = 0.05, \sigma = 0.4, k^* = 0.2$ and $T = 1$. The optimal strategy is illustrated by Fig. 1. Note that the optimal portfolio π^* in the *crash-free* standard model equals $\pi^* = 0,9375$. It is way above the optimal portfolio process in view of a crash. Given the above input data this is not surprising. For such a small time horizon the crash risk dominates the potential of the stock return compared to bond investment. To understand how the strategy works note that as long as no crash has happened the investor has to keep his portfolio on the lower line. Directly after the crash he can switch to π^*.

6 Further recent developments and aspects of future research

The area of continuous-time portfolio optimization is still a very active area of research and results on various different levels of generality and applicability have

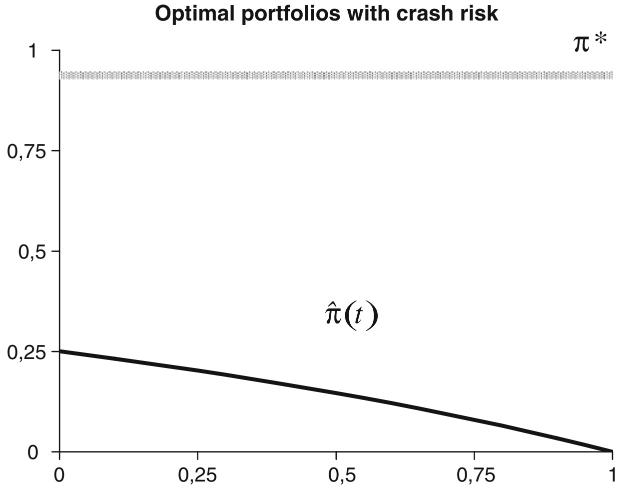

Fig. 1 Optimal investment for insurers with exponential utility

been achieved recently. On the side of the theory the introduction of duality methods to solve constrained portfolio optimization problems (such as in Cvitanic and Karatzas 1992 or—even more general—in Kramkov and Schachermayer 1999) have generalized the standard scope of continuous-time portfolio optimization by a large amount. Further, considering risk measures in portfolio optimization is an intensively researched area (see e.g. Föllmer and Schied 2002 for an introduction into the subject). On the practical side, however, the applications of continuous-time portfolio methods in reality are still rare. The main reason is that strictly following the optimal strategies of continuous trading lead to an enormous amount of transaction costs. Therefore, it is substantial to find good ways to transform continuous-time strategies to real life applications. The work by Rogers (2001) is an excellent starting point for further studies. Explicit results on portfolio optimization under transaction costs are still not available. Easily implementable strategies taking transaction costs into account are a necessary ingredient for the breakthrough of the application of continuous-time methods in reality.

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