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Numerical solutions to coupled-constraint (or generalised Nash) equilibrium problems

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Abstract This paper is about games where the agents face constraints in the combined strategy space (unlike in standard games where the action sets are defined separately for each player) and about computational methods for solutions to such games. The motivation examples for such games include electricity generation problems with transmission capacity constraints, environmental management to control pollution and internet switching to comply to buffers of bounded capacity. In each such problem a regulator may aim at compliance to standards or quotas through taxes or charges. The relevant solution concept for these games has been known under several names like *generalised Nash equilibrium*, *coupled constraint equilibrium* and more. Existing numerical methods converging to such an equilibrium will be explained. Application examples of use of NIRA, which is a suite of Matlab routines that implement one of the methods, will be provided.

Keywords Computational economics · Compliance problems · Coupled constraint games · Eeneralised Nash Equilibrium · Nikado–Isoda function · Relaxation algorithm (NIRA) · Quasi-variational inequalities

JEL Classification C63 · C72 · C88 · D78 · E62

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1 Introduction

The aim of this paper is to coach the managerial and economic communities to apply *generalised* equilibrium as a solution concept for games with constraints in the *combined* strategy space of all agents. The benefits of use of this equilibrium include a learned design of policies that can enforce constraints' satisfaction in a competitive environment and an analysis of economic agents' behaviour subjected to constraints.

Computation of Pigouvian taxes *with-a-base-line*,[1] see e.g. Randall (1987), for a one-agent compliance problem is classical in environmental economics. However, a typical feature of an environmental compliance problem is that it is multi-agent. This suggests a game theory context but, the game must be special rather than traditional. It needs to be a game where the set of options available to one agent depends on the other agents' choices. E.g., if an environmental standard on a beach's cleanliness, which *many* agents pollute, is (newly) added to the game then a joint constraint in the *combined* strategy space of all agents needs to be imposed. If all agents act simultaneously, no traditional solution is available to such a game.

The feature of handling games with a constrained strategy space is also of prime importance for electricity market modeling. In a typical problem of electricity generation and distribution, the competing economic agents' strategy space is coupled. This is (mainly) due to capacity constraints and Kirchhoff's laws, and signifies that the set of options available to an agent depends on the other agents' choices. No traditional noncooperative game theory concept can be used to solve such a game.

Neither does a "classical" game theory problem solution exist to an internet traffic problem if a selfish user exhausts a constrained capacity of the buffer.

All three situations[2] described above have the same structure: each is a multi-person non-cooperative game with constraints whose satisfaction depends on the actions undertaken by *all* agents. Moreover, it is possible to envisage that there is an "umpire" (or regulator) in each game capable of levying taxes on agents. If the taxes are customarily designed then they can modify the agents' payoffs so that their maximising actions satisfy the constraint.

The interest in games that allow for constraints in the combined action space is almost as old as Nash equilibrium, see Debreu (1952). However, after a controversial (or misunderstood) remark made in Ichiishi (1983) that the *generalised* Nash equilibrium concept "…is **only** useful as a mathematical tool to establish existence theorems in various **applied contexts**[3]", references to that topic became rare in the "main-stream" economics literature. However, the

[1] I.e., where taxes are applied after a constraint is violated.

[2] For environmental-management coupled-constraint games see Haurie and Krawczyk (1997), Krawczyk (2005) (and other papers listed there) and Tidball and Zaccour (2005); for such games "played" on electricity markets, see Contreras et al. (2004) and Hobbs and Pang (2006); for constrained internet traffic models see Kesselman et al. (2005).

[3] Stressed by JK.

remark did not stop the mathematical operations-research (OR) researchers from applying and developing this solution concept, see e.g., Harker (1991) and Robinson (1993). The concept has also been used in the politico–economic context of environmental management (see e.g., Haurie and Krawczyk (1997) or Krawczyk and Uryasev (2000)).

In fact, the remark quoted above seems contradictory and was bound to be misunderstood: it questions the importance of the constrained solution concept but, it also says that the concept helps to establish solution existence in applied contexts; so, presumably, the remark's author saw importance of the constrained games and their solutions for applications.

Following is a brief outline of what this paper contains. In Sect. 2 a class of games with coupled constraint action sets will be defined. The solution approaches to such games will be discussed in Sect. 3. Application examples of use of NIRA, which is a suite of Matlab routines that implement one of the methods, will be provided in Sect. 4. The paper ends with concluding remarks.

2 Coupled constraint equilibrium

2.1 Name and history

So, we need a game model for situations where the players' strategy sets are not disjoint. We said that an umpire might be naturally included in such games. If we think of a democracy with a (local) government empowered to levy taxes and charges on players then the government may play the role of the umpire. The latter will (usually) want to induce the agents to act in a *socially* acceptable manner. This might be why Debreu (1952) (see also Arrow and Debreu 1954) called the solution concept for a game with constraints a *social equilibrium*.

It was noticed in McKenzie (1959) that an equilibrium under constraints generalises the "classical" Nash equilibrium and called the concept *general equilibrium*. Other names for the constrained equilibrium concept have been *pseudo-Nash equilibrium*, *generalised Nash equilibrium* , *normalised equilibrium* and *coupled constraint equilibrium*.

The last name is due to Rosen (1965). Perhaps this is the most informative name since it refers explicitly to the coupling character of constraints in equilibrium. At present, this name (i.e., *coupled constraint equilibrium*) and *generalised Nash equilibrium* seem to prevail: the former in politico-economic contexts, the latter in the OR literature.

2.2 The class of games with feasibility sets interactions

There have been several mathematical operations research papers on coupled constraint games applications[4] and methodology published in the last 15 years.

[4] Robinson (1993) is interested in a military problem that he modelled and analysed as constrained equilibrium problem.

It was Harker (1991) who used the enlighting term for this class of mathematical problems: games with *feasibility sets interactions*. He and many other authors developed (and are developing) methodology for these games in conjunction with *variational inequalities*. See Pang and Fukushima (2005) for a modern treatment of generalised Nash equilibria in the context of *quasi*-variational inequalities.[5] A reason for the link between constrained equilibria and variational inequalities is in that existence of this equilibrium can be established as a "byproduct" of the solution existence to a variational inequality problem, see Sect. 3.3 later in this paper.

As said, it is the paper by Rosen (1965) that has a politico-economic appeal. This is because one of Rosen's concerns is induction of agents to chose to "play" a constrained equilibrium. Once the Karush-Kuhn-Tucker multipliers associated with the common constraints are computed the equilibrium implementation is straightforward: the multipliers need to be used as penalty tax rates for the constraints' violation (see Haurie and Krawczyk 1997). If the players allow for the taxes in their payoffs then they will play a real time game whose solution is the desired equilibrium, see Contreras et al. (2004) and Krawczyk (2005).

Rosen (1965) allows for a discriminatory treatment of players and provides a criterion for the solution uniqueness (which is easy to apply for smooth models). The former permits us to modify the Karush-Kuhn-Tucker multipliers to adjust the tax rates among players in case the regulator wants to favour certain players. The latter (i.e., uniqueness) is of prime importance for applications. By introducing an environmental tax, the regulator will "move" the agents from an existing equilibrium. Therefore, knowledge that a new equilibrium exists and is *unique* is crucial for the tax legislation and implementation. It suffices to say that without the equilibrium uniqueness, the tax effectiveness could not be examined.

For the symmetric information case (adopted in this paper), the charges computed in the method will be never paid. In other words, the KKT multiplier is the "correct" shadow price, given which the agents choose to modify their actions so that their output and/or externalities are not exceeding the constraint.

2.3 Mathematical model

We need the following definitions and theorems to establish the existence and uniqueness conditions of a Nash equilibrium in coupled constraint games. The proofs of the theorems can be found in Rosen (1965).

Consider a *concave game* i.e., such where the n players have payoff functions[6] $(\phi_i)_{i=1,\ldots n}$ continuous in $\mathbf{x} = (x_1, x_2, \ldots x_n)' \in X \subset \mathcal{R}^m$ and concave in $x_i \in \mathcal{R}^{m_i}$, and such that strategy set X is a convex, closed and bounded subset

[5] See Nagurney (1993) for a complete introduction to variational inequalities.

[6] We will assume that the payoffs are interpersonally comparable e.g., all expressed in \$ terms.

of $\mathcal{R}^m$. The solution to the game is $\mathbf{x}^*$ that satisfies

$$\phi_i(\mathbf{x}^*) = \max_{y_i|\mathbf{x}^* \in X} \phi_i(y_i|\mathbf{x}^*) \tag{1}$$

where $y_i|\mathbf{x} \equiv (x_1, x_2, \dots y_i, \dots x_n)'$ denotes a collection of actions when the i-th agent "tries" y_i while the remaining agents are playing $x_j, j = 1, 2, \dots i-1, i+1, \dots n$. At $\mathbf{x}^*$ no player can improve their own payoff by a unilateral change in his (or her) strategy which keeps the combined vector in X.

Game (1) shall be called a *coupled constraint game* (à la Rosen 1965). The *coupling* refers to the fact that one player's action affects what the other players' actions can be. In the special case where $X = X_1 \times \cdots \times X_n$ i.e., each player's action is individually constrained, the game is said to have uncoupled constraints.

Notice that (1) can be interpreted as a model for a solution to each *compliance* problem alluded to above (environmental, electrical and switching). Functions $(\phi_i)_{i=1,\dots,n}$, are the players' payoffs. The set X will comprise strategies that fulfil the game standards imposed by the government or the network requirements. In particular, set X will be convex, closed and bounded if the standards to comply with are defined as concave functions of agent strategies.

We want now to specify the mathematical conditions for the possibly unique existence of coupled-constraint equilibrium.

Let us first consider uncoupled-constraint games.

Theorem 1 *An equilibrium point exists for every concave n-person game.*

Therefore, we know that if each player has a payoff function which is continuous in all players' actions and concave[7] with respect to his own strategy while the other players' strategies remain fixed, then the game must have at least one Nash equilibrium. The payoff concavity assumption is common for many economic activities' models.

If the equilibrium is unique than regulator can enforce it. The conditions for uniqueness rely on the concept of diagonal strict concavity (DSC) of the *joint payoff function*

$$f(\mathbf{x}, \mathbf{r}) \equiv \sum_{i=1}^{n} r_i \phi_i(\mathbf{x}), \quad \mathbf{r} \in \mathcal{R}_+^n. \tag{2}$$

Vector $\mathbf{r}$ is composed of weights r_i with which the regulator appraises an individual agent's payoff (e.g., from a community's point of view). We will see that the weights $\mathbf{r}$ will help the regulator to adjust the levels of agents' responsibility for the constraint satisfaction. If all agents are treated equally *i.e.*, the burden of responsibility is distributed evenly, then $\mathbf{r} = [1\ 1\ \dots]$.

[7] These assumptions can be weakened, see e.g., Harker (1991) or Fudenberg and Tirole (1991).

Definition 1 *The joint payoff function* $f(\mathbf{x},\mathbf{r}) = \sum_{i=1}^{n} r_i\phi_i(\mathbf{x})$, $\mathbf{r} \in \mathcal{R}_+^n$, *is called* diagonally strictly concave *(*DSC*) for* $\mathbf{x} \in X$ *and fixed* $\mathbf{r}$ *if for every* $\mathbf{x}^1, \mathbf{x}^2 \in X$ *we have*

$$(\mathbf{x}^2 - \mathbf{x}^1)' g(\mathbf{x}^1, \mathbf{r}) + (\mathbf{x}^1 - \mathbf{x}^2)' g(\mathbf{x}^2, \mathbf{r}) > 0 \tag{3}$$

where $g(\mathbf{x},\mathbf{r})$ *is the pseudogradient of*

$$g(\mathbf{x},\mathbf{r}) = \begin{bmatrix} r_1 \nabla_1 \phi_1(\mathbf{x}) \\ \vdots \\ r_n \nabla_n \phi_n(\mathbf{x}) \end{bmatrix}. \tag{4}$$

Notice that this definition relies on differentiability of payoff functions $\phi_i(\mathbf{x})$, $i = 1, \dots n$. For twice-continuously differentiable functions, a criterion for DSC is straightforward and consists of checking whether the Jacobian of $g(\mathbf{x},\mathbf{r})$ (or, equivalently, the pseudo-Hessian of $f(\mathbf{x},\mathbf{r})$) is negative definite. If so, the main diagonal terms, which reflect concavity of an agent's payoff in his own actions, dominate the other terms that indicate the strength of the other agents interactions.

Now, the economic interpretation of DSC becomes clear. A game whose joint payoff is DSC (or, for shortness, a game which is DSC), is one in which each player has more control over his own payoff than the other players have over it. This is also a common, and desired, feature of many economic models.

Later in Sect. 3.4 we will see that an approximated equilibrium existence can be established through weaker conditions that DSC.

Theorem 2 *In a game with uncoupled constraints, if the joint payoff function* $f(\mathbf{x},\mathbf{r})$ *is* DSC *for some* $\mathbf{r} > 0$, *then there exists a unique Nash equilibrium.*

When the constraints are *coupled*, there are no such guarantees, and a special type of equilibrium must be defined.

For that purpose, assume that the constraint set X is defined through a collection of functions $h : \mathcal{R}^m \to \mathcal{R}^L$, where m is the dimension of the collective strategy space (n is the number of players so, $m \geq n$)

$$X = \{\mathbf{x} : h(\mathbf{x}) \geq 0\}. \tag{5}$$

Each component of h represents a constraint and there may be L of them. In the case of a concave game, each $h_\ell(\mathbf{x}), \ell = 1, \dots L$, is a concave function of $\mathbf{x}$ (suppose continuously differentiable in $\mathbf{x}$ and such that the constraint qualification conditions are satisfied).

Denote the constraint shadow price vector for player i by $\boldsymbol{\lambda}_i^* \in \mathcal{R}^L$. Then, $\mathbf{x}^* \in X$ is a *coupled constraint equilibrium* point if and only if it satisfies the

Karush-Kuhn-Tucker conditions:

$$h(\mathbf{x}^*) \geq 0 \tag{6}$$

$$(\boldsymbol{\lambda}_i^*)^T h(\mathbf{x}^*) = 0 \tag{7}$$

$$\phi_i(\mathbf{x}^*) \geq \phi_i(y_i|\mathbf{x}^*) + (\boldsymbol{\lambda}_i^*)^T h(y_i|\mathbf{x}^*) \tag{8}$$

for all $i = 1, \ldots n$ and where $y_i|\mathbf{x}$ was defined in (1). In general terms, conditions (6)–(8) define $\mathbf{x}^*$ as a vector of non improvable strategies if $\mathbf{x}^* \in X$.

The above conditions establish a solution to (1) under the adopted differentiability and qualification assumptions. In general, the multipliers will not be related to each other. However, we shall consider a special kind of equilibrium, which can reflect the different levels of agent responsibility for the constraint satisfaction (expressed by the vector $\mathbf{r}$) and is unique.

Definition 2 *An equilibrium point* $\mathbf{x}^*$ *is a Rosen–Nash* normalised equilibrium point *if, for some vectors* $\mathbf{r} > 0$ *and* $\boldsymbol{\lambda}^* \geq 0$*, conditions* (6)–(8) *determine* $\mathbf{x}^*$ *and are satisfied for*

$$\boldsymbol{\lambda}_i^* = \frac{\boldsymbol{\lambda}^*}{r_i} \tag{9}$$

for each i.[8]

For shortness we have dropped *coupled constraint* from the equilibrium definition.

The number of components of vector $\boldsymbol{\lambda}^*$ (and of $\boldsymbol{\lambda}_i^*$) is obviously the same as the "length" of $h(\mathbf{x})$. So, $\boldsymbol{\lambda}^* = [\lambda^{*,1}, \ldots \lambda^{*,\ell}, \ldots \lambda^{*,L}]'$ where each $\lambda^{*,\ell}$, $\ell = 1, \ldots L$ is a single constraint's Lagrange multiplier.

We can better understand the role of weights r_i now. If an agent's weight r_i, in the joint payoff function (2), is greater than those of his (or her) competitors, then his (or her) Lagrange multipliers are lessened, relative to the competitors'. This can be interpreted as a concession gained by this agent to pollute more (or transmit more energy) than their competitors. Paraphrasing, the vector $\mathbf{r}$ tells us of how the regulator has distributed the burden of the constraints' satisfaction among the agents.

Theorem 3 *There exists a normalised equilibrium point to a concave n-person game for every specified* $\mathbf{r} > 0$.

The wording of the following theorem crucial for coupled-constraint games is a bit stronger than in Rosen (1965), see Haurie and Krawczyk (2002).

Theorem 4 *Let the weighting* $\bar{\mathbf{r}} \in Q$ *be given where* Q *is a convex subset of* $\mathcal{R}^n_+$*. Let* $f(\mathbf{x}, \bar{\mathbf{r}})$ *be diagonally strictly concave on the convex set* X *and such that the Kuhn–Tucker multipliers exist. Then, for the weighting* $\bar{\mathbf{r}}$*, there is a unique normalised equilibrium point.*

[8] We could say that $\boldsymbol{\lambda}^*$ are the "objective" shadow prices while $\boldsymbol{\lambda}_i^*$ are the "subjective" ones.

This means that, if a game is DSC for a feasible distribution of the constraint satisfaction responsibilities, then the game possesses a unique coupled constraint equilibrium for each such distribution.

The regulator can enforce the equilibrium through the modification of each agent's payoff by a penalty term

$$T_i(\boldsymbol{\lambda}^*, r_i, \mathbf{x}) \equiv \frac{\boldsymbol{\lambda}^*}{r_i} \min\big(0, h(\mathbf{x})\big); \tag{10}$$

obviously, $T_i(\boldsymbol{\lambda}^*, r_i, \mathbf{x}) < 0$ for $\mathbf{x} \notin X$.

We can say that the computation of the Lagrange multipliers *uncouples* the equilibrium strategies $\mathbf{x}^*$ (or that the game becomes uncoupled). This means that the Rosen–Nash normalised equilibrium strategies $\mathbf{x}^*$ are *also* Nash equilibrium strategies of the following uncoupled game

$$\phi_i(\mathbf{x}^{**}) = \max_{y_i|\mathbf{x}^{**} \in \mathcal{R}^n} \big(\phi_i(y_i|\mathbf{x}^{**}) + T_i(\boldsymbol{\lambda}^*, r_i, y_i|\mathbf{x}^{**})\big) \tag{11}$$

i.e.,

$$\mathbf{x}^{**} = \mathbf{x}^*.$$

In other words, if the charge scheme (10) (which includes information on coupled-constraint-equilibrium shadow prices $\frac{\boldsymbol{\lambda}^*}{r_i}$) is communicated to agents, then the agents play an uncoupled game whose unique solution satisfies the constraints.

This is important for applications. If the game at hand is DSC for $\bar{\mathbf{r}} \in Q$, the regulator can be certain of what the agents' reaction is going to be to a tax signal defined through the Lagrange multipliers $\boldsymbol{\lambda}_i^*$. After DSC is established (which implies uniqueness), the game still needs to be solved. In particular, the Lagrange multipliers $\boldsymbol{\lambda}_i^*$ as well as the equilibrium strategies $\mathbf{x}^*$ have to be computed. This might be a difficult procedure that requires computational support.

Notwithstanding the computational difficulty of calculating $\boldsymbol{\lambda}^*$, the policy maker needs to observe $h(\mathbf{x})$ but not individual $x_i, i = 1, \ldots n$ to determine $T_i(\boldsymbol{\lambda}^*, r_i, \mathbf{x})$. This is an attractive feature of regulation based on coupled constraint equilibria in situations where tracing a constraint's violation to the offender is difficult or impossible.

3 Numerical solutions to coupled constraint games

3.1 Numerical approach

Numerical methods for solutions to games with joint constraints in the strategy space can be grouped in three categories:

- methods exploiting "explicit" use of Rosen's (1965) theorems and conditions;
- methods based on solutions to quasi-variational inequalities à la Pang and Fukushima (2005);
- min-maximisation of a bi-variate function like the Nikaido–Isoda function[9] (see Uryasev and Rubinstein 1994).[10]

The methods will be described below albeit the latter will receive most attention due to author's extensive experience with this method.

3.2 Rosen's algorithm

Rosen's (1965) projected gradient algorithm consists of constrained minimisation of a pseudo-gradient norm (i.e., norm of (4)). The algorithm can be itemised as follows:

(i) Begin.
(ii) Compute the payoffs' pseudo-gradients at feasible points $\mathbf{x}^{(m)}$, where m is the iteration number and $j = 1, 2, \ldots, n$:

$$\boldsymbol{\gamma}_j^{(m)} = r_j \cdot \nabla_j \phi_j(\mathbf{x})\big|_{\mathbf{x}^{(m)}}$$

(iii) Make a step in the direction of the pseudo-gradient

$$\mathbf{x}^{(m+1)} = \mathbf{x}^{(m)} + \kappa^{(m)} \boldsymbol{\gamma}^{(m)}$$

where $\kappa^{(m)}$ is such that $\mathbf{x}^{(m+1)}$ attains either the first active constraint or the pseudo-gradient norm is minimised.
Repeat (ii) and (iii) until the pseudo-gradient norm is sufficiently small.
(iv) Determine the Karush-Kuhn-Tucker multipliers.

As for any gradient algorithm, convergence of Rosen's algorithm depends on concavity of the underlying function (here, on joint payoff $f(\mathbf{x}, \mathbf{r})$, see (2)). For games that are DSC, convergence is quick and reliable. In Haurie and Krawczyk (1997), the above algorithm was used to solve a three-player River Basin Pollution game.[11]

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[9] Or Ky Fan, see Aubin (1993).

[10] Also see papers by this author on coupled constraint equilibria.

[11] The game was later treated as a test problem in Krawczyk and Uryasev (2000) and Krawczyk (2005).

3.3 Quasi-variational inequalities

A variational inequality problem[12] consists of finding a vector $\mathbf{x}^o \in K$ such that

$$(\mathbf{z} - \mathbf{x}^o)^T F(\mathbf{x}^o) \leq 0, \quad \forall \mathbf{z} \in K \tag{12}$$

where F is a vector map $F : \mathcal{R}^n \to \mathcal{R}^n$ and $K \subset \mathcal{R}^n$. Notice that $\mathbf{x}^o$ (if it exists) is such that $F(\mathbf{x}^o)$ is perpendicular to the (convex) set K at $\mathbf{x}^o$. Should $F(\cdot)$ be the gradient of a maximised function $\varphi(\cdot)$ then a solution to (12) $\mathbf{x}^o$ solves a constrained maximisation problem

$$\mathbf{x}^o = \arg \max_{x \in K} \varphi(x). \tag{13}$$

Hence, if we have a numerical method that solves a variational inequality then we can use this method to compute a solution to the constrained maximisation problem represented as the variational inequality. As a matter of fact, numerical methods for variational inequalities are readily available and utilised as optimisation "tools", see e.g. Facchinei and Pang (2003).

The link between variational inequalities and equilibria is immediate if we think of $F(\cdot)$ as of the pseudo-gradient of the joint payoff function (2) with $\mathbf{r} \equiv 1$.

The constraint set which was generically represented by symbol X in Sect. 2.3, can now be denoted

$$X \equiv \bigcup_i K_i(\mathbf{x}_{(-i)})$$

where $K_i(\mathbf{x}_{(-i)})$ is the action set for player i. (This notation highlights the fact that the actions available to player i depend on the other players' choices.) Hence, player i maximisation problem (1) can be represented as:

$$\max \phi_i(y_i|\mathbf{x}_{(-i)}) \quad \text{s.t.} \quad y_i \in K_i(\mathbf{x}_{(-i)}), \quad i = 1, \ldots, n. \tag{14}$$

If $F(\mathbf{x})$ represents (again) the pseudo-gradient of the joint payoff function (2) with $\mathbf{r} \equiv 1$ then $\mathbf{x}^o$ that satisfies

$$(\mathbf{z} - \mathbf{x}^o)^T F(\mathbf{x}^o) \leq 0, \quad \forall \mathbf{z} \in K(\mathbf{x}^o) \tag{15}$$

is a coupled constraint equilibrium (or a generalised Nash equilibrium, see Pang and Fukushima 2005). Inequality (15) is called a quasi-variational inequality and is a generalisation of (12) because of the constraint set $K(\cdot)$ dependence on strategy $\mathbf{x}$.

[12] In the mathematical OR literature, variational inequalities are usually formulated for minimisation problems see e.g., Pang and Fukushima (2005) or Hobbs and Pang (2006). Consequently, their inequality.

One can formulate the theorem (see e.g., Haurie and Krawczyk 2002) that if the game is DSC then (15) is the necessary and sufficient condition for $\mathbf{x}^o$ to be a unique Nash equilibrium.

The diagonal strict concavity (DSC) assumption is then equivalent to the property of strong monotonicity of the pseudo-gradient operator $F(x)$ in the parlance of variational inequality theory.

On the other hand, Pang and Fukushima (2005) formulate the theorem that guarantees existence of a solution to the quasi-variational inequality problem (15). Requested is continuity of F, and also continuity of $K(x)$ where $K(\cdot)$ is understood as a correspondence[13] between point $x \in \mathcal{R}^n$ and a subset of $\mathcal{R}^n$; moreover, for every $x \in T \subset \mathcal{R}^n$, $K(x) \neq \emptyset$ needs to be a convex and closed subset of a compact and convex T. Obviously, DSC (backed by convexity of $K(x)$) is a sufficient condition under which a solution to (15) is the generalised Nash equilibrium (or coupled constraint equilibrium).

3.4 The NIRA approach

3.4.1 What is NIRA*?*

NIRA is the acronym used to denote an equilibrium search method based on the (bivariate) Nikaido–Isoda function and a relaxation algorithm. Min-maxmisation of this function can deliver a solution to a coupled constraint game. The Nikaido–Isoda function and its min-maxmisation relaxation algorithm are economically interpretable. The algorithm convergence conditions do not have to rely on smoothness of the payoff functions.

However, should the payoff functions be twice continuously differentiable, verification of the convergence theorem's hypotheses becomes much easier. Moreover, for such *smooth* games, the conditions under which NIRA converges to a unique Nash equilibrium imply diagonal strict concavity of the game at hand. This means that if the game satisfies the NIRA convergence conditions and the algorithm has converged, then the convergence point is a coupled constraint equilibrium. Moreover, NIRA delivers the Lagrange multipliers as an integral part of the solution procedure. The multipliers can then be used to decouple the game.

NIRA was used in Berridge and Krawczyk (1997), Contreras et al. (2004) and Krawczyk and Uryasev (2000) for the solution of several infinite games of varying complexity. An open-loop dynamic River Pollution game was solved through NIRA in Krawczyk (2005). In that paper, managerial and economic aspects of a transition process, from a polluted environment state to one in which the agents comply to the standards, were discussed; economics of a non-conservative behaviour of the agents was explained.

[13] Or, point-to-set mapping.

3.4.2 Nikaido–Isoda function

Let us now introduce the Nikaido–Isoda function (see Nikaido and Isoda 1955). This function transforms an equilibrium problem into an optimisation problem and has an interesting economic interpretation.

Definition 3 *Let ϕ_i be the payoff function for player i and X a collective strategy set as in* (1). *Then the* Nikaido–Isoda function $\Psi : X \times X \to \mathcal{R}$ *is defined as*

$$\Psi(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} [\phi_i(y_i|\mathbf{x}) - \phi_i(\mathbf{x})]. \tag{16}$$

Result 1 *Uryasev and Rubinstein* (1994)

$$\Psi(\mathbf{x}, \mathbf{x}) \equiv 0 \quad \mathbf{x} \in X. \tag{17}$$

Each summand in (16) can be thought of as the *improvement in payoff* a player will receive by changing his or her action from x_i to y_i while all other players continue to play according to $\mathbf{x}$. The function thus represents the sum of these improvements in payoff. Note that the *maximum* value this function can take for a given $\mathbf{x}$ is always nonnegative, owing to Result 1 above. Also, the function is nonpositive everywhere when either $\mathbf{x}$ or $\mathbf{y}$ is a Nash equilibrium point, since in an equilibrium situation no player can make a unilateral improvement to their payoff, and so each summand in this case can be at most zero.

From here, the conclusion is reached that when the Nikaido–Isoda function cannot be made (significantly) positive for a given $\mathbf{y}$, we have (approximately) reached the Nash equilibrium point. This observation is useful in constructing a termination condition for our algorithm; that is, choose an $\varepsilon > 0$ such that, when $\max_{\mathbf{y} \in X} \Psi(\mathbf{x}^s, \mathbf{y}) < \varepsilon$, where $\mathbf{x}^s \in X$ is computed at the current iteration s, the equilibrium has been reached to a sufficient degree of precision.

3.4.3 Mathematics of NIRA

The next three definitions (see *e.g.*, Nurminski 1982 or Uryasev and Rubinstein 1994) are about weak convexity and concavity[14] of a bivariate function. As the following Theorem 5 (the convergence theorem) will document it, weak convex–concavity of a function is a crucial assumption needed for convergence of a relaxation algorithm to a coupled constraint equilibrium.

Let X be a convex closed subset of the Euclidean space $\mathcal{R}^m$ and f a continuous function $f : X \to \mathcal{R}$.

[14] Recall the following elementary definition: a function is "just" *convex* $\iff$

$$\alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \;\geq\; f\Big(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}\Big).$$

Definition 4 *A function of one argument $f(\mathbf{x})$ is* weakly convex *on X if there exists a function $r(\mathbf{x},\mathbf{y})$ such that $\forall \mathbf{x},\mathbf{y} \in X$*

$$\begin{aligned} &\alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \geq f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) + \alpha(1-\alpha) r(\mathbf{x},\mathbf{y}) \\ &0 \leq \alpha \leq 1, \text{ and } \frac{r(\mathbf{x},\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|} \to 0 \text{ as } \|\mathbf{x}-\mathbf{y}\| \to 0 \quad \forall \mathbf{x} \in X. \end{aligned} \tag{18}$$

Definition 5 *A function of one argument $f(\mathbf{x})$ is* weakly concave *on X if there exists a function $\mu(\mathbf{x},\mathbf{y})$ such that, $\forall \mathbf{x},\mathbf{y} \in X$*

$$\begin{aligned} &\alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \leq f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) + \alpha(1-\alpha) \mu(\mathbf{x},\mathbf{y}) \\ &0 \leq \alpha \leq 1, \text{ and } \frac{\mu(\mathbf{x},\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|} \to 0 \quad \text{as} \quad \|\mathbf{x}-\mathbf{y}\| \to 0 \ \forall \mathbf{x} \in X. \end{aligned} \tag{19}$$

Example The convex function $f(x) = x^2$ is weakly concave. See Krawczyk and Uryasev (2000).

Let $\Psi : X \times X \to \mathcal{R}$ be a function defined on a product $X \times X$, where X is a convex closed subset of the Euclidean space $\mathcal{R}^m$.

Definition 6 *A function of two vector arguments, $\Psi(\mathbf{x},\mathbf{y})$ is referred to as* weakly convex–concave *if it satisfies weak convexity with respect to its first argument and weak concavity with respect to its second argument.*

That is, for fixed $\mathbf{z} \in X$,

$$\begin{aligned} &\alpha\Psi(\mathbf{x},\mathbf{z}) + (1-\alpha)\Psi(\mathbf{y},\mathbf{z}) \geq \Psi(\alpha\mathbf{x} + (1-\alpha)\mathbf{y},\mathbf{z}) + \alpha(1-\alpha) r(\mathbf{x},\mathbf{y};\mathbf{z}) \\ &\mathbf{x},\mathbf{y} \in X, \ 0 \leq \alpha \leq 1, \text{ and } \frac{r(\mathbf{x},\mathbf{y};\mathbf{z})}{\|\mathbf{x}-\mathbf{y}\|} \to 0 \quad \text{as} \quad \|\mathbf{x}-\mathbf{y}\| \to 0 \ \forall \mathbf{x} \in X \end{aligned} \tag{20}$$

and

$$\begin{aligned} &\alpha\Psi(\mathbf{z},\mathbf{x}) + (1-\alpha)\Psi(\mathbf{z},\mathbf{y}) \leq \Psi(\mathbf{z},\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) + \alpha(1-\alpha) \mu(\mathbf{x},\mathbf{y};\mathbf{z}) \\ &\mathbf{x},\mathbf{y} \in X, \ 0 \leq \alpha \leq 1, \text{ and } \frac{\mu(\mathbf{x},\mathbf{y};\mathbf{z})}{\|\mathbf{x}-\mathbf{y}\|} \to 0 \quad \text{as} \quad \|\mathbf{x}-\mathbf{y}\| \to 0 \ \forall \mathbf{x} \in X \end{aligned} \tag{21}$$

$r(\mathbf{x},\mathbf{y};\mathbf{z})$ and $\mu(\mathbf{x},\mathbf{y};\mathbf{z})$ are referred to as the residual terms.

The functions $r(\mathbf{x},\mathbf{y};\mathbf{z})$ and $\mu(\mathbf{x},\mathbf{y};\mathbf{z})$ were introduced with the concept of weak convex–concavity. Notice that smoothness of $\Psi(\mathbf{z},\mathbf{y})$ is not required. However, if $\Psi(\mathbf{x},\mathbf{y})$ is twice continuously differentiable with respect to both arguments on $X \times X$, the residual terms satisfy (see e.g., Krawczyk and Uryasev 2000)

$$r(\mathbf{x},\mathbf{y};\mathbf{y}) = \frac{1}{2}\langle A(\mathbf{x},\mathbf{x})(\mathbf{x}-\mathbf{y}),\mathbf{x}-\mathbf{y}\rangle + o_1(\|\mathbf{x}-\mathbf{y}\|^2) \tag{22}$$

and

$$\mu(\mathbf{y},\mathbf{x};\mathbf{x}) = \frac{1}{2}\langle B(\mathbf{x},\mathbf{x})(\mathbf{x}-\mathbf{y}),\mathbf{x}-\mathbf{y}\rangle + o_2(\|\mathbf{x}-\mathbf{y}\|^2) \tag{23}$$

where $A(\mathbf{x},\mathbf{x}) = \Psi_{\mathbf{xx}}(\mathbf{x},\mathbf{y})|_{\mathbf{y}=\mathbf{x}}$ is the Hessian of the Nikaido–Isoda function with respect to the first argument and $B(\mathbf{x},\mathbf{x}) = \Psi_{\mathbf{yy}}(\mathbf{x},\mathbf{y})|_{\mathbf{y}=\mathbf{x}}$ is the Hessian of the Nikaido–Isoda function with respect to the second argument, both evaluated at $\mathbf{y} = \mathbf{x}$.

To prove the inequality of condition (v) of Theorem 5 (the convergence theorem, below) under the assumption that $\Psi(\mathbf{x},\mathbf{y})$ is twice continuously differentiable, it suffices to show that

$$Q(\mathbf{x},\mathbf{x}) = A(\mathbf{x},\mathbf{x}) - B(\mathbf{x},\mathbf{x}) \tag{24}$$

is strictly positive.

Definition 7 *The relaxation algorithm's* optimum response function*, possibly multi-valued, at point* $\mathbf{x}$ *is (see Başar* 1987*, Uryasev and Rubinstein* 1994*)*

$$Z(\mathbf{x}) = \arg\max_{\mathbf{y}\in X} \Psi(\mathbf{x},\mathbf{y}),\ \ \mathbf{x}, Z(\mathbf{x}) \in X. \tag{25}$$

Suppose we wish to find a Nash equilibrium of a game and we have some initial estimate of it, say $\mathbf{x^0}$, and $Z(\mathbf{x})$ is single-valued; the relaxation algorithm is given by the following formula

$$\mathbf{x}^{s+1} = (1-\alpha_s)\mathbf{x}^s + \alpha_s Z(\mathbf{x}^s), \quad s = 0, 1, 2, \dots \quad , \tag{26}$$

where $0 < \alpha_s \leq 1$. The iterate at step $s+1$ is constructed as a weighted average of the improvement point $Z(\mathbf{x}^s)$ and the current point $\mathbf{x}^s$. This averaging ensures convergence of the algorithm under certain conditions as stated in the theorem below. See Krawczyk and Uryasev (2000) for a proof.

Theorem 5 (Convergence theorem.) *There exists a unique normalised Nash equilibrium point to which the algorithm* (26) *converges if:*

- (i) *X is a convex compact subset of $\mathcal{R}^m$,*
- (ii) *the Nikaido–Isoda function $\Psi : X \times X \to \mathcal{R}$ is a weakly convex-concave function and $\Psi(\mathbf{x},\mathbf{x}) = 0$ for $\mathbf{x} \in X$,*
- (iii) *the optimum response function $Z(\mathbf{x})$ is single valued and continuous on X,*
- (iv) *the residual term $r(\mathbf{x},\mathbf{y};\mathbf{z})$ is uniformly continuous on X w.r.t. $\mathbf{z}$ for all $\mathbf{x},\mathbf{y} \in X$,*
- (v) *the residual terms satisfy*

$$r(\mathbf{x},\mathbf{y};\mathbf{y}) - \mu(\mathbf{y},\mathbf{x};\mathbf{x}) \geq \beta(\|\mathbf{x}-\mathbf{y}\|) \quad \mathbf{x},\mathbf{y} \in X\ , \tag{27}$$

where $\beta(0) = 0$ and β is a strictly increasing function (i.e., $\beta(t_2) > \beta(t_1)$ *if* $t_2 > t_1$),

(vi) *the relaxation parameters* α_s *satisfy*
- *either (non-optimised step)*
  (a) $\alpha_s > 0$,
  (b) $\sum_{s=0}^{\infty} \alpha_s = \infty$,
  (c) $\alpha_s \to 0$ *as* $s \to \infty$.
- *or (optimised step)*

$$\alpha_s = \arg \min_{\alpha \in [0,1)} \left\{ \max_{\mathbf{y} \in X} \Psi(\mathbf{x}^{s+1}(\alpha), \mathbf{y}) \right\}. \tag{28}$$

Remark 1 It is interesting to note that one can consider the relaxation algorithm as either performing a static optimisation or calculating successive actions of players in convergence to an equilibrium in a real time process. If all payoff functions are known, one can directly find the Nash equilibrium using the relaxation algorithm. However, if only one player's payoff function and all players' past actions can be accessed, or computed, then at each stage in the real time process one calculates the optimum response for that player, assuming that the other players will play as they had in the previous period. In this way, convergence to the Nash normalised equilibrium will occur as $s \to \infty$. By taking sufficiently many iterations of the algorithm, the Nash equilibrium will be determined with a specified precision.

4 Management and economics of coupled constraint equilibria

The managerial aspect of enforcing compliance to standards, and the enforcement process economics will be illustrated using the solution to the River Basin Pollution Game (see Krawczyk 2005, compare Haurie and Krawczyk 1997). For a discussion of the same aspects in an energy generation context see Contreras et al. (2004).

4.1 Formulation of a River Basin Pollution Problem

Three players $j = 1, 2, 3$ are located along a river. Each of them is engaged in an economic activity (paper pulp producing, say) at a chosen level x_j. Their joint production externalities (pollution) must satisfy environmental constraints set by a local authority.

It is assumed that one pollutant is produced in the quantity that is a linear (not crucial for the algorithm's convergence) function of agent's output x_j i.e.,

$$\text{pollution} = e_j x_j$$

where e_j is the emission coefficient of player j, given in Table 1. The pollution is expelled into the river, where it disperses, decays and, finally, reaches a

Table 1 Constants for the River Basin Pollution game

Player j	c_{1j}	c_{2j}	e_j	δ_{j1}	δ_{j2}
1	0.10	0.01	0.50	6.5	4.583
2	0.12	0.05	0.25	5.0	6.250
3	0.15	0.01	0.75	5.5	3.750

monitoring station in the amount of $\sum_j \delta_{j\ell} e_j x_j$, where $\delta_{j\ell}$ is the decay-and-transportation coefficient from player j to location ℓ. In the game, two monitoring stations $\ell = 1, 2$ are located along the river, at which the local authority has set maximum pollutant concentration levels K_ℓ. So, the constraint on pollution that is imposed by the local authority at location ℓ is

$$q_\ell(\mathbf{x}) = \sum_{j=1}^{3} \delta_{j\ell} e_j x_j \leq K_\ell, \quad \ell = 1, 2. \tag{29}$$

Each player j is supposed to maximise the net profit (also referred to as payoff)

$$\phi_j(\mathbf{x}) = \underbrace{[d^{(1)} - d^{(2)}(x_1 + x_2 + x_3)]x_j}_{\text{revenue}} - \underbrace{(c_{1j} + c_{2j}x_j)x_j}_{\text{cost}}. \tag{30}$$

The economic constants $d^{(1)}$ and $d^{(2)}$ determine the inverse demand law and are set to 3.0 and 0.01, respectively. The values for the cost function coefficients c_{1j} and c_{2j} are given in Table 1; finally $K_\ell = 100, \ \ell = 1, 2$. Notice that Player 3 is "inefficient" in that his emission and cost (see coefficients e_j and c_{13}) are the largest.

4.2 Equilibrium solution

The game,

$$\phi_j(\mathbf{x}^*) = \max_{q_\ell(y_j|\mathbf{x}^*) \leq K_\ell, \ell=1,2} \phi_j(y_j|\mathbf{x}^*), \quad j = 1, 2, 3 \tag{31}$$

in which agents maximise profits (30) subject to actions satisfying jointly linear constraints (29), is a coupled constraint game. The NIRA approach will be used to compute an equilibrium to this game.

Although our game is diagonally strictly concave, exploiting this feature is not needed here. Instead, it is shown below that the Nikaido–Isoda function is *weakly convex–concave* for the game at hand. This is sufficient for the relaxation algorithm to converge to an equilibrium of the river basin pollution game.

We assume that the players share the responsibility for satisfying the constraints evenly or “in solidarity” i.e., $r_i = 1$, $i = 1, 2, 3$. The Nikaido-Isoda function in this case is

$$\begin{aligned}\Psi(\mathbf{x}, \mathbf{y}) &= \sum_{j=1}^{3} (\phi_j(y_i|\mathbf{x}) - \phi_j(\mathbf{x})) \\ &= [d^{(1)} - d^{(2)}(y_1 + x_2 + x_3) - c_{11} - c_{21}y_1]y_1 \\ &\quad + [d^{(1)} - d^{(2)}(x_1 + y_2 + x_3) - c_{12} - c_{22}y_2]y_2 \\ &\quad + [d^{(1)} - d^{(2)}(x_1 + x_2 + y_3) - c_{13} - c_{23}y_3]y_3 - \sum_{j=1}^{3} \phi_j(\mathbf{x}). \end{aligned} \tag{32}$$

Notice that the set defined by equations (29) is convex. Condition (v) of the convergence theorem (Theorem 5) follows from strict positive-definiteness of the matrix $Q(\mathbf{x}, \mathbf{x})$ (see (24))

$$\begin{aligned} Q(\mathbf{x}, \mathbf{x}) &= \Psi_{\mathbf{xx}}(\mathbf{x}, \mathbf{y})|_{\mathbf{y}=\mathbf{x}} - \Psi_{\mathbf{yy}}(\mathbf{x}, \mathbf{y})|_{\mathbf{y}=\mathbf{x}} \\ &= \begin{pmatrix} 4c_{21} + 4d^{(2)} & 2d^{(2)} & 2d^{(2)} \\ 2d^{(2)} & 4c_{22} + 4d^{(2)} & 2d^{(2)} \\ 2d^{(2)} & 2d^{(2)} & 4c_{23} + 4d^{(2)} \end{pmatrix} \\ &= 2\,d^{(2)} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + 2 \begin{pmatrix} 2c_{21} + d^{(2)} & 0 & 0 \\ 0 & 2c_{22} + d^{(2)} & 0 \\ 0 & 0 & 2c_{23} + d^{(2)} \end{pmatrix}. \end{aligned} \tag{33}$$

The other assumptions of Theorem 5 also are satisfied.

Using a starting guess of $\mathbf{x} = (0, 0, 0)$ in NIRA’ suite of MATLAB programs (see Berridge and Krawczyk 1997) the convergence result was obtained

$$\mathbf{x}^* = (21.14, 16.03, 2.73),$$

which yields net profits $\phi^* = (48.41, 26.92, 6.61)$. The first constraint is active *i.e.*, $q_1(\mathbf{x}) = K_1 = 100$ and the corresponding equilibrium Lagrange multiplier[15] $\lambda_1 = 0.5744$. The second constraint is non active ($q_2(\mathbf{x}) = 81.17$ hence the slack is 18.83) and $\lambda_2 = 0$.

[15] In NIRA, the Lagrange multipliers are computed as the constraints’ shadow prices by a constrained optimisation routine (`fmincon`) used in min-maximisation of $\Psi(\mathbf{x}, \mathbf{y})$ *s.t.* $(\mathbf{x}, \mathbf{y}) \in X$.

4.3 Management of equilibria

4.3.1 A self-enforced constrained equilibrium

We can compel the players to obey an equilibrium solution by applying Pigouvian taxes. This is straightforward to show for the *equilibrium* solution $\mathbf{x}^*$.

Use the Lagrange multipliers $\lambda_1 = 0.5744$, $\lambda_2 = 0$ to create a new, uncoupled (or unconstrained) game. For the active constraint, place a charge on each player in the amount of

$$T(\mathbf{x}) \equiv T_{j,1}(\lambda_1, 1, \mathbf{x}) = \lambda_1 \max(0, q_1(\mathbf{x}) - K_1), \tag{34}$$

where $\lambda_1 = 0.5744$, is the penalty coefficient for violating the first constraint. Since $T(\mathbf{x})$ is a non-smooth penalty function [see (34), compare(10)], there will always exist coefficients λ_1 sufficiently large to ensure that agents adhere to the environmental constraints (29) (see Shor 1985). In other words, for "large" λ_1, the pollution produced by agents' optimal solutions will satisfy the environmental standard. Applying the taxes as above modifies the payoff functions, which now become $\tilde{\phi}_j$

$$\tilde{\phi}_j(\mathbf{x}) = R_j(\mathbf{x}) - F_j(\mathbf{x}) - \sum_{\ell} T(\mathbf{x})$$

or

$$\begin{aligned}\tilde{\phi}(\mathbf{x}) = [d^{(1)} - d^{(2)}(x_1 + x_2 + x_3) - c_{1j} - c_{2j}x_j]x_j \\ - \lambda_1 \max\left(0, \sum_{j=1}^{3} \delta_{j1} e_j x_j - K_1\right), \quad j = 1, 2, 3.\end{aligned} \tag{35}$$

Where the parameters are as in Table 1.

The new equilibrium problem with payoff functions $\tilde{\phi}_j$ and *uncoupled* constrains has the Nash equilibrium point x^{**} defined by the equation

$$\tilde{\phi}_j(\mathbf{x}^{**}) = \max_{y_j \geq 0} \tilde{\phi}_j(y_j|\mathbf{x}^{**}) \quad j = 1, \ldots n. \tag{36}$$

Following Krawczyk and Uryasev (2000), a conjecture has been made based on the general theory of non-smooth optimisation (see, for example Shor 1985) that, for the environmental constraint satisfaction, the penalty coefficient λ_1 should be greater than or equal to the Lagrange multiplier corresponding to first of the constraints (29). In the numerical experiments, λ_1 was set to *equal* the "final" Lagrange multiplier that was observed during the calculation of the constrained equilibrium by algorithm (26). For this setup, the unconstrained equilibrium $\mathbf{x}^{**}$ was approximately *equal* to the constrained equilibrium $\mathbf{x}^{*}$.[16]

In Krawczyk and Uryasev (2000) cross-sectional graphs were produced[17] to plot the modified payoff functions for each player. Each payoff function appears non-smooth and achieves its maximum at point $\mathbf{x}^{*}$.

4.3.2 A social planner's solutions

It is interesting to observe that the third player's production is very small as compared to the others'. As said, this player is obviously "inefficient" in terms of his (or her) cost function as well as (poor) abatement technology (see high c_{13} and e_3). The inefficiency of the third player is even more evident from a *cooperative* or social planner's problem solution. If the social planner's problem is defined as

$$\hat{\mathbf{x}} = \arg \max_{q_\ell(\mathbf{x}) \le K_\ell, \ell=1,2} \{\phi_1 + \phi_2 + \phi_3\} \tag{37}$$

the optimal solution is

$$\hat{\mathbf{x}} = (24.63, 14.66, 0.4)\,,$$

the slack on the second constraint is larger than before ($q_2(\mathbf{x}) = 80.45$) and the corresponding joint profit $\sum_{i=1}^{3} \phi_1(\hat{\mathbf{x}}) = 82.21$, which is more then the sum of the competitive equilibrium profits $\sum_{i=1}^{3} \phi_1(\mathbf{x}^{*}) = 81.94\,.$

It is also possible to compute the taxes that compel the players to produce (and pollute) at the cooperative (or *efficient*) solution levels $\hat{\mathbf{x}}$. A new set of Lagrange multipliers has to be computed and a new distribution of the burden of satisfying the constraints needs to be decided. The latter requirement means that weights $r_i, i = 1, 2, 3$ in formulae (9) and (10) *cannot* be [1, 1, 1] and have to be calculated.

The weights and the Lagrange multipliers could be established through the solution of the following problem

$$\text{compute } r_1, r_2, r_3 \quad \text{s.t.} \tag{38}$$

$$\left.\begin{aligned} \hat{\mathbf{x}} &= \arg\max\, r_1\phi_1(y_1|\hat{\mathbf{x}}) \\ \hat{\mathbf{x}} &= \arg\max\, r_2\phi_2(y_2|\hat{\mathbf{x}}) \\ \hat{\mathbf{x}} &= \arg\max\, r_3\phi_3(y_3|\hat{\mathbf{x}}) \end{aligned}\right\} \tag{39}$$

[16] In general, the new unconstrained Nash equilibrium $\mathbf{x}^{**}$ may not equal exactly the constraint equilibrium $\mathbf{x}^{*}$, see Shor (1985).

[17] Repeated in Krawczyk (2005).

where each maximisation is constrained by pollution $q_\ell(y_j|\mathbf{x}^*) \leq K_\ell$, $\ell = 1,2$, $j = 1,2,3$. A solution to (38), (39) (which is *not* a coupled constrained game[18]) was obtained in Krawczyk (2005) as

$$\bar{\mathbf{r}} = [1.001\ 0.778\ 0.91818]; \quad \lambda_1 = 0.54341,\ \lambda_2 = 0.$$

The modified payoff functions diminished by the taxes can now be defined:

$$\begin{aligned}\bar{\phi}_j(\mathbf{x}) &= [d^{(1)} - d^{(2)}(x_1 + x_2 + x_3) - c_{1j} - c_{2j}x_j]x_j \\ &\quad - \frac{\lambda_1}{r_j} \max\left(0, \sum_{j=1}^{3} \delta_{j1} e_j x_j - K_1\right), \quad j = 1,2,3 \end{aligned} \tag{40}$$

and the equilibrium of the uncoupled game with the modified payoffs

$$\bar{\phi}_j(\bar{\mathbf{x}}) = \max_{y_j \geq 0} \bar{\phi}_j(y_j|\bar{\mathbf{x}}), \quad j = 1, \ldots n. \tag{41}$$

can be computed. This game was solved in Krawczyk (2005) and the solution $\bar{\mathbf{x}}$ was the same as the social planner's solution $\hat{\mathbf{x}}$. This confirms that, through a set of specially designed Pigouvian taxes, an efficient solution can be made a Nash equilibrium.

4.4 Economics of constrained equilibria

Solutions to coupled constraint games can give us information on how an economy might react to an introduction of (new) standards of production, externalities, transmission capacities *etc.* Accordingly, the socio-economic implications of the standards can be assessed. Some implications are discussed below in relation to the River Basin Game considered in previous sections.

Of importance to the regulator are *efficiency* issues. The sum of payoffs (*wealth*) generated by the efficient solution $\hat{\mathbf{x}}$ happens to be numerically close to the competitive equilibrium $\mathbf{x}^*$. This would suggest that rather than enforcing the former through a somewhat arbitrary weight modification (from [1 1 1] to $\bar{\mathbf{r}}$) the regulator will prefer to rely on the equilibrium solution, in which all agents are treated as equal.

As said, the data in Table 1 suggests that Player 3 is inefficient. In the constrained equilibrium, this agent is ascribed 8% of the total payoff. Rather expectedly, under the efficient solution this agent receives much less (which is about 0.5% of the total payoff). This might raise questions regarding efficiency of the "generalised" Nash equilibrium.

In general, no Nash equilibrium is Pareto efficient. While the observation about assigning 8%, instead of 0.5%, of payoff to a player who *appears*

[18] Hence does not have to have a unique solution.

inefficient might be worrisome for this game, it is worthwhile to note that in the *unconstrained* equilibrium (i.e., if $\mathbf{x} \in \mathcal{R}^n$) this player's participation level in wealth creation would be about 44%. This is because the unconstrained equilibrium of this game is

$$\mathbf{x}^0 = (55.3497, 14.9139, 53.6841) \tag{42}$$

which generates the payoffs to players [61.272, 13.346, 57.636]. We may conclude that for this game, "playing" a coupled constrained game improves *efficiency* with respect to the unconstrained game.

5 Concluding remarks

This paper deals with models suitable to study an important class of games arising in politico-economic contexts of compliance to standards or quotas. Such contexts comprise problems of environmental management, electricity generation and transmission and internet switching. Their common features is that the players' strategy spaces are coupled.

In broad terms, numerical solutions to such games can be obtained through gradient projection, quasi-variational inequalities and min-maximisation of the Nikaido–Isoda function.

The Nikaido–Isoda function was utilised to compute a coupled constraint equilibrium of a game representative for coupled constraint contexts. Managerial aspects of making a constrained equilibrium a self-enforced Nash equilibrium (or *decoupling* the game) were highlighted and economics of compliance to standards was discussed. In general terms, decoupling is straightforward for suitably concave games and requires players' payoffs modifications based on the constraints' Lagrange multipliers. "Playing" a coupled constraint equilibrium can improve the solution's *relative* efficiency.

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