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Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games

Jong-Shi Pang[1,⋆] **and Masao Fukushima**[2,⋆⋆]

[1] Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, New York 12180-3590, USA (e-mail: pangj@rpi.edu)
[2] Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan (e-mail: fuku@i.kyoto-u.ac.jp)

Abstract. The noncooperative multi-leader-follower game can be formulated as a generalized Nash equilibrium problem where each player solves a nonconvex mathematical program with equilibrium constraints. Two major deficiencies exist with such a formulation: One is that the resulting Nash equilibrium may not exist, due to the nonconvexity in each player's problem; the other is that such a nonconvex Nash game is computationally intractable. In order to obtain a viable formulation that is amenable to practical solution, we introduce a class of remedial models for the multi-leader-follower game that can be formulated as generalized Nash games with convexified strategy sets. In turn, a game of the latter kind can be formulated as a quasi-variational inequality for whose solution we develop an iterative penalty method. We establish the convergence of the method, which involves solving a sequence of penalized variational inequalities, under a set of modest assumptions. We also discuss some oligopolistic competition models in electric power markets that lead to multi-leader-follower games.

Keywords: Quasi-variational inequalities, leader-follower games, Nash equilibrium, electric power market modeling, oligopolistic competition, mathematical program with equilibrium constraints

1 Introduction

It is by now a well-known fact that the Nash equilibrium problem where each player solves a convex program can be formulated and solved as a finite-dimensional

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variational inequality [11, 19]. The generalized Nash game is a Nash game in which each player's strategy set depends on the other players' strategies. The connection between the generalized Nash games and quasi-variational inequalities (QVIs) was recognized by Bensoussan [4] as early as 1974 who studied these problems with quadratic functionals in Hilbert space. Harker [18] revisited these problems in Euclidean spaces. Robinson [39, 40] discussed an application of a generalized Nash problem in a two-sided game model of combat. Kocvara and Outrata [28] discussed a class of QVIs with applications to engineering. Wei and Smeers [46] introduced a QVI formulation of a spatial oligopolistic electricity model with Cournot generators and regulated transmission prices. Pang [34] recently analyzes the computational resolution of the generalized Nash game by the Josephy-Newton method [25, 11].

Unlike the VI, which has a long history and an extensive literature (see the monograph [11], which presents a state-of-the-art study of finite-dimensional variational inequalities and complementarity problems), the study of the QVI to date is in its infancy at best. In particular, there is only a handful of papers that address the QVI in finite dimensions in terms of existence of solutions [8, 38, 47] (see also [11, Section 2.8]) and in solution methods [28, 34]. As such, it is of independent interest to develop efficient computational methods for solving QVIs. This is one motivation for the present research.

More importantly, another motivation of the present paper stems from some recent oligopolistic competition models in the electricity power markets, which can be formulated as a unified Nash equilibrium problem, where each player solves a nonconvex optimization problem, called a "subgame equilibrium". One such model [7, 22] describes the bidding strategy of a dominant firm (the leader) in a competitive electricity market, whose goal is to maximize its profit subject to a set of price equilibrium constraints. The mathematical formulation for this model with a single dominant firm is thus a **M**athematical **P**rogram with **E**quilibrium **C**onstraints, or simply, an MPEC. Due to the disjunctive complementarity conditions in its constraints, an MPEC is a difficult, nonconvex optimization problem [30, 33]; efficient methods for computing globally optimal solutions are to date not available. In practice, the multi-dominant firm problem is of greater importance; and yet, as a Nash game, the latter problem can have no equilibrium solution in the standard sense. An example due to Hobbs [21] which we reproduce later illustrates this non-existence of solution. In game-theoretic language, while the problem of a single dominant firm can be identified with a Stackelberg leader-follower game, whose relation with the MPEC is known [30], that of multi-dominant firms corresponds to a noncooperative game with multiple leaders and several followers, which as a Nash problem is defined by players solving nonconvex subgame equilibrium problems. A major objective of the present paper is to propose QVI relaxations for a "nonconvex Nash problem" of the latter kind.

A related Nash problem with nonconvex subgame equilibria arises from an alternative competitive electricity model. Originated from Hobbs [20] and further developed in several subsequent papers [31, 37, 36], the model aims at understanding the strategic behavior of several firms in dealing with the presence of an arbitrager in the market. There are two behavioral approaches considered; exogenous versus endogenous arbitrage. In the exogenous-arbitrage approach, the arbitrager is considered a Nash player who maximizes its profit and occupies a role in the game just like all other players, including the firms. This approach leads to a standard Nash equilibrium problem that can be formulated as a linear or nonlinear complementarity problem, depending on other model characteristics. In the endogenous-arbitrage approach, the firms anticipate the behavior of the arbitrager and therefore include the optimality conditions and the associated variables of the arbitrager's profit maximization problem in their own optimization problems. In the cited references, the arbitrager's problem is a simple equality-constrained linear program whose optimality conditions is a system of linear equations. Therefore, there is no great technical difficulty including the latter equations in the constraints of the firms' problems. Nevertheless, a tremendous challenge arises with the endogenous-arbitrage approach when there are inequality constraints present in the arbitrager's problem. The optimality conditions of the latter problem involve complementarity conditions, which when put into the constraints of the firms' optimization problems turn the latter into MPECs. Once again, a Nash problem with nonconvex subgame equilibria is obtained. To the best of our knowledge, the latter problem has not been formulated in the literature; in Subsection 5.2, we present a formal description of the problem and explain how it fits the framework of a multi-leader-follower game.

2 The generalized Nash game as a QVI

The generalized Nash game with N players may be defined in the following abstract way. For $\nu = 1, \ldots, N$, let $K^\nu : \Re^{n_{-\nu}} \to \Re^{n_\nu}$ be a given set-valued map, where each n_ν is a positive integer,

$$n \equiv \sum_{\nu=1}^{N} n_\nu \quad \text{and} \quad n_{-\nu} \equiv n - n_\nu;$$

thus for each $x^{-\nu} \in \Re^{n_{-\nu}}$, $K^\nu(x^{-\nu})$ is a subset of $\Re^{n_\nu}$, which is the strategy set of player ν. We write a vector $x \in \Re^n$ in the partitioned form:

$$x \equiv (x^\nu)_{\nu=1}^N \quad \text{with each } x^\nu \in \Re^{n_\nu}.$$

For the most part in the paper, we assume that $K^\nu(x^{-\nu})$ is finitely representable by convex inequalities. To describe such a representation and each player's optimization problem, let $g^\nu : \Re^n \to \Re^{m_\nu}$, $h^\nu : \Re^{n_\nu} \to \Re^{\ell_\nu}$, and $\theta_\nu : \Re^n \to \Re$ be given functions, where m_ν and ℓ_ν are positive integers. We make the following blanket convexity assumption for each $\nu = 1, \ldots, N$.

Convexity assumption. Each function h_j^ν is continuously differentiable and convex on $\Re^{n_\nu}$ for all $j = 1, \ldots, \ell_\nu$; moreover, for each $x^{-\nu} \in \Re^{n_{-\nu}}$, the functions

$\theta_\nu(x^{-\nu}, \cdot)$ and $g_i^\nu(x^{-\nu}, \cdot)$ are continuously differentiable and convex in the argument x^ν for each $i = 1, \ldots, m_\nu$.

The functions g^ν and h^ν are distinguished by their arguments; the former is a function of the entire vector $x \in \Re^n$ whereas the latter is a function of only the subvector x^ν. This distinction is motivated by the application in the generalized Nash game where x^ν is the strategy vector of player ν and x is the strategy vector of all the N players in the game. We then define

$$K^\nu(x^{-\nu}) \equiv \{ x^\nu \in \Re^{n_\nu} : g^\nu(x) \leq 0, \; h^\nu(x^\nu) \leq 0 \}; \tag{1}$$

thus $K^\nu(x^{-\nu})$ is a closed convex subset of $\Re^{n_\nu}$. Here we see that player ν's strategies are constrained in two ways: those that are dependent on other players' strategies $g^\nu(x) \leq 0$ and those that are dependent solely on player ν's strategies $h^\nu(x^\nu) \leq 0$. This distinction is useful for both the treatment of the existence of solution and for the penalty method described subsequently.

The generalized Nash game is to find a tuple $x^* \equiv (x^{*,\nu}) \in \Re^n$, called a generalized Nash equilibrium (GNE), such that for each $\nu = 1, \ldots, N$, $x^{*,\nu}$ is an optimal solution of the convex optimization problem in the variable x^ν with $x^{-\nu}$ fixed at $x^{*,-\nu}$:

$$\begin{aligned} &\text{minimize } \theta_\nu(x^{*,-\nu}, x^\nu) \\ &\text{subject to } x^\nu \in K^\nu(x^{*,-\nu}). \end{aligned} \tag{2}$$

Defining

$$K(x) \equiv \prod_{\nu=1}^{N} K^\nu(x^{-\nu}) \quad \text{for } x \equiv (x^\nu)_{\nu=1}^N \in \Re^n \tag{3}$$

and

$$F(x) \equiv (\nabla_{x^\nu} \theta_\nu(x))_{\nu=1}^N \in \Re^n, \tag{4}$$

we see that x^* is a GNE if and only if $x^* \in K(x^*)$ and

$$(y - x^*)^T F(x^*) \geq 0, \quad \forall\, y \in K(x^*).$$

The latter problem is an instance of the QVI, which is formally defined in the next subsection. For each tuple $x^{-\nu} \in \Re^{n_{-\nu}}$, let $S_\nu(x^{-\nu}) \subset \Re^{n_\nu}$ denote the optimal solution set of player ν's optimization problem (2) parameterized by its rivals' strategy vector $x^{-\nu}$. A GNE is thus a tuple x^* such that $x^{*,\nu} \in S_\nu(x^{*,-\nu})$ for all ν.

2.1 Existence of a solution to a QVI

Formally, given a point-to-point map F from $\Re^n$ into itself and a point-to-set map K from $\Re^n$ into subsets of $\Re^n$, the QVI (K, F) is to find a vector $x^* \in K(x^*)$ such that

$$(y - x^*)^T F(x^*) \geq 0, \quad \forall\, y \in K(x^*).$$

When $K(x)$ is independent of x, say, $K(x) = K$ for all x, the QVI becomes the VI of finding $x \in K$ such that

$$(y - x)^T F(x) \geq 0, \quad \forall\, y \in K.$$

We recall that a set-valued map $\Phi : \Re^n \to \Re^m$ is *continuous* at a vector x if Φ is both upper and lower semicontinuous at x. Upper semicontinuity at x means that for every $\varepsilon > 0$ there exists an open neighborhood $\mathcal{N}$ of x such that

$$\bigcup_{y \in \mathcal{N}} \Phi(y) \subseteq \Phi(x) + \mathbb{B}(0, \varepsilon),$$

where $\mathbb{B}(0, r)$ denotes the open Euclidean ball centered at the origin and with radius $r > 0$; lower semicontinuity at x means that for every open set $\mathcal{U}$ such that $\Phi(x) \cap \mathcal{U} \neq \emptyset$, there exists an open neighborhood $\mathcal{N}$ of x such that for each $y \in \mathcal{N}$, $\Phi(y) \cap \mathcal{U} \neq \emptyset$. We say that Φ is continuous in a domain if it is continuous at every point in the domain. For excellent references on set-valued maps, see [1, 41].

In the existence result below, we postulate that the set-valued map K is compact-valued and continuous. We do not impose any special structure on the sets $K(x)$. This general framework is presented herein for the sake of completeness.

Theorem 1 Let F be a continuous point-to-point map from $\Re^n$ into itself and let K be a point-to-set map from $\Re^n$ into subsets of $\Re^n$. If there exists a compact convex set $T \subset \Re^n$ such that

(a) for every $x \in T$, $K(x)$ is a nonempty, closed, convex subset of T;
(b) K is continuous at every point in T,

then the QVI (K, F) has a solution.

Proof. It is clear that

$$K(T) \equiv \bigcup_{x \in T} K(x) \subseteq T.$$

Thus the theorem follows from Corollary 3.1 in [8]. □

It is possible to refine the above theorem by relaxing the boundedness of the sets $K(x)$ and replacing it by strengthened assumptions on F (such as some form of coercivity), for our purpose in this paper, Theorem 1 is sufficient.

2.2 Existence of a GNE

Theorem 1 can certainly be applied to the generalized Nash problem, provided that the strategy map K given by (3) is continuous. While such a continuity condition (or its refinement) is appropriate in the general framework where no special structure is assumed on each set $K^\nu(x^{-\nu})$, when the latter is finitely representable by convex inequalities, as in (1), it is possible to give a more direct treatment of the existence of a GNE that takes advantage of the explicit representation of the set $K^\nu(x^{-\nu})$. The treatment still employs Kakutani's fixed-point theorem.

Before proceeding further, we contrast our treatment of this problem with a similar treatment employed by Robinson [40]. While both use Kakutani's fixed-point theorem, the main difference between our approach and Robinson's lies in how the "rival-dependent constraints" are being handled. Whereas Robinson's approach is based on convex analysis and the theory of functional epiconvergence (see also [26]), which is fairly general and requires no particular representation of the strategy sets $K^\nu(x^{-\nu})$ nor the differentiability of the players' objective functions, our approach below is based on constraint qualifications on the latter sets and on the KKT systems of the players' optimization problems. The approach is therefore more "computation-friendly" and the required assumptions are generally easier to verify than those in the epiconvergence approach.

To begin, let us write down the Karush-Kuhn-Tucker (KKT) system of player ν's optimization problem (2):

$$\begin{aligned} \nabla_{x^\nu} \theta_\nu(x) + \sum_{i=1}^{m_\nu} \lambda_i^\nu \nabla_{x^\nu} g_i^\nu(x) + \sum_{j=1}^{\ell_\nu} \mu_j^\nu \nabla h_j^\nu(x^\nu) &= 0 \\ 0 \le \lambda^\nu \perp g^\nu(x) &\le 0 \\ 0 \le \mu^\nu \perp h^\nu(x^\nu) &\le 0 \end{aligned} \tag{5}$$

where $\lambda^\nu \in \Re^{m_\nu}$ and $\mu^\nu \in \Re^{\ell_\nu}$ are the KKT multipliers of the constraints $g^\nu(x) \le 0$ and $h^\nu(x^\nu) \le 0$, respectively, in the set $K^\nu(x^{-\nu})$, and $a \perp b$ means two vectors a and b satisfy $a^T b = 0$. We postulate the following **S**equentially **B**ounded **C**onstraint **Q**ualification for every player $\nu = 1, \ldots, N$.

(SBCQ) For every bounded sequence $\{x^k\} \subset \Re^n$ such that $x^{k,\nu} \in S_\nu(x^{k,-\nu})$ for every k, there exists a bounded sequence $\{(\lambda^{k,\nu}, \mu^{k,\nu})\}$ such that the pair $(\lambda^{k,\nu}, \mu^{k,\nu})$ satisfies the KKT system (5) corresponding to x^k for every k.

The SBCQ was introduced in [30] for the study of the MPEC; it is a unification of various well-known CQs such as the Mangasarian-Fromovitz CQ and the constant-rank CQ of Janin. In particular, the SBCQ plays an important role in the existence of an optimal solution to an MPEC; such a role persists in the present context of a GNE. Rather than repeating all the special cases of the SBCQ (for which the details can be found in the cited reference), we consider the important case where both g^ν and h^ν are affine functions and present the proof to show that such linear constraints satify the SBCQ easily. Letting

$$\begin{aligned} \alpha_\nu(x) &\equiv \{ i : g_i^\nu(x) = 0 \} \\ \gamma_\nu(x^\nu) &\equiv \{ j : h_j^\nu(x^\nu) = 0 \} \end{aligned}$$

be the active constraints at x in player ν's problem (2), we can rewrite the first equation in (5) as

$$\nabla_{x^\nu} \theta_\nu(x) + \sum_{i \in \alpha_\nu(x)} \lambda_i^\nu \nabla_{x^\nu} g_i^\nu(x) + \sum_{j \in \gamma_\nu(x^\nu)} \mu_j^\nu \nabla h_j^\nu(x^\nu) = 0;$$

moreover we have

$$\lambda_i^\nu = 0, \quad \forall i \notin \alpha_\nu(x)$$

and

$$\mu_j^\nu = 0, \quad \forall j \notin \gamma_\nu(x^\nu).$$

Since the gradients $\nabla_{x^\nu} g_i^\nu(x)$ and $\nabla h^\nu(x^\nu)$ are constant vectors, it follows that if x^ν belongs to $S_\nu(x^{-\nu})$, then by polyhedral theory, there exist nonnegative λ^ν and μ^ν that satisfy the above three sets of equations. Moreover, (λ^ν, μ^ν) is represented as

$$\begin{pmatrix} \lambda^\nu \\ \mu^\nu \end{pmatrix} = A_\nu(x) \nabla_{x^\nu} \theta_\nu(x), \tag{6}$$

where $A_\nu(x)$ is a linear operator depending only on $\alpha_\nu(x)$ and $\gamma_\nu(x)$, rather than on x. Since there are only finitely many index sets $\alpha_\nu(x)$ and $\gamma_\nu(x^\nu)$ (even though there is a continuum of values for x), (6) shows that (λ^ν, μ^ν) is bounded as long as x belongs to a bounded set. This indicates that the SBCQ holds for the linearly constrained generalized Nash problem.

We need a compactness assumption and a feasibility assumption for each $\nu = 1, \ldots, N$. Let

$$X^\nu \equiv \{ x^\nu \in \Re^{n_\nu} : h^\nu(x^\nu) \le 0 \} \quad \text{and} \quad X^{-\nu} \equiv \prod_{\nu \neq \nu' = 1}^{N} X^{\nu'}.$$

Compactness assumption. The set X_ν is nonempty and bounded.

Feasibility assumption. For each $x^{-\nu} \in X^{-\nu}$, the set $K^\nu(x^{-\nu})$ is nonempty.

The compactness and feasibility assumptions together imply that $S_\nu(x^{-\nu})$ is nonempty for all $x^{-\nu} \in X^{-\nu}$.

Theorem 2 Under the convexity, compactness, and feasibility assumptions and the SBCQ, there exists a GNE.

Proof. Let

$$\mathbf{X} \equiv \prod_{\nu=1}^{N} X^\nu,$$

which, by assumption, is a nonempty, compact, convex subset of $\Re^n$. Define the set-valued mapping $\Phi : \mathbf{X} \to \mathbf{X}$, where for each $x \equiv (x^\nu)_{\nu=1}^N \in \mathbf{X}$,

$$\Phi(x) \equiv \prod_{\nu=1}^{N} S_\nu(x^{-\nu}).$$

For each $x \in \mathbf{X}$, $\Phi(x)$ is a nonempty, compact, convex subset of $\mathbf{X}$. It suffices to show that Φ is a closed point-to-set map. In turn, we need to show that if $\left\{ x^k \equiv \left(x^{k,\nu} \right)_{\nu=1}^N \right\}$ and $\left\{ y^k \equiv \left(y^{k,\nu} \right)_{\nu=1}^N \right\}$ are sequences in $\Re^n$ such that

$$\lim_{k\to\infty} x^{k,\nu} = x^{\infty,\nu} \quad \text{and} \quad \lim_{k\to\infty} y^{k,\nu} = y^{\infty,\nu}$$

for all $\nu = 1, \ldots, N$ and that $y^{k,\nu} \in S_\nu(x^{k,-\nu})$ for all k and all ν, it then follows that $y^{\infty,\nu}$ is an element of $S_\nu(x^{\infty,-\nu})$ for all ν. Fix an arbitrary ν and define, for each k, $z^k \equiv \left(z^{k,\nu'} \right)_{\nu'=1}^N$ by

$$z^{k,\nu'} \equiv \begin{cases} y^{k,\nu} & \text{if } \nu' = \nu \\ x^{k,\nu'} & \text{otherwise.} \end{cases}$$

The sequence $\{z^k\}$ converges to the limit $z^\infty \equiv \left(z^{\infty,\nu'} \right)_{\nu'=1}^N$, where

$$z^{\infty,\nu'} \equiv \begin{cases} y^{\infty,\nu} & \text{if } \nu' = \nu \\ x^{\infty,\nu'} & \text{otherwise.} \end{cases}$$

For each k, there exist $\lambda^{k,\nu}$ and $\mu^{k,\nu}$ satisfying the KKT system:

$$\nabla_{x^\nu} \theta_\nu(z^k) + \sum_{i=1}^{m_\nu} \lambda_i^{k,\nu} \nabla_{x^\nu} g_i^\nu(z^k) + \sum_{j=1}^{\ell_\nu} \mu_j^{k,\nu} \nabla h_j^\nu(z^{k,\nu}) = 0$$

$$0 \leq \lambda^{k,\nu} \perp g^\nu(z^k) \leq 0$$

$$0 \leq \mu^{k,\nu} \perp h^\nu(z^{k,\nu}) \leq 0;$$

moreover, by the SBCQ, we may assume, without loss of generality, that the sequence $\{(\lambda^{k,\nu}, \mu^{k,\nu})\}$ converges to a pair $(\lambda^{\infty,\nu}, \mu^{\infty,\nu})$. Passing to the limit $k \to \infty$, we conclude readily that $y^{\infty,\nu} \in S_\nu(x^{\infty,-\nu})$ as desired, by the convexity assumption. □

3 A Sequential penalty VI approach to QVIs

To take advantage of the computational advances for solving VIs [11], we propose a penalty approach for solving a QVI of the following kind: For given continuous mappings $F : \Re^n \to \Re^n$, $G : \Re^{2n} \to \Re^m$, and $H : \Re^n \to \Re^\ell$, find a vector $x \in K(x)$ such that

$$(y - x)^T F(x) \geq 0, \quad \forall\, y \in K(x),$$

where

$$K(x) \equiv \{ y \in \Re^n : G(x, y) \leq 0,\ H(y) \leq 0 \}.$$

We denote this problem by QVI (F, G, H). In the application to the GNE, the mapping F is given by (4), the mapping G is given by

$$G(x, y) \equiv \begin{pmatrix} g^1(x^{-1}, y^1) \\ g^2(x^{-2}, y^2) \\ \vdots \\ g^N(x^{-N}, y^N) \end{pmatrix} \in \Re^m, \quad \text{with} \quad m \equiv \sum_{\nu=1}^N m_\nu, \tag{7}$$

for $x \equiv (x^\nu)_{\nu=1}^n$ and $y \equiv (y^\nu)_{\nu=1}^n$ both in $\Re^n$; and the mapping H is given by

$$H(y) \equiv \begin{pmatrix} h^1(y^1) \\ h^2(y^2) \\ \vdots \\ h^N(y^N) \end{pmatrix} \in \Re^\ell, \quad \text{with} \quad \ell \equiv \sum_{\nu=1}^N \ell_\nu. \tag{8}$$

Consistent with the compactness assumption, we assume that the set

$$\mathbf{X} \equiv \{ x \in \Re^n : H(x) \leq 0 \}$$

is nonempty and bounded. Moreover, we assume that each component function H_j is convex; thus $\mathbf{X}$ is a nonempty, compact, convex set. Note that, by (8), the set $\mathbf{X}$ can be written as the Cartesian product of the sets

$$X^\nu \equiv \{ x^\nu \in \Re^{n_\nu} : h^\nu(x^\nu) \leq 0 \}, \tag{9}$$

that is

$$\mathbf{X} = \prod_{\nu=1}^N X^\nu. \tag{10}$$

Although many iterative algorithms for solving mixed complementarity problems can in principle be applied to the equivalent KKT system of the above QVI

(F, G, H), the regularity conditions that are needed for the convergence of these algorithms (see [11, Chapter 10]) are in jeopardy, due to the dependence of the function G on two arguments x and y. Instead of deriving restrictive conditions on the triple (F, G, H) to satisfy these conditions, we adopt a different approach for solving the QVI (F, G, H), which is inspired by the augmented Lagrangian approach for nonlinear programming [5, 6]. The key idea is to penalize the non-standard constraint $G(x, y) \leq 0$ via a penalty term and to solve a sequence of penalized VIs on the set $\mathbf{X}$. Details of the resulting algorithm are described below.

A penalty method. Let $\{\rho_k\}$ be a sequence of positive scalars satisfying $\rho_k < \rho_{k+1}$ and tending to ∞. Let $\{u^k\}$ be a given sequence of vectors. Generate a sequence of iterates $\{x^k\}$ as follows: For each k, x^k is a solution of the VI_k, which is to find $x \in \mathbf{X}$ such that for all $x' \in \mathbf{X}$,

$$(x' - x)^T \left[F(x) + \sum_{i=1}^{m} \max(0, u_i^k + \rho_k G_i(x, x)) \nabla_y G_i(x, x) \right] \geq 0.$$

Since $\mathbf{X}$ is a compact convex set, each iterate x^k exists and stays in $\mathbf{X}$. Hence the sequence $\{x^k\}$ is bounded. By imposing an appropriate condition on a limit point x^∞ of such a sequence, we show that x^∞ is a solution of the QVI (F, G, H).

Theorem 3 Let $F : \Re^n \to \Re^n$ be continuous. Let each $H_j : \Re^n \to \Re$ be a continuously differentiable, convex function. Suppose that the set $\mathbf{X}$ is nonempty and compact. Assume that $G : \Re^{2n} \to \Re^m$ is continuous and that for each $x \in \mathbf{X}$, the function $G_i(x, \cdot)$ is continuously differentiable and convex. Let $\{\rho_k\}$ be a sequence of positive scalars satisfying $\rho_k < \rho_{k+1}$ and tending to ∞. Let $\{u^k\}$ be a bounded sequence of vectors. Let x^∞ be the limit of a convergent subsequence $\{x^k : k \in \kappa\}$. If the following implication holds:

$$\left.\begin{aligned} \sum_{i \in \alpha} \lambda_i \nabla_y G_i(x^\infty, x^\infty) + \sum_{j \in \gamma} \mu_j \nabla H_j(x^\infty) &= 0 \\ \lambda_i \geq 0, \quad \forall\, i &\in \alpha \\ \mu_j \geq 0, \quad \forall\, j &\in \gamma \end{aligned}\right\} \tag{11}$$

$$\Longrightarrow \quad \lambda_i = \mu_j = 0, \quad \forall\, (i, j) \in \alpha \times \gamma,$$

where

$$\alpha \equiv \{ i : G_i(x^\infty, x^\infty) \geq 0 \} \quad \text{and} \quad \gamma \equiv \{ j : H_j(x^\infty) = 0 \},$$

then x^∞ is a solution of the QVI (F, G, H).

Proof. We claim that, for all $k \in \kappa$ sufficiently large, a multiplier μ^k exists such that

$$F(x^k) + \sum_{i=1}^{m} \max(0, u_i^k + \rho_k G_i(x^k, x^k)) \nabla_y G_i(x^k, x^k) + \sum_{j=1}^{\ell} \mu_j^k \nabla H_j(x^k) = 0$$

$$0 \leq \mu^k \perp H(x^k) \leq 0.$$

In turn, this holds under the condition that for all $k \in \kappa$ sufficiently large,

$$\left.\begin{array}{l} \sum_{j \in \gamma_k} \mu_j^k \nabla H_j(x^k) = 0 \\ \mu_j^k \geq 0, \quad \forall j \in \gamma_k \end{array}\right\} \implies \mu_j^k = 0, \quad \forall j \in \gamma_k,$$

where

$$\gamma_k \equiv \{ j : H_j(x^k) = 0 \},$$

the latter condition being equivalent to the Mangasarian-Fromovitz constraint qualification for the VI_k at the solution x^k. Note that $\gamma_k \subseteq \gamma$ for all k sufficiently large. Assume that the displayed implication fails to hold for infinitely many k's in κ. It then follows that a nonempty index set $\gamma_\infty \subseteq \gamma$ exists such that for every k in an infinite index subset κ' of κ, the following system of linear inequalities has a solution in μ_j^k:

$$\sum_{j \in \gamma_\infty} \mu_j^k \nabla H_j(x^k) = 0$$
$$\sum_{j \in \gamma_\infty} \mu_j^k = 1$$
$$\mu_j \geq 0, \quad \forall j \in \gamma_\infty.$$

We may assume without loss of generality that for each $j \in \gamma_\infty$, the sequence $\{\mu_j^k : k \in \kappa'\}$ converges to a limit μ_j^∞, which must satisfy

$$\sum_{j \in \gamma_\infty} \mu_j^\infty = 1.$$

In view of the fact that $\gamma_\infty \subseteq \gamma$, the above expression contradicts the assumed implication (11). Consequently, the claim at the opening of the proof holds. For all $k \in \kappa$ sufficiently large, we can rewrite the first displayed equation as

$$F(x^k) + \sum_{i \in \alpha} \max(0, u_i^k + \rho_k G_i(x^k, x^k)) \nabla_y G_i(x^k, x^k) + \sum_{j \in \gamma} \mu_j^k \nabla H_j(x^k) = 0. \tag{12}$$

The sequences

$$\{ \max(0, u_i^k + \rho_k G_i(x^k, x^k) \} \quad \text{and} \quad \{ \mu_j^k \}$$

must be bounded for every $i \in \alpha$ and $j \in \gamma$; otherwise, we would easily get a contradiction to (11). This implies that $G(x^\infty, x^\infty) \leq 0$ and that there exist nonnegative λ_i^* and μ_j^* satisfying

$$F(x^\infty) + \sum_{i \in \alpha} \lambda_i^* \nabla_y G_i(x^\infty, x^\infty) + \sum_{j \in \gamma} \mu_j^* \nabla H_j(x^\infty) = 0.$$

This shows that the triple $(x^\infty, \lambda^*, \mu^*)$, where $\lambda_i^* = \mu_j^* = 0$ for all $i \notin \alpha$ and $j \notin \gamma$, satisfies the KKT system of the QVI (F, G, H). Equivalently, x^* is a solution of the QVI. □

In Theorem 3, the sequence $\{u^k\}$ is somewhat arbitrary and only assumed to be bounded. By analogy with the augmented Lagrangian method, we may update u^k successively by the formula

$$u_i^{k+1} \equiv \max(0, u_i^k + \rho_k G_i(x^k, x^k)), \quad i = 1, \ldots, m. \tag{13}$$

The proof of the theorem shows that the sequence $\{u^k\}$ defined in this way is indeed bounded, provided that (11) holds at every accumulation point of the sequence $\{x^k\}$. Indeed, with the above definition of u_i^{k+1}, we can write (12) as:

$$F(x^k) + \sum_{i \in \alpha} u_i^{k+1} \nabla_y G_i(x^k, x^k) + \sum_{j \in \gamma} \mu_j^k \nabla H_j(x^k) = 0.$$

A standard normalization/limiting argument will establish the boundedness of $\{u^{k+1}\}$ defined by (13).

3.1 Specialization to a generalized Nash game

When specialized to a generalized Nash game, the above penalty method has the following game-theoretic interpretation. First let $u^k \equiv (u^{k,\nu})_{\nu=1}^N \in \Re^m$ with $u^{k,\nu} \in \Re^{m_\nu}$ for each $\nu = 1, \ldots, N$. Then, from (4), (7) and (10), the VI_k solved at each iteration of the penalty method can be restated as follows: For each $\nu = 1, \ldots, N$, find a vector $x^\nu \in X^\nu$ such that for all $x^{\nu'} \in X^\nu$

$$\begin{aligned}
&(x^{\nu'} - x^\nu)^T \\
&\times \left[\nabla_{x^\nu} \theta_\nu(x^{-\nu}, x^\nu) + \sum_{i=1}^{m_\nu} \max(0, u_i^{k,\nu} + \rho_k g_i^\nu(x^{-\nu}, x^\nu)) \nabla_{x^\nu} g_i^\nu(x^{-\nu}, x^\nu) \right] \\
&\geq 0.
\end{aligned}$$

This together with (9) implies that, at the k-th iteration, each player $\nu = 1, \ldots, N$, taking the other players' strategies $x^{-\nu}$ as exogenous variables, solves the optimization problem over its variable x^ν:

$$\begin{aligned}
&\text{minimize } \theta_\nu(x^{-\nu}, x^\nu) + \frac{1}{2\rho_k} \sum_{i=1}^{m_\nu} \max(0, u_i^{k,\nu} + \rho_k g_i^\nu(x^{-\nu}, x^\nu))^2 \\
&\text{subject to } h^\nu(x^\nu) \leq 0.
\end{aligned} \tag{14}$$

The overall iterate $x^k \equiv (x^{k,\nu})_{\nu=1}^N$ is thus a standard Nash equilibrium where each player has a modified objective function given by

$$\theta_\nu(x^{-\nu}, \cdot) + \frac{1}{2\rho_k} \sum_{i=1}^{m_\nu} \max(0, u_i^{k,\nu} + \rho_k g_i^\nu(x^{-\nu}, \cdot))^2,$$

which remains convex, and a convex strategy set

$$X^\nu = \{ x^\nu \in \Re^{n_\nu} : h^\nu(x^\nu) \leq 0 \}$$

that is independent of its rivals' strategies. The convergence of this sequential Nash equilibrium approach to a generalized Nash equilibrium is ensured under the assumptions in Theorem 3.

In principle, the above algorithmic approach does not require the differentiability of the functions involved. Nevertheless, in the absence of any function differentiability, the practical implementation of the approach by VI methods is in serious jeopardy. In fact, even in the case of twice continuously differentiable data functions, the objective function in (14), due to the second summand, is only SC^1, meaning that it is once continuously differentiable with a "semismooth" gradient. As such, in applying the VI methods that require derivatives [11] for solving the Nash equilibrium subproblems, one has to take note of such weak smoothness in the resulting functions that define the equivalent VIs.

There are two ways to deal with the lack of twice differentiability in the quadratic max-penalty in the objective function of (14). One way is to employ an exponential penalty instead [5, 45], which has a more favorable property with respect to differentiability. Specifically, the optimization problem (14) for each player ν may be replaced by

$$\begin{aligned} &\text{minimize } \theta_\nu(x^{-\nu}, x^\nu) + \frac{1}{\rho_k} \sum_{i=1}^{m_\nu} u_i^{k,\nu} \exp\left(\rho_k g_i^\nu(x^{-\nu}, x^\nu) \right) \\ &\text{subject to } h^\nu(x^\nu) \leq 0, \end{aligned} \tag{15}$$

which, in the setting of the original QVI (F, G, H), amounts to solving the following VI: Find a vector $x \in \mathbf{X}$ such that for all $x' \in \mathbf{X}$,

$$(x' - x)^T \left[F(x) + \sum_{i=1}^{m} u_i^k \exp\left(\rho_k G_i(x, x)\right) \nabla_y G_i(x, x) \right] \geq 0.$$

Problem (15) is twice continuously differentiable whenever the data functions are so. Moreover it is not difficult to see that the convergence result stated in Theorem 3 remains valid for the exponential penalty method. Other examples of smooth penalty functions may be found, for example, in [2, 3, 5].

Keeping the quadratic max-penalty, we can still obtain a smooth formulation for the penalized Nash subproblems by employing an equivalent complementarity statement of the max term. Specifically, observing that the squared max function is once continuously differentiable, we can write the KKT conditions of (14) as:

$$\begin{aligned} \nabla_{x^\nu} \theta_\nu(x^{-\nu}, x^\nu) + \sum_{i=1}^{m_\nu} \max(0, u_i^{k,\nu} + \rho_k g_i^\nu(x^{-\nu}, x^\nu)) \nabla_{x^\nu} g_i^\nu(x^{-\nu}, x^\nu) \\ + \sum_{j=1}^{\ell_\nu} \mu_j^\nu \nabla h_j^\nu(x^\nu) = 0 \end{aligned}$$

$$0 \le \mu^\nu \perp h^\nu(x^\nu) \le 0,$$

which are clearly equivalent to the following system:

$$\nabla_{x^\nu}\theta_\nu(x^{-\nu}, x^\nu) + \sum_{i=1}^{m_\nu} \lambda_i^\nu \nabla_{x^\nu} g_i^\nu(x^{-\nu}, x^\nu) + \sum_{j=1}^{\ell_\nu} \mu_j^\nu \nabla h_j^\nu(x^\nu) = 0$$

$$0 \le \mu^\nu \perp h^\nu(x^\nu) \le 0$$

$$0 \le \lambda^\nu \perp \lambda^\nu - u^{k,\nu} - \rho_k g^\nu(x^{-\nu}, x^\nu) \ge 0.$$

Concatenating the latter conditions for all $\nu = 1, \ldots, N$, we obtain a smooth mixed complementarity problem in the variables (x, λ, μ), provided that the model functions are smooth. To the best of our knowledge, this complementarity reformulation of a penalty method has not been investigated before, even in the context of a standard nonlinear program. A detailed investigation of this suggestion is, however, beyond the scope of this paper.

4 Multi-leader-follower games

The generalized Nash problem provides a mathematical model for a noncooperative game in which each player takes no leadership position over its rivals. In the case where one or more players assume the role of leader(s) in the game, then a multi-leader-follower game arises, the simplest of which is the Stackelberg game in which there is one leader and multiple followers who react to the leader's strategies. A mathematical model for the Stackelberg game is the MPEC, for which there already exists an extra layer of complexity over the generalized Nash problem presented above.

For simplicity, we consider a game with two leaders, labelled I and II, and N followers, labelled $\nu = 1, \ldots, N$; we further assume a leader's strategy set is independent of its rival's strategies. Let $X^{\mathrm{I}} \subseteq \Re^{n_{\mathrm{I}}}$ and $X^{\mathrm{II}} \subseteq \Re^{n_{\mathrm{II}}}$ denote the strategy sets of leaders I and II, respectively. The leaders' objective functions are denoted by $\varphi_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}, y)$ and $\varphi_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}, y)$, respectively. The notation suggests that each leader's objective is a function of its own and the rival leader's strategies and also of the followers' strategies that are collectively denoted by the vector y. The followers respond to the leaders' strategies in the following way. For each $\nu = 1, \ldots, N$, let $\theta_\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, y)$ and $K^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, y^{-\nu})$ denote, respectively, follower ν's objective function and strategy set that depends on the pair of strategies $(x^{\mathrm{I}}, x^{\mathrm{II}}) \in X^{\mathrm{I}} \times X^{\mathrm{II}}$. For each such pair $(x^{\mathrm{I}}, x^{\mathrm{II}})$, the followers' problem is modeled by a generalized Nash game parameterized by the leaders' strategies; let $Y(x^{\mathrm{I}}, x^{\mathrm{II}})$ denote the set of such GNEs, which is not necessarily a singleton. Specifically, each element $\tilde{y} \in Y(x^{\mathrm{I}}, x^{\mathrm{II}}) \subseteq \Re^n$ is a tuple $(\tilde{y}^\nu)_{\nu=1}^N$, where

$$n \equiv \sum_{\nu=1}^N n_\nu,$$

such that each $\tilde{y}^\nu$ is an optimal solution of follower ν's optimization problem:

$$\begin{aligned} &\text{minimize } \theta_\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, \tilde{y}^{-\nu}, y^\nu) \\ &\text{subject to } y^\nu \in K^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, \tilde{y}^{-\nu}), \end{aligned} \tag{16}$$

in which the tuple $(x^{\mathrm{I}}, x^{\mathrm{II}}, \tilde{y}^{-\nu})$ is exogenous to the minimization problem (16) and y^ν is the primary variable to be computed.

We can now define a concept of Nash equilibrium for the above 2-leader-multi-follower game. Specifically, a pair $(x^{*,\mathrm{I}}, x^{*,\mathrm{II}}) \in X^{\mathrm{I}} \times X^{\mathrm{II}}$ is said to be a *L/F Nash equilibrium*, where L/F means leader-follower, if there exists $(y^{*,\mathrm{I}}, y^{*,\mathrm{II}})$ such that $(x^{*,\mathrm{I}}, y^{*,\mathrm{I}})$ is an optimal solution of leader I's problem, which is to seek $(x^{\mathrm{I}}, y^{\mathrm{I}})$ to

$$\begin{aligned} &\text{minimize } \varphi_{\mathrm{I}}(x^{\mathrm{I}}, x^{*,\mathrm{II}}, y^{\mathrm{I}}) \\ &\text{subject to } x^{\mathrm{I}} \in X^{\mathrm{I}} \\ &\text{and } \qquad y^{\mathrm{I}} \in Y(x^{\mathrm{I}}, x^{*,\mathrm{II}}) \end{aligned} \tag{17}$$

and $(x^{*,\mathrm{II}}, y^{*,\mathrm{II}})$ is an optimal solution of leader II's problem, which is to seek $(x^{\mathrm{II}}, y^{\mathrm{II}})$ to

$$\begin{aligned} &\text{minimize } \varphi_{\mathrm{II}}(x^{*,\mathrm{I}}, x^{\mathrm{II}}, y^{\mathrm{II}}) \\ &\text{subject to } x^{\mathrm{II}} \in X^{\mathrm{II}} \\ &\text{and } \qquad y^{\mathrm{II}} \in Y(x^{*,\mathrm{I}}, x^{\mathrm{II}}). \end{aligned} \tag{18}$$

Notice that in the above definition, the followers' equilibrium strategies, $y^{*,\mathrm{I}}$ and $y^{*,\mathrm{II}}$, while both being elements of the equilibrium response set $Y(x^{*,\mathrm{I}}, x^{*,\mathrm{II}})$, are not required to be equal. The reason for this flexibility is that since $y^{*,\mathrm{I}}$ and $y^{*,\mathrm{II}}$ are leader I's and II's anticipation of the followers' collective response to the pair $(x^{*,\mathrm{I}}, x^{*,\mathrm{II}})$, and since such anticipation may be different among the leaders (due to the non-uniqueness of the followers' equilibrium responses), it seems reasonable not to require the equality between $y^{*,\mathrm{I}}$ and $y^{*,\mathrm{II}}$. Of course, this issue disappears when $Y(x^{*,\mathrm{I}}, x^{*,\mathrm{II}})$ is a singleton. One could also define a variation of the above problem by stipulating a market clearing mechanism that enforces $y^{*,\mathrm{I}} = y^{*,\mathrm{II}}$. As we see from the example below, a L/F Nash equilibrium may not exist even when $Y(x^{\mathrm{I}}, x^{\mathrm{II}})$ is a singleton for all $(x^{\mathrm{I}}, x^{\mathrm{II}})$.

Individually, problems (17) and (18) are constrained optimization problems with equilibrium constraints; i.e., MPECs. A major difficulty associated with such an MPEC is the nonconvexity of its feasible region, which in turn is due to the disjunction that is the result of the complementary slackness condition in the KKT system describing the response set $Y(x^{\mathrm{I}}, x^{\mathrm{II}})$. Although there is much progress in research on the MPEC [10, 14, 15, 16, 12, 13, 17, 23, 24, 29, 32, 35, 43, 42, 44], the computation of global solutions to MPECs remains elusive, if not impossible. Consequently, each leader's problem is by itself already not easy to deal with.

Notwithstanding the technical difficulty in the leaders' individual optimization problems, the above leader-follower problem is a genuine noncooperative game with the leaders being the dominant players and with a well-defined equilibrium concept. Nevertheless, due to the nonconvexity of the response set $Y(x^{\mathrm{I}}, x^{\mathrm{II}})$ in general, the existence of a L/F Nash equilibrium is in jeopardy; see the examples to follow. Even in the favorable case where such an equilibrium exists, its complete characterization remains a daunting, if not impossible, task. (This is unlike the standard Nash problem that has an equivalent VI formulation.) Finally, any rigorous attempt to compute a L/F Nash equilibrium (if it exists) is presently out of the reach of existing methods.

In summary, although the multi-leader-follower problem is a sensible mathematical model with a well-defined solution concept, its high level of complexity and technical hardship make it a computationally intractable problem. Subsequently, we present a proposal for the remedial resolution of this game problem. Before doing so, we describe a numerical example of a nonconvex Nash problem that has no equilibrium. (A word about notation; we use subscripts to denote scalar quantities.)

Example 4 Consider a 2-leader noncooperative game with $X_{\mathrm{I}} = X_{\mathrm{II}} = [0, 1] \subset \Re$ and

$$\varphi_{\mathrm{I}}(x_{\mathrm{I}}, x_{\mathrm{II}}, y) \equiv \tfrac{1}{2}\, x_{\mathrm{I}} + y \quad \text{and} \quad \varphi_{\mathrm{II}}(x_{\mathrm{I}}, x_{\mathrm{II}}, y) \equiv -\tfrac{1}{2}\, x_{\mathrm{II}} - y.$$

The follower's optimization problem is a nonnegatively constrained quadratic program: For fixed but arbitrary $(x_{\mathrm{I}}, x_{\mathrm{II}}) \in [0, 1]^2 \in \Re^2$, find $y \in \Re$ to

$$\begin{aligned} &\text{minimize } \ y(-1 + x_{\mathrm{I}} + x_{\mathrm{II}}) + \tfrac{1}{2}\, y^2 \\ &\text{subject to } y \geq 0. \end{aligned}$$

Thus the follower response function is given by

$$y(x_{\mathrm{I}}, x_{\mathrm{II}}) \equiv \max(\, 0, 1 - x_{\mathrm{I}} - x_{\mathrm{II}}\,)$$

Leader I's problem is therefore written as follows: For fixed but arbitrary $x_{\mathrm{II}} \in [0, 1]$, find x_{I} to

$$\begin{aligned} &\text{minimize } \max\left(\tfrac{1}{2}\, x_{\mathrm{I}},\ 1 - \tfrac{1}{2} x_{\mathrm{I}} - x_{\mathrm{II}}\right) \\ &\text{subject to } 0 \leq x_{\mathrm{I}} \leq 1. \end{aligned}$$

It is easy to see that the optimal solution set to this problem is a singleton, which we write as

$$E_{\mathrm{I}}(x_{\mathrm{II}}) = \{\, 1 - x_{\mathrm{II}}\, \}, \qquad \forall\, x_{\mathrm{II}} \in [0, 1].$$

Unlike leader I's problem, which is convex, leader II's optimization problem is written as the following nonconvex problem: For fixed but arbitrary $x_{\mathrm{I}} \in [0, 1]$, find x_{II} to

$$\begin{aligned} &\text{minimize } \min\left(-\tfrac{1}{2}\, x_{\mathrm{II}},\ -1 + x_{\mathrm{I}} + \tfrac{1}{2} x_{\mathrm{II}}\right) \\ &\text{subject to } 0 \leq x_{\mathrm{II}} \leq 1. \end{aligned}$$

It is easy to see that the optimal solution set to leader II's problem is given by

$$E_{\text{II}}(x_{\text{I}}) \equiv \begin{cases} \{0\} & \text{if } x_{\text{I}} \in [0, \frac{1}{2}) \\ \{0, 1\} & \text{if } x_{\text{I}} = \frac{1}{2} \\ \{1\} & \text{if } x_{\text{I}} \in (\frac{1}{2}, 1]. \end{cases}$$

The graphs of the two optimal sets are

$$\begin{aligned} \text{gph}\, E_{\text{I}} &= \left\{ (x_{\text{I}}, x_{\text{II}}) \in [0,1]^2 \,:\, x_{\text{I}} + x_{\text{II}} = 1 \right\} \\ \text{gph}\, E_{\text{II}} &= ([0, 1/2] \times \{0\}) \cup ([1/2, 1] \times \{1\}); \end{aligned}$$

it is easy to see that these two graphs do not intersect. Hence there exists no L/F Nash equilibrium to this game. Note that gph E_{II} is nonconvex. □

Part of the culprit for the non-existence of a L/F Nash equilibrium in the above example is the non-convexity of the optimal solution set $E_{\text{II}}(1/2)$, which in turn is due to the non-convexity of leader II's optimization problem. This situation is typical of a multi-leader-follower game and is reminiscent of the well-known fact that a two-person, zero-sum matrix game need not have an equilibrium in pure strategies. In this elementary case, randomization of the pure strategies, which leads to the notion of mixed strategies, provides a remedy to the original difficulty. In a nonconvex multi-leader-follower game, the idea of randomization has to be broadened to mean *convexification*. Details are explained next.

4.1 Remedial models

There are in general two ways to convexify a non-convex multi-leader-follower game. One way is to convexify each leader's optimal solution set for each of the rival leaders' strategies. Consider Example 4 for instance. If we define the set-valued map $\tilde{E}_{\text{II}}$ pointwise by

$$\widetilde{E}_{\text{II}}(x_{\text{I}}) \equiv \text{conv}\, E_{\text{II}}(x_{\text{I}}), \quad \forall\, x_{\text{I}} \in [0, 1],$$

where the abbreviation "conv" refers to the "convex hull of", then gph $E_{\text{I}} \cap$ gph $\widetilde{E}_{\text{II}}$ is equal to the singleton $\left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\}$, which could then be taken as a randomized L/F equilibrium solution. While conceptually very simple, this naive convexification is not expected to be practical in any realistic situation; this is due to the fact that if a leader's optimization problem is nonconvex, it is already very difficult to compute a globally optimal solution, let alone convexifying the entire optimal set.

The alternative convexification idea can be compared to that of the pure strategies by mixed strategies in a matrix game. Nevertheless, unlike this simple case where the pure strategies are finite and known explicitly, a straightforward convexification of the graph of the response multifunction Y, which has the role of the pure strategies, may not lead to a computationally feasible resolution because this graph is in general a continuum and can be quite complex. Instead, we prefer an algebraic relaxation/restriction approach. Specifically, our proposal to remedy the non-existence of a L/F Nash equilibrium is to exploit the KKT representation of the graph of Y and to consider various relaxations/restrictions of such a representation.

To describe our proposal, we assume that for each $(x^{\mathrm{I}}, x^{\mathrm{II}})$ in $X^{\mathrm{I}} \times X^{\mathrm{II}}$ and each $\nu = 1, \ldots, N$,

$$K^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, y^{-\nu}) \equiv \{y^\nu \in \Re^{n_\nu}, : g^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, y) \le 0, h^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, y^\nu) \le 0\} \neq \emptyset,$$

where $g_\nu : \Re^{n_{\mathrm{I}}+n_{\mathrm{II}}+n} \to \Re^{m_\nu}$ and $h_\nu : \Re^{n_{\mathrm{I}}+n_{\mathrm{II}}+n_\nu} \to \Re^{\ell_\nu}$ are continuously differentiable. Expressed in terms of the KKT conditions of the followers' optimization problems (16), leader I's optimization problem is written as follows: With x^{II} as an exogenous variable, find $(x^{\mathrm{I}}, y^{\mathrm{I}}, \lambda^{\mathrm{I},\nu}, \mu^{\mathrm{I},\nu})$ in $\Re^{n_{\mathrm{I}}+n+m_\nu+\ell_\nu}$ to

$$\begin{aligned}
&\text{minimize } \varphi_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}, y^{\mathrm{I}})\\
&\text{subject to } x^{\mathrm{I}} \in X^{\mathrm{I}}\\
&\text{and}\\
&\left.\begin{aligned}
&\nabla_{y^\nu}\theta_\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, y^{\mathrm{I}}) + \sum_{i=1}^{m_\nu} \lambda_i^{\mathrm{I},\nu}\nabla_{y^\nu} g_i^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, y^{\mathrm{I}})\\
&\quad + \sum_{j=1}^{\ell_\nu} \mu_j^{\mathrm{I},\nu}\nabla_{y^\nu} h_j^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, y^{\mathrm{I},\nu}) = 0\\
&\quad 0 \le \lambda^{\mathrm{I},\nu} \perp g^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, y^{\mathrm{I}}) \le 0\\
&\quad 0 \le \mu^{\mathrm{I},\nu} \perp h^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, y^{\mathrm{I},\nu}) \le 0
\end{aligned}\right\} \quad \forall\, \nu = 1, \ldots, N.
\end{aligned}$$

In general, the constraints of the above problem are highly nonlinear and nonconvex in its variables. In what follows, we restrict to an important special case that covers a broad class of applied models which we will describe subsequently. Extension to the general case is straightforward.

Specifically, we assume that the functions θ_ν, g^ν, and h^ν are given as follows:

$$\theta_\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, y) \equiv \tfrac{1}{2}\,(y^\nu)^T M^\nu y^\nu + (c^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, y^{-\nu}))^T y^\nu + \psi_\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, y^{-\nu})$$

$$g^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, y) \equiv A^{\mathrm{I},\nu}x^{\mathrm{I}} + A^{\mathrm{II},\nu}x^{\mathrm{II}} + \sum_{\nu'=1}^{N} B^{\nu,\nu'} y^{\nu'}$$

$$h^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, y^\nu) \equiv C^{\mathrm{I},\nu}x^{\mathrm{I}} + C^{\mathrm{II},\nu}x^{\mathrm{II}} + D^\nu y^\nu,$$

for some matrices $M^\nu \in \Re^{n_\nu \times n_\nu}$, which are symmetric positive semidefinite, $A^{\mathrm{I},\nu} \in \Re^{m_\nu \times n_{\mathrm{I}}}$, $A^{\mathrm{II},\nu} \in \Re^{m_\nu \times n_{\mathrm{II}}}$, $B^{\nu,\nu'} \in \Re^{m_\nu \times n_{\nu'}}$, $C^{\mathrm{I},\nu} \in \Re^{\ell_\nu \times n_{\mathrm{I}}}$, $C^{\mathrm{II},\nu} \in \Re^{\ell_\nu \times n_{\mathrm{II}}}$, $D^\nu \in \Re^{\ell_\nu \times n_\nu}$, affine functions $c^\nu : \Re^{n_{\mathrm{I}}+n_{\mathrm{II}}+n_{-\nu}} \to \Re^{n_\nu}$, and arbitrary real-valued functions $\psi_\nu : \Re^{n_{\rm I}+n_{\rm II}+n_{-\nu}} \to \Re$. With these specifications, leader I's optimization problem can be written as follows: With $x^{\rm II}$ as an exogenous variable, find $(x^{\rm I}, y^{\rm I}, \lambda^{{\rm I},\nu}, \mu^{{\rm I},\nu}) \in \Re^{n_{\rm I}+n+m_\nu+\ell_\nu}$ to

$$\begin{array}{l}\text{minimize } \varphi_{\rm I}(x^{\rm I}, x^{\rm II}, y^{\rm I})\\ \text{subject to } x^{\rm I} \in X^{\rm I}\\ \text{and}\\ \left.\begin{array}{l} c^\nu(x^{\rm I}, x^{\rm II}, y^{{\rm I},-\nu}) + M^\nu y^{{\rm I},\nu} + (B^{\nu,\nu})^T \lambda^{{\rm I},\nu} + (D^\nu)^T \mu^{{\rm I},\nu} = 0\\ 0 \le \lambda^{{\rm I},\nu} \perp A^{{\rm I},\nu} x^{\rm I} + A^{{\rm II},\nu} x^{\rm II} + \displaystyle\sum_{\nu'=1}^{N} B^{\nu,\nu'} y^{{\rm I},\nu'} \le 0\\ 0 \le \mu^{{\rm I},\nu} \perp C^{{\rm I},\nu} x^{\rm I} + C^{{\rm II},\nu} x^{\rm II} + D^\nu y^{{\rm I},\nu} \le 0 \end{array}\right\} \quad \forall \nu = 1, \ldots, N.\end{array}$$

Except for the set $X^{\rm I}$, which may contain nonlinear (but convex) constraints, and the orthogonality conditions

$$(\lambda^{{\rm I},\nu})^T \left[A^{{\rm I},\nu} x^{\rm I} + A^{{\rm II},\nu} x^{\rm II} + \sum_{\nu'=1}^{N} B^{\nu,\nu'} y^{{\rm I},\nu'} \right] = 0 \tag{19}$$

and

$$(\mu^{{\rm I},\nu})^T (C^{{\rm I},\nu} x^{\rm I} + C^{{\rm II},\nu} x^{\rm II} + D^\nu y^{{\rm I},\nu}) = 0, \tag{20}$$

the remaining constraints in leader I's problem are linear. Informally, our proposal to deal with these nonconvex constraints could be interpreted in the following abstract manner. Each leader has access to only partial information of the followers' responses, which are described by certain "favorable" sets $Z^{\rm I}(x^{\rm I}, x^{\rm II})$ and $Z^{\rm II}(x^{\rm I}, x^{\rm II})$, respectively, and subject to which the leaders optimize their objective functions. In turn, these incomplete response sets could be due to imperfect market conditions, or to a certain market mechanism that "regulates" the followers' reactions, or to the withholding of complete information by the followers. In general, such partial information could be of one of two kinds, restricted or relaxed; restricted information for leader I means

$$Z^{\rm I}(x^{\rm I}, x^{\rm II}) \subseteq Y(x^{\rm I}, x^{\rm II}) \subseteq \Re^n \tag{21}$$

and relaxed information for player I means

$$Y(x^{\rm I}, x^{\rm II}) \subseteq Z^{\rm I}(x^{\rm I}, x^{\rm II}) \subseteq \Re^n. \tag{22}$$

Notice that a restricted response must necessarily be an equilibrium response, whereas a relaxed response is not necessarily an equilibrium response. Similar classifications apply to the partial responses available to leader II. By separating the responses available to the two leaders, we allow the possibility that one leader has only restricted followers' responses, whereas the other leader has relaxed followers' responses.

In mathematical terms, our proposal is to replace the two nonconvex orthogonality conditions (19) and (20) in leader I's constraints by the following (possibly disequilibrium) conditions:

$$(x^{\mathrm{I}}, y^{\mathrm{I}}, \lambda^{\mathrm{I},\nu}) \in W^{\mathrm{I},\nu}(x^{\mathrm{II}}) \quad \text{and} \quad (x^{\mathrm{I}}, y^{\mathrm{I},\nu}, \mu^{\mathrm{I},\nu}) \in V^{\mathrm{I},\nu}(x^{\mathrm{II}}), \tag{23}$$

where $W^{\mathrm{I},\nu}(x^{\mathrm{II}}) \subseteq \Re^{n_{\mathrm{I}}+n+m_\nu}$ and $V^{\mathrm{I},\nu}(x^{\mathrm{II}}) \subseteq \Re^{n_{\mathrm{I}}+n_\nu+\ell_\nu}$ are appropriate polyhedral sets such that together they represent either a restriction or a relaxation of the complementarity constraints in $Y(x^{\mathrm{I}}, x^{\mathrm{II}})$. Let $Z^{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}})$ be the set of all tuples $y^{\mathrm{I}} = \left(y^{\mathrm{I},\nu}\right)_{\nu=1}^{N}$ for which there exists $(\lambda^{\mathrm{I}}, \mu^{\mathrm{I}})$ satisfying, for all $\nu = 1, \ldots, N$,

$$\begin{gathered} c^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, y^{\mathrm{I},-\nu}) + M^\nu y^{\mathrm{I},\nu} + (B^{\nu,\nu})^T \lambda^{\mathrm{I},\nu} + (D^\nu)^T \mu^{\mathrm{I},\nu} = 0 \\ 0 \leq \lambda^{\mathrm{I},\nu}, \quad A^{\mathrm{I},\nu} x^{\mathrm{I}} + A^{\mathrm{II},\nu} x^{\mathrm{II}} + \sum_{\nu'=1}^{N} B^{\nu,\nu'} y^{\mathrm{I},\nu'} \leq 0 \\ 0 \leq \mu^{\mathrm{I},\nu}, \quad C^{\mathrm{I},\nu} x^{\mathrm{I}} + C^{\mathrm{II},\nu} x^{\mathrm{II}} + D^\nu y^{\mathrm{I},\nu} \leq 0 \\ \text{and (23).} \end{gathered} \tag{24}$$

We call the elements of $Z^{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}})$ the followers' *partial responses* anticipated by (or available to) leader I and classify such responses as *restricted* or *relaxed* according to the satisfaction of (21) or (22), respectively. In terms of the partial responses, leader I's optimization problem is then written as follows: With x^{II} as an exogenous variable, find $(x^{\mathrm{I}}, y^{\mathrm{I}}) \in \Re^{n_{\mathrm{I}}+n}$ to

$$\begin{aligned} &\text{minimize } \varphi_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}, y^{\mathrm{I}}) \\ &\text{subject to } x^{\mathrm{I}} \in X^{\mathrm{I}} \\ &\text{and} \qquad (x^{\mathrm{I}}, y^{\mathrm{I}}) \in \operatorname{gph} Z^{\mathrm{I}}(\cdot, x^{\mathrm{II}}). \end{aligned} \tag{25}$$

Similarly, associated with the surrogate complementarity conditions

$$(x^{\mathrm{II}}, y^{\mathrm{II}}, \lambda^{\mathrm{II},\nu}) \in W^{\mathrm{II},\nu}(x^{\mathrm{I}}) \quad \text{and} \quad (x^{\mathrm{II}}, y^{\mathrm{II},\nu}, \mu^{\mathrm{II},\nu}) \in V^{\mathrm{II},\nu}(x^{\mathrm{I}}) \tag{26}$$

for leader II, we may define the set $Z^{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}})$ of partial responses y^{II} anticipated by leader II and further classify such responses as restricted or relaxed according to whether $Z^{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}})$ is contained in, or contains the true response set $Y(x^{\mathrm{I}}, x^{\mathrm{II}})$. In terms of the partial responses, leader II's optimization problem is then written as follows: With x^{I} as an exogenous variable, find $(x^{\mathrm{II}}, y^{\mathrm{II}}) \in \Re^{n_{\mathrm{I}}+n}$ to

$$\begin{aligned} &\text{minimize } \varphi_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}, y^{\mathrm{II}}) \\ &\text{subject to } x^{\mathrm{II}} \in X^{\mathrm{II}} \\ &\text{and} \qquad (x^{\mathrm{II}}, y^{\mathrm{II}}) \in \operatorname{gph} Z^{\mathrm{II}}(x^{\mathrm{I}}, \cdot). \end{aligned} \tag{27}$$

We say that a pair $(x^{*,\mathrm{I}}, x^{*,\mathrm{II}})$ is a *remedial L/F Nash equilibrium* if there exists $(y^{*,\mathrm{I}}, y^{*,\mathrm{II}})$ such that $(x^{*,\mathrm{I}}, y^{*,\mathrm{I}})$ and $(x^{*,\mathrm{II}}, y^{*,\mathrm{II}})$ constitute a GNE of the two leaders' surrogate optimization problems (25) and (27), respectively.

The following result establishes the existence of remedial L/F Nash equilibria. For simplicity, we assume that various sets involved are bounded. Such boundedness condition can be somewhat relaxed; the details are omitted.

Theorem 5 Let X^{I} and X^{II} be nonempty, bounded polyhedra. Assume

(a) for each $(x^{\mathrm{I}}, x^{\mathrm{II}}) \in X^{\mathrm{I}} \times X^{\mathrm{II}}$, the functions $\varphi_{\mathrm{I}}(\cdot, x^{\mathrm{II}}, \cdot)$ and $\varphi_{\mathrm{II}}(x^{\mathrm{I}}, \cdot, \cdot)$ are convex and continuously differentiable;
(b) for all $\nu = 1, \ldots, N$, the function $c^{\nu}(x^{\mathrm{I}}, x^{\mathrm{II}}, y^{\mathrm{I},-\nu})$ is affine, and the graphs of the four set-valued maps $W^{\mathrm{I},\nu}$, $V^{\mathrm{I},\nu}$, $W^{\mathrm{II},\nu}$, and $V^{\mathrm{II},\nu}$ are polyhedra;
(c) for each $(x^{\mathrm{I}}, x^{\mathrm{II}}) \in X^{\mathrm{I}} \times X^{\mathrm{II}}$, $Z^{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}})$ and $Z^{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}})$ are nonempty;
(d) $Z^{\mathrm{I}}(X^{\mathrm{I}}, X^{\mathrm{II}})$ and $Z^{\mathrm{II}}(X^{\mathrm{I}}, X^{\mathrm{II}})$ are bounded.

Then there exists a remedial L/F Nash equilibrium.

Proof. Under the given assumptions, the remedial L/F Nash equilibrium problem is a linearly constrained generalized Nash game that satisfies the compactness and feasibility assumption in Subsection 2.2. The convexity assumption holds trivially because of the linearity of the system (24). Consequently, the existence of a remedial L/F Nash equilibrium follows from Theorem 2. □

There are many remedial L/F models corresponding to different ways to relax or restrict the complementarity conditions (19) and (20) and their counterparts for leader II. Below we mention several of these choices and illustrate them with the game in Example 4. We consider only the set $W^{\mathrm{I},\nu}(x^{\mathrm{II}})$ because the others are similar. Corresponding to any pair of partitioning index subsets α and $\bar{\alpha}$ of $\{1, \ldots, m_\nu\}$, we can let $W^{\mathrm{I},\nu}(x^{\mathrm{II}})$ be the set of tuples $(x^{\mathrm{I}}, y^{\mathrm{I}}, \lambda^{\mathrm{I},\nu})$ satisfying

$$0 \leq \lambda^{\mathrm{I},\nu}_{\alpha}, \left[A^{\mathrm{I},\nu} x^{\mathrm{I}} + A^{\mathrm{II},\nu} x^{\mathrm{II}} + \sum_{\nu'=1}^{N} B^{\nu,\nu'} y^{\mathrm{I},\nu'} \right]_{\alpha} = 0$$

$$0 = \lambda^{\mathrm{I},\nu}_{\bar{\alpha}}, \left[A^{\mathrm{I},\nu} x^{\mathrm{I}} + A^{\mathrm{II},\nu} x^{\mathrm{II}} + \sum_{\nu'=1}^{N} B^{\nu,\nu'} y^{\mathrm{I},\nu'} \right]_{\bar{\alpha}} \leq 0.$$

This choice yields a set of restricted responses. For a set of relaxed responses, we can let $W^{\mathrm{I},\nu}(x^{\mathrm{II}})$ be a box defined by simple bounds on the tuples $(x^{\mathrm{I}}, y^{\mathrm{I}}, \lambda^{\mathrm{I},\nu})$ that are implied by the followers' equilibrium conditions.

Example 4 continued. The full version of the L/F Nash game is stated as follows: Leader I's problem is to

$$\begin{aligned} &\text{minimize } \tfrac{1}{2} x_{\mathrm{I}} + y_{\mathrm{I}} \\ &\text{subject to } 0 \leq x_{\mathrm{I}} \leq 1 \\ &\text{and} \qquad 0 \leq y_{\mathrm{I}} \perp -1 + x_{\mathrm{I}} + x_{\mathrm{II}} + y_{\mathrm{I}} \geq 0; \end{aligned}$$

and leader II's problem is to

$$\begin{aligned}&\text{minimize } -\tfrac{1}{2}\,x_{\mathrm{II}} - y_{\mathrm{II}}\\ &\text{subject to } 0 \le x_{\mathrm{II}} \le 1\\ &\text{and} \qquad 0 \le y_{\mathrm{II}} \perp -1 + x_{\mathrm{I}} + x_{\mathrm{II}} + y_{\mathrm{II}} \ge 0.\end{aligned}$$

Since the follower's equilibrium strategy y is naturally bounded by 0 and 1 because of the same restriction on x_{I} and x_{II}, we may consider the following relaxed L/F Nash game. Leader I's problem is to

$$\begin{aligned}&\text{minimize } \tfrac{1}{2}\,x_{\mathrm{I}} + y_{\mathrm{I}}\\ &\text{subject to } 0 \le x_{\mathrm{I}} \le 1\\ &\text{and} \qquad 0 \le y_{\mathrm{I}} \le 1, \quad -1 + x_{\mathrm{I}} + x_{\mathrm{II}} + y_{\mathrm{I}} \ge 0;\end{aligned}$$

and leader II's problem is to

$$\begin{aligned}&\text{minimize } -\tfrac{1}{2}\,x_{\mathrm{II}} - y_{\mathrm{II}}\\ &\text{subject to } 0 \le x_{\mathrm{II}} \le 1\\ &\text{and} \qquad 0 \le y_{\mathrm{II}} \le 1, \quad -1 + x_{\mathrm{I}} + x_{\mathrm{II}} + y_{\mathrm{II}} \ge 0.\end{aligned}$$

We leave it to the reader to verify that $(x_{\mathrm{I}}, y_{\mathrm{I}}) = (0, 0)$ and $(x_{\mathrm{II}}, y_{\mathrm{II}}) = (1, 1)$ constitute the unique GNE to the above relaxed L/F Nash game.

One could argue that the above GNE is not a desirable equilibrium solution to the 2-leader noncooperative game. The reason is that the two leaders are perceiving drastically different follower responses. As an alternative, consider the following restricted L/F Nash game where leader I's problem is

$$\begin{aligned}&\text{minimize } \tfrac{1}{2}\,x_{\mathrm{I}} + y_{\mathrm{I}}\\ &\text{subject to } 0 \le x_{\mathrm{I}} \le 1\\ &\text{and} \qquad 0 \le y_{\mathrm{I}} \le 1, \quad -1 + x_{\mathrm{I}} + x_{\mathrm{II}} + y_{\mathrm{I}} = 0;\end{aligned}$$

and leader II's problem is

$$\begin{aligned}&\text{minimize } -\tfrac{1}{2}\,x_{\mathrm{II}} - y_{\mathrm{II}}\\ &\text{subject to } 0 \le x_{\mathrm{II}} \le 1\\ &\text{and} \qquad 0 \le y_{\mathrm{II}} \le 1, \quad -1 + x_{\mathrm{I}} + x_{\mathrm{II}} + y_{\mathrm{II}} = 0.\end{aligned}$$

Again, the reader can verify that $(x_{\mathrm{I}}, y_{\mathrm{I}}) = (1, 0)$ and $(x_{\mathrm{II}}, y_{\mathrm{II}}) = (0, 0)$ constitute the unique GNE to the above restricted L/F Nash game. In this equilibrium solution, both leaders arrive at the same follower response.

In summary, we conclude that for this simple 2-leader noncooperative game, if both leaders include the follower's exact response function in their optimization problems, there exists no equilibrium solution. The remedial models are not all desirable; while they always have equilibria, a careful choice of a remedial model leads to a sensible equilibrium solution, whereas a not-so-careful choice could lead to an equilibrium solution where the leaders have entirely different expectations on the follower's behavior. □

5 Multi-L/F games in electricity markets

The subject of noncooperative competition in electricity power markets is of immense contemporary interest due to the privatization and restructuring of this industry that are presently taking place all over the world. In this section, we discuss two approaches in the analysis of electricity power market competition naturally leading to mathematical models that are special cases of the multi-leader-follower game-theoretic framework presented in the last section. The approaches outlined below represent different market designs that describe the electricity firms' strategic behavior under the forces of competition from rival firms.

5.1 Instance I: A competitive bidding problem

While there are many variations of this approach (see e.g. [7, 27, 22]), in what follows, we present a simple model that contains the main idea of this class of spatial market models. There is a finite set of firms, indexed by the elements in the finite set $\mathcal{F}$, which are competing for market power in an electricity network with node set $\mathcal{N}$. Each firm f submits a bid function $b_f(q^f, \rho^f)$ to a market maker who is an independent system operator (ISO); this function depends on the (vector) quantity $q^f \equiv (q_i^f)_{i\in\mathcal{N}}$ of supplies by the firm and a parameter ρ^f. The ISO employs a market clearing mechanism to determine the price p_i at each region and sets the firms' supplies accordingly. One such mechanism is via the solution of an optimization problem as follows. Assume an affine demand curve that yields the price p_i as a function of the total regional demand quantity D:

$$p_i(D) \equiv \alpha_i - \beta_i D,$$

where α_i and β_i are given positive constants. The market clearing process is then determined by the solution of the following optimization problem whose objective function is the ISO's revenue less the bid costs:

$$\begin{aligned} &\text{maximize} \sum_{i\in\mathcal{N}} \left(\alpha_i Q_i - \frac{\beta_i}{2} Q_i^2 \right) - \sum_{f\in\mathcal{F}} b_f(q^f, \rho^f) \\ &\text{subject to } Q_i = \sum_{f\in\mathcal{F}} q_i^f, \quad \forall\, i \in \mathcal{N} \\ &\text{and} \qquad (q^f)_{f\in\mathcal{F}} \in \mathcal{Q}, \end{aligned} \tag{28}$$

where $\mathcal{Q}$ is the set of feasible supplies of the firms. (In many cases, $\mathcal{Q}$ is the Cartesian product of $|\mathcal{F}|$ sets of lower dimensions, each being the supply set of an individual firm.) The firms' bid parameters ρ^f are exogenous to the optimization problem (28), whose optimal solution set we denote $\bar{\mathcal{Q}}(\rho) \subseteq \mathcal{Q}$. The latter set represents the market maker's determination of the firms' supplies as per their bids. Assuming that $b_f(\cdot, \rho^f)$ is a continuously differentiable, convex function in its first argument for all $f \in \mathcal{F}$, an optimal solution $(q^f)_{f\in\mathcal{F}}$ of (28) is characterized by the variational inequality: For all $(s^f)_{f\in\mathcal{F}} \in \mathcal{Q}$,

$$\sum_{f\in\mathcal{F}} (s^f - q^f)^T [-\alpha + \mathrm{diag}(\beta)\, Q + \nabla_{q^f} b_f(q^f, \rho^f)] \geq 0,$$

where $\alpha \equiv (\alpha_i) \in \Re^{|\mathcal{N}|}$, $\mathrm{diag}(\beta)$ is the diagonal matrix whose diagonal entries are the β_i for $i \in \mathcal{N}$, and

$$Q \equiv (Q_i)_{i\in\mathcal{N}} = \sum_{f\in\mathcal{F}} q^f \in \Re^{|\mathcal{N}|}.$$

All the firms are aware of the market clearing process; hence they will take problem (28) as part of the constraints in their own profit maximization problems. Assume for simplicity that each firm's profit is its bid function. Taking the rivals' bid parameters ρ^{-f} as exogenous, each firm f's problem is to determine its bid parameter ρ^f in its admissible set Ω^f along with all the supply quantities $q^{f,t}$ for $t \in \mathcal{F}$ by solving

$$\begin{aligned} &\text{maximize } b_f(q^{f,f}, \rho^f) \\ &\text{subject to } \rho^f \in \Omega^f \\ &\text{and} \qquad (q^{f,t})_{t\in\mathcal{F}} \in \bar{\mathcal{Q}}(\rho^{-f}, \rho^f). \end{aligned}$$

The overall competition problem among all firms can be seen to be a multi-leader single-follower game in which the firms are the leaders and the ISO is the follower. The identification of the variables with those in Section 4 is as follows. For the case of two firms $\mathcal{F} = \{\mathrm{I}, \mathrm{II}\}$, the bids ρ^{I} and ρ^{II} are x^{I} and x^{II}, respectively; the firms' supplies q^{I} and q^{II} are the follower's responses y^{I} and y^{II}, respectively; the supply set $\bar{\mathcal{Q}}(\rho^{\mathrm{I}}, \rho^{\mathrm{II}})$ is the follower's response multifunction $Y(x^{\mathrm{I}}, x^{\mathrm{II}})$. The following simple example illustrates this game.

Example 6 (Hobbs) Consider the case of two firms, which we label as I and II, competing in a single region; thus $\mathcal{N} = \{1\}$. Each firm has a feasible supply set and an admissible bid set that are, respectively, the one-dimensional interval $[0, 1/2]$ and $[0, 1]$. Let $\alpha_1 = \beta_1 = 1$. For $f =$ I and II, let $b_f(q_f, \rho_f) \equiv \rho_f q_f$. In this case, problem (28) takes the explicit form:

$$\begin{aligned} &\text{maximize } [\, q_{\mathrm{I}} + q_{\mathrm{II}} - 0.5\,(q_{\mathrm{I}} + q_{\mathrm{II}})^2\,] - \rho_{\mathrm{I}}\, q_{\mathrm{I}} - \rho_{\mathrm{II}}\, q_{\mathrm{II}} \\ &\text{subject to } 0 \leq q_{\mathrm{I}} \leq 0.5 \quad \text{and} \quad 0 \leq q_{\mathrm{II}} \leq 0.5. \end{aligned}$$

Notice that this optimization problem has a unique solution whenever $\rho_{\rm I} \neq \rho_{\rm II}$ and multiple solutions otherwise. The KKT system of the problem can be written in the following complementarity form: For $f = $ I, II,

$$\begin{aligned} 0 \,\leq\, q_f \qquad & \perp -1 + q_{\rm I} + q_{\rm II} + \rho_f + \gamma_f \,\geq\, 0 \\ 0 \,\leq\, 0.5 - q_f & \perp \gamma_f \,\geq\, 0. \end{aligned}$$

Notice that with $\rho_f \geq 0$, the multiplier γ_f is bounded above by 0.5. Taking this implied bound into consideration (which is redundant in the full formulation below), firm I's problem is then stated as the following MPEC: For fixed but arbitrary $\rho_{\rm II} \in [0, 1]$, determine $\rho_{\rm I}$, $q_{\rm I,I}$, $q_{\rm I,II}$, $\gamma_{\rm I,I}$, and $\gamma_{\rm I,II}$ to

$$\begin{aligned} & \text{maximize } \rho_{\rm I}\, q_{\rm I,I} \\ & \text{subject to } 0 \,\leq\, \rho_{\rm I} \,\leq\, 1 \\ & \qquad 0 \,\leq\, q_{\rm I,I} \,\perp\, -1 + q_{\rm I,I} + q_{\rm I,II} + \rho_{\rm I} + \gamma_{\rm I,I} \,\geq\, 0 \\ & \qquad 0 \,\leq\, q_{\rm I,II} \,\perp\, -1 + q_{\rm I,I} + q_{\rm I,II} + \rho_{\rm II} + \gamma_{\rm I,II} \,\geq\, 0 \\ & \qquad 0 \,\leq\, 0.5 - q_{\rm I,I} \,\perp\, \gamma_{\rm I,I} \,\geq\, 0 \\ & \qquad 0 \,\leq\, 0.5 - q_{\rm I,II} \,\perp\, \gamma_{\rm I,II} \,\geq\, 0 \\ & \qquad \gamma_{\rm I,I} \,\leq\, 0.5 \quad \text{and} \quad \gamma_{\rm I,II} \,\leq\, 0.5. \end{aligned}$$

(Note: $q_{\rm I,I}$ is firm I's supply and $q_{\rm I,II}$ is firm II's supply anticipated by firm I.) Although the objective function $\rho_{\rm I} q_{\rm I,I}$ is not concave, from the complementarity constraints, we easily deduce

$$\rho_{\rm I} q_{\rm I,I} \,=\, -\rho_{\rm II} q_{\rm I,II} + q_{\rm I,I} + q_{\rm I,II} - (q_{\rm I,I} + q_{\rm I,II})^2 - 0.5(\,\gamma_{\rm I,I} + \gamma_{\rm I,II}\,)$$

and the right side is indeed a concave function of the variables $q_{\rm I,I}$, $q_{\rm I,II}$, $\gamma_{\rm I,I}$, and $\gamma_{\rm I,II}$ because $\rho_{\rm II}$ is taken to be exogenous in firm I's problem. Consequently, firm I's problem can be equivalently stated as follows: With $\rho_{\rm II}$ as an exogenous variable,

$$\begin{aligned} & \text{minimize } (q_{\rm I,I} + q_{\rm I,II})^2 - q_{\rm I,I} - q_{\rm I,II} + \rho_{\rm II} q_{\rm I,II} + 0.5\,(\,\gamma_{\rm I,I} + \gamma_{\rm I,II}\,) \\ & \text{subject to } 0 \,\leq\, \rho_{\rm I} \,\leq\, 1 \\ & \qquad 0 \,\leq\, q_{\rm I,I} \,\perp\, -1 + q_{\rm I,I} + q_{\rm I,II} + \rho_{\rm I} + \gamma_{\rm I,I} \,\geq\, 0 \\ & \qquad 0 \,\leq\, q_{\rm I,II} \,\perp\, -1 + q_{\rm I,I} + q_{\rm I,II} + \rho_{\rm II} + \gamma_{\rm I,II} \,\geq\, 0 \\ & \qquad 0 \,\leq\, 0.5 - q_{\rm I,I} \,\perp\, \gamma_{\rm I,I} \,\geq\, 0 \\ & \qquad 0 \,\leq\, 0.5 - q_{\rm I,II} \,\perp\, \gamma_{\rm I,II} \,\geq\, 0 \\ & \qquad \gamma_{\rm I,I} \,\leq\, 0.5 \quad \text{and} \quad \gamma_{\rm I,II} \,\leq\, 0.5. \end{aligned}$$

Similarly, firm II's problem is equivalent to the following MPEC: With $\rho_{\rm I}$ as an exogenous variable,

$$\begin{aligned}
&\text{minimize } (q_{\rm II,I} + q_{\rm II,II})^2 - q_{\rm II,I} - q_{\rm II,II} + \rho_{\rm I} q_{\rm II,I} + 0.5\,(\,\gamma_{\rm II,I} + \gamma_{\rm II,II}\,) \\
&\text{subject to } 0 \;\leq\; \rho_{\rm II} \;\leq\; 1 \\
&\qquad 0 \;\leq\; q_{\rm II,I} \;\perp\; -1 + q_{\rm II,I} + q_{\rm II,II} + \rho_{\rm I} + \gamma_{\rm II,I} \;\geq\; 0 \\
&\qquad 0 \;\leq\; q_{\rm II,II} \;\perp\; -1 + q_{\rm II,I} + q_{\rm II,II} + \rho_{\rm II} + \gamma_{\rm II,II} \;\geq\; 0 \\
&\qquad 0 \;\leq\; 0.5 - q_{\rm II,I} \;\perp\; \gamma_{\rm II,I} \;\geq\; 0 \\
&\qquad 0 \;\leq\; 0.5 - q_{\rm II,II} \;\perp\; \gamma_{\rm II,II} \;\geq\; 0 \\
&\qquad \gamma_{\rm II,I} \;\leq\; 0.5 \quad \text{and} \quad \gamma_{\rm II,II} \;\leq\; 0.5.
\end{aligned}$$

The reader can verify that $(\rho_{\rm I}^*, \rho_{\rm II}^*) = (0.5, 0.5)$ is a L/F Nash equilibrium with the firms's unique supplies as follows: $(q_{\rm I,I}^*, q_{\rm I,II}^*) = (0.5, 0)$ and $(q_{\rm II,I}^*, q_{\rm II,II}^*) = (0, 0.5)$. Again, this is not a desirable equilibrium because there is no common pair of supplies for both firms.

Given the fact that the above formulation does not lead to a L/F Nash equilibrium with $(q_{\rm I,I}^*, q_{\rm I,II}^*) = (q_{\rm II,I}^*, q_{\rm II,II}^*)$, the question is whether there exists a formulation that would yield a desirable equilibrium of some sort. The answer is affirmative if we consider a different pair of optimization problems for the firms. The rationale is as follows. Since each firm realizes that it cannot control (as opposed to "anticipate") the rival firm's supply, each firm would take its rival's variables as exogenous to its own optimization problem, while still adhering to the part of the market clearing process that pertains to its supplies. Adopting this mode of optimization, firm I's problem is the following MPEC: With $q_{\rm II}$, $\rho_{\rm II}$, and $\gamma_{\rm II}$ as exogenous variables, find $q_{\rm I}$, $\rho_{\rm I}$, and $\gamma_{\rm I}$ to

$$\begin{aligned}
&\text{minimize } (q_{\rm I} + q_{\rm II})^2 - q_{\rm I} - q_{\rm II} + 0.5\,\gamma_{\rm I} \\
&\text{subject to } 0 \;\leq\; \rho_{\rm I} \;\leq\; 1 \\
&\qquad 0 \;\leq\; q_{\rm I} \;\perp\; -1 + q_{\rm I} + q_{\rm II} + \rho_{\rm I} + \gamma_{\rm I} \;\geq\; 0 \\
&\qquad\qquad -1 + q_{\rm I} + q_{\rm II} + \rho_{\rm II} + \gamma_{\rm II} \;\geq\; 0 \\
&\qquad 0 \;\leq\; 0.5 - q_{\rm I} \;\perp\; \gamma_{\rm I} \;\geq\; 0 \\
&\text{and} \qquad \gamma_{\rm I} \;\leq\; 0.5.
\end{aligned}$$

Notice that except for the term $-q_{\rm II}$, the two exogenous terms $\rho_{\rm II} q_{\rm II}$ and $0.5\gamma_{\rm II}$ are excluded from the objective function; the kept term is useful to balance the squared term in the objective. Similarly, firm II's alternative problem is the following MPEC:

With $q_{\rm I}$, $\rho_{\rm I}$, and $\gamma_{\rm I}$ as exogenous variables, find $q_{\rm II}$, $\rho_{\rm II}$, and $\gamma_{\rm II}$ to

$$\begin{array}{ll}\text{minimize} & (q_{\rm I} + q_{\rm II})^2 - q_{\rm I} - q_{\rm II} + 0.5\,\gamma_{\rm II} \\ \text{subject to} & 0 \,\leq\, \rho_{\rm II} \,\leq\, 1 \\ & \qquad\qquad -1 + q_{\rm I} + q_{\rm II} + \rho_{\rm I} + \gamma_{\rm I} \,\geq\, 0 \\ & 0 \,\leq\, q_{\rm II} \,\perp\, -1 + q_{\rm I} + q_{\rm II} + \rho_{\rm II} + \gamma_{\rm II} \,\geq\, 0 \\ & 0 \,\leq\, 0.5 - q_{\rm II} \,\perp\, \gamma_{\rm I} \,\geq\, 0 \\ \text{and} & \gamma_{\rm II} \,\leq\, 0.5. \end{array}$$

In terms of the latter two optimization problems (which define a generalized Nash model), a L/F Nash equilibrium is given by $(\rho_{\rm I}^*, \rho_{\rm II}^*) = (0.5, 0.5)$, $(\gamma_{\rm I}^*, \gamma_{\rm II}^*) = (0, 0)$, and any $(q_{\rm I}^*, q_{\rm II}^*)$ that satisfies

$$\begin{aligned} q_{\rm I} + q_{\rm II} \,&=\, 0.5 \\ (\,0, 0\,) \,\leq\, (\,q_{\rm I}, q_{\rm II}\,) \,&\leq\, (\,0.5, 0.5\,). \end{aligned}$$

In summary, this example illustrates that while one version of the 2-firm bidding game has a L/F equilibrium that does not yield a desirable resolution to the game, a modified version of the rules of the game leads to a plausible equilibrium. □

5.2 Instance II: A model with endogenous arbitrage

An alternative model in electric power markets also leads to a multi-leader-follower game. Originated from [20] and extended in several subsequent papers [9, 31, 36, 37], the model considers several electricity firms competing in spatially separated markets along with an arbitrager, whose goal is to exploit price differentials between regions to maximize profit. Included in the firms' profit maximization problems is the arbitrager's full maximization problem; this leads to a multi-leader-follower game, where each leader (firm) solves an MPEC whose equilibrium constraint is the optimality condition of the arbitrager's optimization problem expressed as a linear complementarity system.

There are four main components in the model; the ISO, the arbitrager, the firms, and a market clearing mechanism. The regions are represented by the nodes in a network. While sharing several key features with the models described in the above cited references, the model below distinguishes itself in that the arbitrager's problem is a nonnegatively constrained optimization problem; these inequality constraints, although simple, is the main source of challenge for the resulting model. We first introduce the basic notation of the model, and then describe each component.

Parameters

$\mathcal{N}$: set of nodes
$\mathcal{F}$: set of firms
c_i^f : cost per unit generation at node i by firm f
P_i^0 : price intercept of sales function at node i
Q_i^0 : quantity intercept of sales function at node i
e_{ij} : ISO's unit cost of shipping from node i to j
CAP_i^f : production capacity at node i for firm f

Variables

s_{ij}^f : amount produced at node i and sold at node j by firm f
y_{ij} : amount of shipment from node i to j
w_{ij} : unit charge of shipping from node i to j, paid by the firms and the arbitrager and received by the ISO
a_{ij} : amount bought by arbitrager at node i and sold at j
S_j : amount of total sales at node j

$$S_j \equiv \sum_{t \in \mathcal{F}} \sum_{i \in \mathcal{N}} s_{ij}^t + \sum_{i \in \mathcal{N}} (a_{ij} - a_{ji}), \quad \forall j \in \mathcal{N} \tag{29}$$

p_j : market price at node j, a linear function of total sales S_j

$$p_j(S_j) \equiv P_j^0 - \frac{P_j^0}{Q_j^0} S_j, \quad \forall j \in \mathcal{N}. \tag{30}$$

The ISO's problem is as follows: Given w_{ij}, $(i,j) \in \mathcal{N} \times \mathcal{N}$, compute y_{ij}, $(i,j) \in \mathcal{N} \times \mathcal{N}$, in order to

$$\begin{aligned} &\text{maximize} \sum_{(i,j) \in \mathcal{N} \times \mathcal{N}} (w_{ij} - e_{ij}) \, y_{ij} \\ &\text{subject to } y_{ij} \geq 0, \quad \forall (i,j) \in \mathcal{N} \times \mathcal{N}. \end{aligned}$$

The optimality conditions of this problem are

$$0 \leq y_{ij} \perp w_{ij} - e_{ij} \leq 0, \quad \forall (i,j) \in \mathcal{N} \times \mathcal{N}. \tag{31}$$

The arbitrager's problem is as follows: Given the prices p_i, $i \in \mathcal{N}$, and charges w_{ij}, $(i,j) \in \mathcal{N} \times \mathcal{N}$, compute a_{ij}, $(i,j) \in \mathcal{N} \times \mathcal{N}$, in order to

$$\begin{aligned} &\text{maximize} \sum_{(i,j) \in \mathcal{N} \times \mathcal{N}} (p_j - p_i - w_{ij}) \, a_{ij} \\ &\text{subject to } a_{ij} \geq 0, \quad \forall (i,j) \in \mathcal{N} \times \mathcal{N}. \end{aligned}$$

The optimality conditions of this problem are

$$0 \leq a_{ij} \perp p_j - p_i - w_{ij} \leq 0, \quad \forall (i,j) \in \mathcal{N} \times \mathcal{N}. \tag{32}$$

The firms' problem is as follows: Given s_{ij}^t, $t\ (\neq f) \in \mathcal{F}$, $(i, j) \in \mathcal{N} \times \mathcal{N}$, and w_{ij}, $(i, j) \in \mathcal{N} \times \mathcal{N}$, find s_{ij}^f and a_{ij}, $(i, j) \in \mathcal{N} \times \mathcal{N}$, in order to

$$\begin{aligned}
\text{maximize} \ & \sum_{j\in\mathcal{N}} \left[p_j(S_j) \sum_{i\in\mathcal{N}} s_{ij}^f \right] - \sum_{(i,j)\in\mathcal{N}\times\mathcal{N}} w_{ij}\, s_{ij}^f - \sum_{i\in\mathcal{N}} c_i^f\, g_i^f \\
\text{subject to} \ & g_i^f \ \equiv \ \sum_{j\in\mathcal{N}} s_{ij}^f \ \leq \ \text{CAP}_i^f, \quad \forall\, i \in \mathcal{N} \\
& 0 \leq a_{ij} \perp p_j(S_j) - p_i(S_i) - w_{ij} \leq 0, \quad \forall\,(i, j) \in \mathcal{N}\times\mathcal{N} \\
& s_{ij}^f \geq 0, \quad \forall\,(i, j) \in \mathcal{N}\times\mathcal{N}.
\end{aligned}$$

Note that the arbitrager's optimality conditions (32) are included as constraints in each firm's optimization problem. Thus, each firm anticipates that the arbitrager will react optimally to the market prices p_i and the transmission charges w_{ij}. The model is completed with the last condition below, which, mathematically, can be thought of as the "dual" condition that is associated with the variable w_{ij}.

The market clearing condition is

$$y_{ij} \ = \ \sum_{f\in\mathcal{F}} s_{ij}^f + a_{ij}, \quad \forall\,(i, j) \in \mathcal{N}\times\mathcal{N}. \tag{33}$$

The overall equilibrium problem is to determine all the variables (summarized above), including the charges w_{ij}, so that the ISO's and the firms' optimization problems are solved according to the Nash equilibrium definition and the market equilibrium condition is satisfied.

At first glance, firm f's objective function is not concave in the firm's variables, because it contains product terms such as $s_{ij}^f s_{kj}^f$ for $i \neq k$. In what follows, similar to Example 6, we derive an equivalent objective function that is concave in the firm's variables. We note that, for each pair $(j, f) \in \mathcal{N} \times \mathcal{F}$,

$$p_j(S_j) \sum_{k\in\mathcal{N}} s_{kj}^f = p_j(S_j) \left[\sum_{k\in\mathcal{N}} s_{kj}^f + \sum_{k\in\mathcal{N}} (a_{kj} - a_{jk}) \right] - p_j(S_j) \sum_{k\in\mathcal{N}} (a_{kj} - a_{jk}).$$

By the complementarity condition (32), we have

$$\sum_{i,j\in\mathcal{N}} a_{ij} \left(p_i(S_i) - p_j(S_j) + w_{ij} \right) \ = \ 0,$$

which implies

$$\sum_{j\in\mathcal{N}} p_j(S_j) \sum_{k\in\mathcal{N}} (a_{kj} - a_{jk}) \ = \ \sum_{i,j\in\mathcal{N}} w_{ij}\, a_{ij}.$$

Therefore,

$$\sum_{j\in\mathcal{N}} p_j(S_j)\sum_{k\in\mathcal{N}} s_{kj}^f = \sum_{j\in\mathcal{N}} p_j(S_j)\left[\sum_{k\in\mathcal{N}} s_{kj}^f + \sum_{k\in\mathcal{N}}(a_{kj}-a_{jk})\right] - \sum_{i,j\in\mathcal{N}} w_{ij}a_{ij}.$$

Firm f's problem can therefore be rewritten as: Given s_{ij}^t, $t\,(\neq f)\in\mathcal{F}$, $(i,j)\in\mathcal{N}\times\mathcal{N}$, and w_{ij}, $(i,j)\in\mathcal{N}\times\mathcal{N}$, find s_{ij}^f, $(i,j)\in\mathcal{N}\times\mathcal{N}$, and a_{ij}, $(i,j)\in\mathcal{N}\times\mathcal{N}$, in order to

$$\begin{aligned}
\text{maximize}\ & \sum_{j\in\mathcal{N}} p_j(S_j)\left[\sum_{k\in\mathcal{N}} s_{kj}^f + \sum_{k\in\mathcal{N}}(a_{kj}-a_{jk})\right] \\
&\qquad - \sum_{i,j\in\mathcal{N}} w_{ij}\,(a_{ij}+s_{ij}^f) - \sum_{i\in\mathcal{N}} c_i^f \sum_{j\in\mathcal{N}} s_{ij}^f \\
\text{subject to}\ & \sum_{j\in\mathcal{N}} s_{ij}^f \le \mathrm{CAP}_i^f, \quad \forall\, i\in\mathcal{N} \\
& 0 \le a_{ij} \perp p_j(S_j) - p_i(S_i) - w_{ij} \le 0, \quad \forall\,(i,j)\in\mathcal{N}\times\mathcal{N} \\
& s_{ij}^f \ge 0, \quad \forall\,(i,j)\in\mathcal{N}\times\mathcal{N}.
\end{aligned}$$

Recognizing from (29) that

$$\sum_{k\in\mathcal{N}} s_{kj}^f + \sum_{k\in\mathcal{N}}(a_{kj}-a_{jk}) = S_j - \sum_{(f\neq)\,t\in\mathcal{F}}\ \sum_{k\in\mathcal{N}} s_{kj}^t, \tag{34}$$

we deduce that firm f's objective function is now a concave function of its variables (s_{ij}^f, a_{ij}) with its rivals' variables fixed. Thus firm f's problem is a constrained maximization problem with a concave quadratic objective function and a linear complementarity constraint together with other linear constraints. Once again, we obtain a multi-leader-follower game.

The above game is an instance of an "equilibrium program with equilibrium constraints[1](EPEC)", which is a class of mathematical programs of significant difficulty in general. There is an "exogenous arbitrage" version of the game in which the firms do not include the arbitrager's optimality condition (32) in their constraints; instead, the firms will take the arbitrage variable a_{ij} as exogenous to their problems. This version of the game becomes a standard Nash problem in which the players are the ISO, the firms, and the arbitrager, along with a fictitious player whose problem is the market clearing condition. The solution to the latter Nash

[1] As a historical note, we mention that this terminology was initially brought to the first author's attention by Ben Hobbs shortly after they completed their joint paper with Carolyn Metzler [22]. The terminology was used by Stefan Scholtes, Danny Ralph, and Yves Smeers in their respective talks in the International Conference on Complementarity Problems held in August 2002 in Cambridge, England.

problem is by concatenating the KKT conditions of the firms' problems together with (31), (32), and the market clearing condition, leading to a large-scale linear complementarity problem (LCP). In contrast, the "endogenous arbitrage" model as described above is challenging because there are no "clean" KKT conditions for the firms' problems, which are themselves MPECs. Presently, there is great need for systematic study of this class of games.

5.3 A numerical experiment

We use a variant of the above endogenous-arbitrage model in an experiment to test the sequential penalty VI method presented in Section 3 for solving a QVI. Specifically, the test problem is obtained by removing the arbitrage from the model in the previous subsection but keeping the price differential constraint $p_j(S_j) - p_i(S_i) - w_{ij} \leq 0$ in firm f's problem, which becomes:

$$
\begin{aligned}
\text{maximize} \; & \sum_{j \in \mathcal{N}} \left[p_j(S_j) \sum_{i \in \mathcal{N}} s_{ij}^f \right] - \sum_{i,j \in \mathcal{N}} w_{ij}\, s_{ij}^f - \sum_{i \in \mathcal{N}} c_i^f \sum_{j \in \mathcal{N}} s_{ij}^f \\
\text{subject to} \; & \sum_{j \in \mathcal{N}} s_{ij}^f \leq \text{CAP}_i^f, \quad \forall\, i \in \mathcal{N} \\
& S_j \equiv \sum_{t \in \mathcal{F}} \sum_{i \in \mathcal{N}} s_{ij}^t, \quad \forall\, j \in \mathcal{N} \\
& p_j(S_j) - p_i(S_i) - w_{ij} \leq 0, \quad \forall\, (i,j) \in \mathcal{N} \times \mathcal{N} \\
& s_{ij}^f \geq 0, \quad \forall\, (i,j) \in \mathcal{N} \times \mathcal{N}.
\end{aligned}
$$

The overall Nash equilibrium problem is a QVI because the rival firms' sales s_{ij}^{-f} and the shipping charges w_{ij} remain in the price differential constraint $p_j(S_j) - p_i(S_i) - w_{ij} \leq 0$. While the reader can debate about the practical merit of the price differential constraint without the arbitrage, we feel that this is a meaningful restriction on the firms' sales s_{ij}^f because after all this constraint is part of the overall price equilibrium conditions. Our intention here is not on the modeling aspect of the problem, but instead, to use it for the purpose of testing the numerical performance of the penalty method on a simple example, as suggested by the referees of the paper.

Firm f's penalized subproblem (14) is:

$$\text{maximize } \theta_f(s^{-f}, s^f, w) + \frac{1}{2\rho_k} \sum_{i,j \in \mathcal{N}} \max(0, u_{ij}^{k,f} + \rho_k(p_j(S_j) - p_i(S_i) - w_{ij}))^2$$

$$\text{subject to } \sum_{j \in \mathcal{N}} s_{ij}^f \leq \text{CAP}_i^f, \quad \forall\, i \in \mathcal{N}$$

$$s_{ij}^f \geq 0, \quad \forall\, (i,j) \in \mathcal{N} \times \mathcal{N},$$

where

$$\theta_f(s^{-f}, s^f, w) \equiv \sum_{j \in \mathcal{N}} \left[p_j(S_j) \sum_{i \in \mathcal{N}} s_{ij}^f \right] - \sum_{i,j \in \mathcal{N}} w_{ij}\, s_{ij}^f - \sum_{i \in \mathcal{N}} c_i^f \sum_{j \in \mathcal{N}} s_{ij}^f.$$

To convert the penalized Nash subproblem as an LCP, we let μ_i^f denote the multiplier of the capacity constraint in firm f's penalized optimization problem, and introduce the slack $v_{ij} \equiv e_{ij} - w_{ij}$, from which we can eliminate the variable w_{ij}. We further write

$$\lambda_{ij}^{k,f} \equiv \max(0,\, u_{ij}^{k,f} + \rho_k\,(\, p_j(S_j) - p_i(S_i) - e_{ij} + v_{ij}\,)),$$

which is in accordance with the smoothed formulation described at the end of Subsection 3.1. We can then concatenate the KKT conditions of the firms' penalized optimization problems and combine them with the ISO's optimality condition (31) and the market clearing condition (33), obtaining the following equivalent LCP: For all $f \in \mathcal{F}$ and $(i, j) \in \mathcal{N} \times \mathcal{N}$,

$$0 \leq s_{ij}^f \quad \perp c_i^f + e_{ij} - p_j(S_j) + \frac{P_j^0}{Q_j^0} \sum_{i \in \mathcal{N}} s_{ij}^f + \mu_i^f + \rho_k \frac{P_j^0}{Q_j^0} \left(\lambda_{ij}^{k,f} - \lambda_{ji}^{k,f} \right) \geq 0$$

$$0 \leq \mu_i^f \quad \perp \text{CAP}_i^f - \sum_{j \in \mathcal{N}} s_{ij}^f \geq 0$$

$$0 \leq v_{ij} \quad \perp \sum_{f \in \mathcal{F}} s_{ij}^f \geq 0$$

$$0 \leq \lambda_{ij}^{k,f} \perp \lambda_{ij}^{k,f} - u_{ij}^{k,f} - \rho_k\,(\, p_j(S_j) - p_i(S_i) - e_{ij} + v_{ij}\,) \geq 0$$

The above LCP is solved recursively by updating $u_{ij}^{k,f}$ according to the rule (13): $u_{ij}^{k+1,f} \leftarrow \lambda_{ij}^{k,f}$.

We apply the method to a simple example consisting of 2 firms, labelled I and II, which are competing in a 3-node network with six arcs $\{(1,2), (2,1), (1,3), (3,1), (2,3), (3,2)\}$. Firm I owns generation plants at nodes 1 and 2 and firm II owns

Table 1. Generation costs, capacities, and price function data

node	firm	c_i^f	CAP_i^f	P_i^0	Q_i^0
1				40	500
1	I	15	100		
1	II	15	0		
2				35	400
2	I	15	50		
2	II	15	100		
3				32	600
3	I	15	0		
3	II	15	50		

Table 2. Firms' sales and nodal prices with data in Table 1

node i	node j	firm	sales s_{ij}^f	price p_i
1				27.46
2				28.46
3				28.35
1	2	I	31.53	
1	3	I	68.47	
2	1	I	50	
2	3	I	0	
3	1	I	0	
3	2	I	0	
1	2	II	0	
1	3	II	0	
2	1	II	100	
2	3	II	0	
3	1	II	43.23	
3	2	II	06.77	

generation plants at nodes 2 and 3. The other data for the example are given in Table 1.

The ISO's unit costs of shipping e_{ij} are all taken to be 1. The initial $u_{ij}^{0,f}$ are chosen to be zero; the initial ρ_0 is set equal to 100; each time the termination test:

$$\max(0, p_j(S_j) - p_i(S_i) - e_{ij} + v_{ij}) \le 10^{-6}, \quad \forall i \ne j,$$

fails, we update $\rho_{k+1} \leftarrow 10\rho_k$. The LCPs are solved by the MATLAB code PATHLCP.M (written and maintained jointly by Michael C. Ferris at the University of Wisconsin, Madison and Todd Munson at Argonne National Laboratory), that is downloaded from the website (ftp://ftp.cs.wisc.edu/math-prog/solvers/path/matlab/). After three iterations, the above termination test is satisfied with $p_j(S_j) - p_i(S_i) - e_{ij} + v_{ij}$ all negative. The firms' equilibrium sales and the nodal prices (rounded to 2 decimals) are summarized in Table 2. The shipping charges w_{ij} are all equal to e_{ij} except for $w_{23} = -0.11$.

We rerun the above problem with an alternative penalty update rule: $\rho_{k+1} \leftarrow 2\rho_k$, obtaining a structurally different set of sales for firm II after 4 iterations; see

Table 3. Alternative sales and nodal prices with data in Table 1

node i	node j	firm	sales s_{ij}^f	price p_i
1				27.79
2				28.79
3				27.92
1	2	I	70.92	
1	3	I	29.08	
2	1	I	50	
2	3	I	0	
3	1	I	0	
3	2	I	0	
1	2	II	0	
1	3	II	0	
2	1	II	52.56	
2	3	II	47.44	
3	1	II	50	
3	2	II	0	

Table 3. In the third and last run, we change firm II's costs c_i^{II} to 20 at all 3 nodes. With the former update rule ($\rho_{k+1} \leftarrow 10\rho_k$), PATHLCP fails to solve the problem in the fifth iteration ($\rho_k = 10^6$); with the latter update rule ($\rho_{k+1} \leftarrow 2\rho_k$), the method terminates successfully after 3 iterations. We omit the results.

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