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Metric propositional neighborhood logics on natural numbers

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Abstract Interval logics formalize temporal reasoning on interval structures over linearly (or partially) ordered domains, where time intervals are the primitive ontological entities and truth of formulae is defined relative to time intervals, rather than time points. In this paper, we introduce and study Metric Propositional Neighborhood Logic (MPNL) over natural numbers. MPNL features two modalities referring, respectively, to an interval that is "met by" the current one and to an interval that "meets" the current one, plus an infinite set of length constraints, regarded as atomic propositions, to constrain the length of intervals. We argue that MPNL can be successfully used in different areas of computer science to combine qualitative and quantitative interval temporal reasoning, thus providing a viable alternative to well-established logical frameworks such as Duration Calculus. We show that MPNL is decidable in double exponential time and expressively complete with respect to a well-defined sub-fragment of the two-variable fragment $FO^2[\mathbb{N}, =, <, s]$ of first-order logic for linear orders with successor function, interpreted over natural numbers. Moreover, we show that MPNL can be extended in a natural way to cover full $FO^2[\mathbb{N}, =, <, s]$, but, unexpectedly, the latter (and hence the former) turns out to be undecidable.

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1 Introduction

Interval temporal logics provide a natural framework for temporal reasoning about interval structures over linearly (or partially) ordered domains. They take time intervals as the primitive ontological entities and define truth of formulae relative to time intervals, rather than time points. Interval logics feature modal operators that correspond to various relations between pairs of intervals. In particular, the well-known logic HS, introduced by Halpern and Shoham [24], features a set of modal operators that makes it possible to express all Allen's interval relations [1].

Interval-based formalisms have been extensively used in various areas of computer science and artificial intelligence, such as, formal specification and verification of complex systems, temporal databases, planning and plan validation, theories of action and change, natural language processing, and constraint satisfaction problems. However, most of them are subjected to severe syntactic and semantic restrictions that considerably weaken their expressive power. Interval temporal logics relax these restrictions, thus allowing one to cope with much more complex application domains and scenarios. Unfortunately, many of them, including HS and the majority of its fragments, turn out to be undecidable (a comprehensive survey can be found in [7]).

One of the few cases of decidable interval logic with a truly interval semantics, that is, not reducible to point-based semantics, is Propositional Neighborhood Logic (PNL), interpreted over various classes of interval structures (all, discrete, and dense linear orders, integers, natural numbers) [21]. PNL is a fragment of HS with only two modalities, corresponding to Allen's relations *meets* and *met by*. Basic logical properties of PNL, such as representation theorems and axiomatic systems, have been investigated by Goranko et al. [21]. The satisfiability problem for PNL has been addressed by Bresolin et al. [10]. NEXPTIME-completeness with respect to the classes of all linearly ordered domains, well-ordered domains, finite linearly ordered domains, and natural numbers has been proved via a reduction to the satisfiability problem for the two-variable fragment of first-order logic for binary relational structures over ordered domains [33]. Finally, a tableau system for the future fragment of PNL, interpreted over the natural numbers, has been developed in [14]; such a system has been later extended to full PNL over the integers [12].

Various metric extensions of point-based temporal logics have been studied in the literature. They include Alur and Henzinger's Timed Propositional Temporal Logic (TPTL) [2], two-sorted metric temporal logics, developed by Montanari et al. [29,30], Quantitative Monadic Logic of Order, proposed by Hirshfeld and Rabinovich [26], and Owakine and Worrell's Metric Temporal Logic [34], which refines and extends Koymans' Metric Temporal Logic [28]. Little work in that respect has been done in the interval logic setting. Among the few contributions, we mention the extension of Allen's Interval Algebra with a notion of distance, developed by Kautz and Ladkin [27]. The most important quantitative interval temporal logic is definitely Duration Calculus (DC) [15,25], an interval logic for real-time systems devised by Chaochen, Hoare, and Ravn [17], based on Moszkowski's ITL [32]. DC is quite expressive, but generally undecidable. A number of variants and fragments of DC have been proposed to model and to reason about real-time processes and systems [5,15,16,18]. Many of them recover decidability by imposing semantic restrictions, such as the *locality* principle, that essentially reduce the interval logical system to a point-based one.

In this paper, we present a family of non-conservative metric extensions of PNL, which allow one to express *metric properties* of interval structures over natural numbers. We mainly focus our attention on the most expressive language in this class, called *Metric* PNL (MPNL, for short). MPNL features a family of special atomic propositions representing integer constraints (equalities and inequalities) on the length of the intervals over which they are evaluated. MPNL is particularly suitable for quantitative interval reasoning, and thus it emerges as a viable alternative to existing logical systems for quantitative temporal reasoning. The future fragment of MPNL has been introduced and studied in [11]. Full MPNL has been considered in [8]—the main precursor of this paper, which extends and strengthens it substantially.

The main contributions of the paper are:

(i) the proposal of a number of extensions of PNL with metric modalities and with interval length constraints, which turn out to be very useful to reason about interval structures over natural numbers;
(ii) expressive completeness of MPNL with respect to $FO^2_r[\mathbb{N}, =, <, s]$, a proper fragment of the two-variable fragment $FO^2[\mathbb{N}, =, <, s]$ of first-order logic with equality, order, successor, and a family of uninterpreted binary relations, interpreted on natural numbers. We also show how to extend MPNL to obtain an interval logic $MPNL^+$ which is expressively complete with respect to full $FO^2[\mathbb{N}, =, <, s]$;
(iii) decidability and complexity of the satisfiability problem for MPNL, and undecidability of the satisfiability problem for $FO^2[\mathbb{N}, =, <, s]$, and thus for $MPNL^+$;
(iv) analysis and classification of all the proposed metric extensions of PNL with respect to their expressive power.

The results in this paper can be compared with analogous results for PNL and $FO^2[=, <]$ (the two-variable fragment of FO with equality on linear orders with a family of uninterpreted binary relations) [10]. Unlike $FO^2[=, <]$, which was already known to be decidable [33], the decidability of $FO^2_r[\mathbb{N}, =, <, s]$ is a consequence of the decidability and expressive completeness results for MPNL. At the best of our knowledge, this result is new and of independent interest.

The paper is organized as follows. In Sect. 2, we first recall the basic features of PNL, and then we present the metric language MPNL. In Sect. 3, we illustrate various possible applications of MPNL. Next, in Sect. 4, we prove the decidability of the logic. Expressive completeness results are given in Sect. 5. Finally, in Sect. 6, we classify various fragments of MPNL with respect to their expressive power. In the conclusions, we provide an assessment of the work and we mention open problems.

2 PNL and MPNL

2.1 Propositional Neighborhood Logics: PNL

The language of PNL consists of a set $\mathcal{AP}$ of atomic propositions, the propositional connectives $\neg$ and $\vee$, and the modal operators $\Diamond_r$ and $\Diamond_l$, corresponding to the Allen's relation *meets* and its inverse *met by* [1]. The other propositional connectives, as well as the logical constants $\top$ (*true*) and $\bot$ (*false*), and the dual modal operators $\Box_r$ and $\Box_l$, are defined as usual. Propositional neighborhood logics have been studied both in the so-called *strict semantics*, which excludes point-intervals, and in the *non-strict* one, which includes them. In the latter case, it is natural to include in the language a special atomic proposition (modal constant), usually denoted by π, to identify point-intervals (the notation PNL$^\pi$ has been sometimes used to make the presence of π explicit; to maintain the notation as simple as possible, in the following, we will usually omit the superscript). The differences in expressive power in the various cases have been systematically studied in [21]. In this paper, we focus our attention on the non-strict semantics. *Formulae* of PNL, denoted by $\varphi, \psi, \ldots$, are generated by the following grammar:

$$\varphi ::= \pi \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond_r\varphi \mid \Diamond_l\varphi.$$

Given a linearly ordered domain $\mathbb{D} = \langle D, < \rangle$, a (*non-strict*) *interval* over $\mathbb{D}$ is any ordered pair $[i, j]$ with $i \leq j$. An *interval structure* is a pair $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$, where $\mathbb{I}(\mathbb{D})$ is the set of all intervals over $\mathbb{D}$. The semantics of PNL is given in terms of *models* of the form $M = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), V \rangle$, where $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$ is an interval structure and $V : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{D})}$ is a valuation function assigning to every interval the set of all atomic propositions that are true on it. We recursively define the truth relation $\Vdash$ as follows (the first clause is used only if π is added to the language):

- $M, [i, j] \Vdash \pi$ iff $i = j$;
- $M, [i, j] \Vdash p$ iff $[i, j] \in V(p)$, for any $p \in \mathcal{AP}$;
- $M, [i, j] \Vdash \neg\varphi$ iff it is not the case that $M, [i, j] \Vdash \varphi$;
- $M, [i, j] \Vdash \varphi \vee \psi$ iff $M, [i, j] \Vdash \varphi$ or $M, [i, j] \Vdash \psi$;
- $M, [i, j] \Vdash \Diamond_l\varphi$ iff there exists $h \leq i$ such that $M, [h, i] \Vdash \varphi$;
- $M, [i, j] \Vdash \Diamond_r\varphi$ iff there exists $h \geq j$ such that $M, [j, h] \Vdash \varphi$.

It is worth pointing out that the operators corresponding to Allen's relations *before* and *later* can be easily expressed by the formulae $\Diamond_r(\neg\pi \wedge \Diamond_r\varphi)$ and $\Diamond_l(\neg\pi \wedge \Diamond_l\varphi)$, respectively. We say that a PNL-formula φ is *satisfiable* if there exists a model M and an interval $[b, e]$ such that $M, [b, e] \Vdash \varphi$.

PNL logics have been investigated in [10,21], where the decidability of their satisfiability problem has been shown. A tableau-based method for deciding the satisfiability problem for the future fragment of PNL has been presented in [14], and subsequently extended to the full PNL in [12]. In this paper, we restrict our attention on PNL interpreted in the interval structure built on the natural numbers $\mathbb{N}$.

2.2 Metric PNL: MPNL

In this section, we introduce metric extensions of PNL interpreted over $\mathbb{N}$. Depending on the choice of the metric operators, a hierarchy of languages can be built. In Sect. 6, we will study the relative expressive power of these languages.

From now on, we denote by $\delta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ the *distance* function on $\mathbb{N}$, defined as $\delta(i, j) = |i - j|$. The results presented here may be suitably rephrased for any function δ satisfying the standard properties of distance over a linear order. The most expressive metric extension of PNL is based on *atomic propositions for length constraints*. These are pre-interpreted atomic propositions referring to the length of the current interval. Such propositions can be seen as the metric generalizations of the modal constant π. For each $\sim \in \{<, \leq, =, \geq, >\}$, we introduce the length constraint $\mathsf{len}_{\sim k}$, with the following semantics:

$$M, [i, j] \Vdash \mathsf{len}_{\sim k} \text{ iff } \delta(i, j) \sim k$$

As a matter of fact, equality and inequality constraints are mutually definable:

$$M, [i, j] \Vdash \mathsf{len}_{=k} \Leftrightarrow M, [i, j] \Vdash \neg\mathsf{len}_{>k}, \text{ for } k = 0$$
$$M, [i, j] \Vdash \mathsf{len}_{=k} \Leftrightarrow M, [i, j] \Vdash \mathsf{len}_{>k-1} \wedge \neg\mathsf{len}_{>k}, \text{ for } k > 0$$
$$M, [i, j] \Vdash \mathsf{len}_{=k} \Leftrightarrow M, [i, j] \Vdash \mathsf{len}_{\geq k} \wedge \neg\mathsf{len}_{\geq k+1}$$
$$M, [i, j] \Vdash \mathsf{len}_{=k} \Leftrightarrow M, [i, j] \Vdash \mathsf{len}_{<k+1} \wedge \neg\mathsf{len}_{<k}$$
$$M, [i, j] \Vdash \mathsf{len}_{=k} \Leftrightarrow M, [i, j] \Vdash \mathsf{len}_{\leq k}, \text{ for } k = 0$$
$$M, [i, j] \Vdash \mathsf{len}_{=k} \Leftrightarrow M, [i, j] \Vdash \mathsf{len}_{\leq k} \wedge \neg\mathsf{len}_{\leq k-1}, \text{ for } k > 0$$
$$M, [i, j] \Vdash \mathsf{len}_{<k} \Leftrightarrow M, [i, j] \Vdash \bot, \text{ for } k = 0$$
$$M, [i, j] \Vdash \mathsf{len}_{<k} \Leftrightarrow M, [i, j] \Vdash \mathsf{len}_{=0} \vee \ldots \vee \mathsf{len}_{=k-1}, \text{ for } k > 0$$
$$M, [i, j] \Vdash \mathsf{len}_{\leq k} \Leftrightarrow M, [i, j] \Vdash \mathsf{len}_{=0} \vee \ldots \vee \mathsf{len}_{=k}$$
$$M, [i, j] \Vdash \mathsf{len}_{>k} \Leftrightarrow M, [i, j] \Vdash \neg\mathsf{len}_{\leq k}$$
$$M, [i, j] \Vdash \mathsf{len}_{\geq k} \Leftrightarrow M, [i, j] \Vdash \neg\mathsf{len}_{<k}$$

In Sect. 4, we will limit ourselves to constraints of form $\mathsf{len}_{=k}$, without taking into account the increase in length of formulae due to the above encoding.

3 MPNL at work

Finding an optimal balance between expressive power and computational complexity is a challenge for every knowledge representation and reasoning formalism. Interval temporal logics are not an exception in this respect. We believe that MPNL offers a good compromise between these two requirements. In Sect. 3.1, we show that MPNL makes it possible to encode (*metric versions* of) basic operators of point-based linear temporal logic (LTL) as well as interval modalities corresponding to Allen's relations. In addition, we show that it allows one to express limited forms of fuzziness. In Sect. 3.2, we show how to apply MPNL to model the distinctive features of some well-known applications (specification of real-time systems, medical guidelines, ambient intelligence).

3.1 Expressing basic temporal properties in MPNL

First, MPNL is expressive enough to encode the strict *sometimes in the future* (resp., *sometimes in the past*) operator of LTL:

$$\Diamond_r(\mathsf{len}_{>0} \wedge \Diamond_r(\mathsf{len}_{=0} \wedge p))$$

Moreover, length constraints allow one to define a metric version of the *until* (resp., *since*) operator. For instance, the condition: '*p is true at a point in the future at distance k from the current interval and, until that point, q is true (pointwise)*' can be expressed as follows:

$$\Diamond_r(\mathsf{len}_{=k} \wedge \Diamond_r(\mathsf{len}_{=0} \wedge p)) \wedge \Box_r(\mathsf{len}_{<k} \rightarrow \Diamond_r(\mathsf{len}_{=0} \wedge q))$$

MPNL can also be used to constrain interval length and to express metric versions of basic interval relations. First, we can constrain the length of the intervals over which a given property holds to be at least (resp., at most, exactly) k. As an example, the following formula constrains p to hold only over intervals of length l, with $k \leq l \leq k'$:

$$[G](p \rightarrow \mathsf{len}_{\geq k} \wedge \mathsf{len}_{\leq k'}), \qquad \text{(bl)}$$

where the *universal modality* $[G]$ (*for all intervals*) is a shorthand for the formula:

$$[G]p \equiv \Box_l \Box_r \Box_r p \wedge \Box_l \Box_l \Box_r p$$

By exploiting such a capability, metric versions of almost all Allen's relations can be expressed (the only exception is the *during* relation). As an example, we can state that: '*p holds only over intervals of length l, with $k \leq l \leq k'$, and any p-interval begins a q-interval*' as follows:

$$\text{(bl)} \wedge [G] \bigwedge_{i=k}^{k'} (p \wedge \mathsf{len}_{=i} \rightarrow \Diamond_l \Diamond_r(\mathsf{len}_{>i} \wedge q))$$

As another example, a metric version of Allen's relation *contains* (the inverse of the *during* relation) can be expressed by pairing (bl) with:

$$[G] \bigwedge_{i=k}^{k'} (p \wedge \mathsf{len}_{=i} \rightarrow \bigvee_{j \neq 0, j+j' < i} (\Diamond_l \Diamond_r(\mathsf{len}_{=j} \wedge \Diamond_r(\mathsf{len}_{=j'} \wedge q))))$$

The general picture is as follows. Allen's relations *meets*, *met by*, *before*, and *later* can be captured (in their full generality) by PNL. Metric versions of the other relations can be given provided that the number of possible positions of at least one endpoint of the target interval is bounded by the length of the current interval. This is the case of all of them but the relation *during*, whose left and right endpoints can be arbitrarily located respectively before and after the current interval.

The relationships between the satisfiability problem for PNL and the consistency problem for Allen's Interval Networks have been studied in [35].[1] In general, the satisfiability problem for an expressive enough interval temporal logic is much harder than the problem of checking the consistency of a constraint network. The higher complexity of the former is balanced by the expressiveness of the interval logic that allows one to deal with, for instance, negative and disjunctive constraints. As an example, in [35], the author exploits the difference operator to simulate *nominals*, which are then used to force two specific intervals to satisfy a given Allen's relation (the difference operator can be defined in PNL, and thus in MPNL; its definition closely resembles that of the universal modality). Notice that there is no contradiction between the limits to PNL expressive power and its ability to encode (the consistency problem for) constraint networks: PNL allows one to capture Allen's relations among a *finite* number of intervals only (you need a nominal for each interval). The addition of a metric dimension makes it possible to avoid the use of nominals, but it forces one to assign a finite set of possible values for the length of the involved intervals (possibly infinitely many). Whenever there exist some natural bounds for the given finite set of intervals, constraint networks involving all but one Allen's relations can be easily encoded in MPNL (the resulting encoding turns out to be much more natural than the one using nominals).

Finally, MPNL makes it possible to express some forms of 'fuzziness'. As an example, the condition: '*p is true over the current interval and q is true over some interval close to it*', where by 'close' we mean that the right endpoint of the p interval is at distance at most k from the left endpoint of the q interval, can be expressed as follows:

$$p \wedge (\Diamond_r \Diamond_l(\mathsf{len}_{<k} \wedge \Diamond_l \Diamond_r q) \vee \Diamond_r(\mathsf{len}_{<k} \wedge \Diamond_r q))$$

3.2 Some applications of MPNL

In the following, we show that MPNL expressive power suffices to capture meaningful requirements of various application domains. To start with, we consider some basic safety conditions that characterize the behavior of a *gasburner*. This is a classical example commonly used to illustrate the modeling capabilities of a specification formalism.

[1] Spatial generalizations of the problem to (metric versions of) Weak Spatial PNL and Rectangle Algebra have been investigated in [9,13].

For instance, a formalization of such an example in Duration Calculus can be found in [15].

Let the atomic proposition Gas (resp., $Flame$, $Leak$) be used to state that gas is flowing (resp., burning, leaking), e.g., $M, [i, j] \Vdash Gas$ means that gas is flowing over the interval $[i, j]$. The formula

$$[G](Leak \leftrightarrow Gas \wedge \neg Flame)$$

states that $Leak$ holds over an interval if and only if gas is flowing and not burning over that interval. The condition: '*it never happens that gas is leaking for more than k time units*' can be expressed as:

$$[G](\neg(\mathsf{len}_{>k} \wedge Leak))$$

Similarly, the condition: '*the gas burner will not leak uninterruptedly for k time units after the last leakage*' can be formalized as:

$$[G](Leak \rightarrow \neg\Diamond_r(\mathsf{len}_{<k} \wedge \Diamond_r Leak))$$

As another example, let us consider the case of a railway signaling system. A systematic analysis of such a case study, together with its formalization in Duration Calculus, has been done by Veludis and Nissanke [38]. One of the distinctive features of this system is the large set of safety requirements it involves. Here, we choose one of them and we show how to encode it in MPNL. Most of the other requirements can be dealt with in a very similar way. The specification basically constrains the relationships between the controlling system and the controlled system, which is equipped with both *sensors* and *activators*. More precisely, let the atomic proposition $ReqToRed_i$ (resp., $ReqToYellow_i$, $ReqToGreen_i$) denote the fact that the controlling system has sent to the *signal* (semaphore) i the request to change the color to red (resp., yellow, green). Similarly, let $SignalOp_i$ denote the fact that the i-th signal is operative, that is, not broken. A typical (functional) requirement of the railway signaling system imposes that, when a request to change its color is sent to a signal, either the signal actually changes it within a fixed amount of time or the signal is declared non-operative. Such a requirement can be formalized in MPNL as follows:

$$[G]((ReqToRed_i \wedge SignalOp_i) \rightarrow$$
$$(\Diamond_r(\mathsf{len}_{\leq k} \wedge \neg\Diamond_r ProceedAspect_i) \vee \Diamond_r\Diamond_r\neg SignalOp_i)),$$

where $ProceedAspect_i$ denotes the fact that the signal i is either yellow or green. Notice that MPNL allows one to possible bound the duration of the time period during which a signal is non-operative.

Finally, let us consider the application of MPNL to the fields of *medical guidelines* and *ambient intelligence*. In the former (see [36]), events with duration, e.g., '*running a fever*', possibly paired with metric constraints, e.g., '*if a patient is running a fever for more than k time units, then administrate him/her drug D*', are quite common. Medical requirements of this kind can be easily encoded in MPNL. As an example, the above condition can be expressed in MPNL as follows:

$$[G]((Fever \wedge \mathsf{len}_{>k}) \rightarrow \Diamond_r DrugD)$$

In general, many relevant conditions in medical guidelines are inherently interval-based as there are no general rules to deduce their occurrence from point-based data. The use of temporal logic in ambient intelligence, specifically in the area of Smart Homes [3,20], has been advocated by Combi et al. [19]. MPNL can be successfully used to express safety requirements referring to situations that can be properly modeled only in terms of time intervals, e.g., '*being in the kitchen*'.

4 Decidability of MPNL

In this section, we use a model-theoretic argument to show that the satisfiability problem for MPNL has the bounded-model property with respect to finitely presentable ultimately periodic models, and it is therefore decidable. From now on, let φ be any MPNL-formula and let $\mathcal{AP}$ be the set of atomic propositions of the language.

Definition 1 The *closure* of φ is the set $CL(\varphi)$ of all subformulae of $\Diamond_r\varphi$ and their negations (we identify $\neg\neg\psi$ with ψ). Let $\odot \in \{\Diamond_r, \Diamond_l, \Box_r, \Box_l\}$. The set of *temporal requests* from $CL(\varphi)$ is the set $TF(\varphi) = \{\odot\,\psi \mid \odot\,\psi \in CL(\varphi)\}$.

Definition 2 A *φ-atom* is a set $A \subseteq CL(\varphi)$ such that for every $\psi \in CL(\varphi)$, $\psi \in A$ iff $\neg\psi \notin A$ and for every $\psi_1 \vee \psi_2 \in CL(\varphi)$, $\psi_1 \vee \psi_2 \in A$ iff $\psi_1 \in A$ or $\psi_2 \in A$.

We denote the set of all φ-atoms by A_φ. One can easily prove that $|CL(\varphi)| \leq 2(|\varphi| + 1)$, $|TF(\varphi)| \leq 2|\varphi|$, and $|A_\varphi| \leq 2^{|\varphi|+1}$. We now introduce a suitable labeling of interval structures based on φ-atoms.

Definition 3 A *(φ-)labeled interval structure* (LIS for short) is a structure $\mathbf{L} = \langle\mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$, where $\langle\mathbb{D}, \mathbb{I}(\mathbb{D})\rangle$ is the interval structure over natural numbers (or over a finite subset of it) and $\mathcal{L} : \mathbb{I}(\mathbb{D}) \rightarrow A_\varphi$ is a *labeling function* such that for every pair of neighboring intervals $[i, j], [j, h] \in \mathbb{I}(\mathbb{D})$, if $\Box_r\psi \in \mathcal{L}([i, j])$, then $\psi \in \mathcal{L}([j, h])$, and if $\Box_l\psi \in \mathcal{L}([j, h])$, then $\psi \in \mathcal{L}([i, j])$.

Notice that every interval model M induces a LIS, whose labeling function is the valuation function:

$$\psi \in \mathcal{L}([i, j]) \text{ iff } M, [i, j] \Vdash \psi.$$

Thus, LIS can be thought of as *quasi-models* for φ, in which the truth of formulae containing neither $\Diamond_r$, $\Diamond_l$ nor length constraints is determined by the labeling (due to the definitions of φ-atom and LIS). To obtain a model, we must also guarantee that the truth of the other formulae is in accordance with the labeling. To this end, we introduce the notion of fulfilling LIS.

Definition 4 A LIS $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ is *fulfilling* iff:

- for every length constraint $\mathsf{len}_{=k} \in CL(\varphi)$ and interval $[i, j] \in \mathbb{I}(\mathbb{D})$, $\mathsf{len}_{=k} \in \mathcal{L}([i, j])$ iff $\delta(i, j) = k$;
- for every temporal formula $\Diamond_r \psi$ (resp., $\Diamond_l \psi$) in $TF(\varphi)$ and interval $[i, j] \in \mathbb{I}(\mathbb{D})$, if $\Diamond_r \psi$ (resp., $\Diamond_l \psi$) in $\mathcal{L}([i, j])$, then there exists $h \geq j$ (resp., $h \leq i$) such that $\psi \in \mathcal{L}([j, h])$ (resp., $\mathcal{L}([h, i])$).

Clearly, every interval model is a fulfilling LIS. Conversely, every fulfilling LIS $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ can be transformed into a model $M(\mathbf{L})$ by defining the valuation in accordance with the labeling. Then, one can prove that for every $\psi \in CL(\varphi)$ and interval $[i, j] \in \mathbb{I}(\mathbb{D})$,

$$\psi \in \mathcal{L}([i, j]) \quad \text{iff} \quad M(\mathbf{L}), [i, j] \models \psi$$

by a routine induction on ψ. Therefore, a formula φ is *satisfied* by a fulfilling LIS if and only if there exists an interval such that its label contains φ.

Let m be $\frac{|TF(\varphi)|}{2}$ and k be the maximum among the natural numbers occurring in the length constraints in φ. For example, if $\varphi = \Diamond_r(\mathsf{len}_{>3} \wedge p \rightarrow \Diamond_l(\mathsf{len}_{>5} \wedge q))$, then $m = 2$ and $k = 5$. We now introduce the fundamental notions of left and right temporal requests at a given point.

Definition 5 Given a LIS $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ and a point $i \in D$, the set of *left* (resp., *right*) *temporal requests* at i, denoted by $REQ^L(i)$ (resp., $REQ^R(i)$), is the set of pairs of the type (τ, s), where τ is a temporal formula of the forms $\Diamond_l \psi$, $\Box_l \psi$ (resp., $\Diamond_r \psi$, $\Box_r \psi$) in $TF(\varphi)$ belonging to the labeling of any interval beginning (resp., ending) at i, and $s = +$, if there exists an interval $[j, i]$ (resp., $[i, j]$) such that $\tau \in \mathcal{L}([j, i])$ (resp., $\tau \in \mathcal{L}([i, j])$) and $\delta(j, i) > k$ (resp., $\delta(i, j) > k$), and $s = -$ otherwise.

For any $i \in D$, we write $REQ(i)$ for $REQ^L(i) \cup REQ^R(i)$. We denote by $REQ(\varphi)$ the set of all possible sets of temporal requests from $CL(\varphi)$; moreover, for the sake of brevity, we write $\tau \in REQ(i)$ when there exists a pair $(\tau, s) \in REQ(i)$. It is easy to show that $|REQ(\varphi)| = 2^{2 \cdot m}$. Moreover, by definition, any set of temporal requests $REQ^R(j)$ (resp., $REQ^L(i)$) can be entirely satisfied using at most m different points greater than j (resp., less than i).

Now, consider any MPNL-formula φ such that φ is satisfiable on a finite model. We have to show that we can restrict our attention to models with a bounded size.

Definition 6 Given any LIS $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$, we say that a *k-sequence in* $\mathbf{L}$ is a sequence of k consecutive points in D. Given a k-sequence σ in $\mathbf{L}$, its *sequence of requests* $REQ(\sigma)$ is defined as the k-sequence of temporal requests at the points in σ. We say that $i \in D$ *starts a k-sequence* σ if the temporal requests at $i, \ldots, i + k - 1$ form an occurrence of $REQ(\sigma)$. Furthermore, the sequence of requests $REQ(\sigma)$ is said to be *abundant* in $\mathbf{L}$ iff it has at least $2 \cdot (m^2 + m) \cdot |REQ(\varphi)| + 1$ disjoint occurrences in D.

Intuitively, when a model for a given formula φ presents an abundant sequence, then the model can be shortened without affecting satisfiability of φ.

Lemma 1 *Let* $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ *be any LIS such that the sequence* $REQ(\sigma)$ *is abundant in it. Then, there exists an index* q *such that for each element* $\mathcal{R} \in \{REQ(d) \mid i_q < d < i_{q+1}\}$, *where* i_q *and* i_{q+1} *begin the* q*-th and the* $q+1$*-th occurrence of* σ, *respectively,* $\mathcal{R}$ *occurs at least* $m^2 + m$ *times before* i_q *and at least* $m^2 + m$ *times after* $i_{q+1} + k - 1$.

Proof To prove this property, we proceed by contradiction. Suppose that $REQ(\sigma)$ is abundant, that is, it occurs $n > 2 \cdot (m^2 + m) \cdot |REQ(\varphi)|$ times in D and, for each q with $1 \leq q \leq n$, there exists a point $d(q)$ with $i_q < d(q) < i_{q+1}$, such that $REQ(d(q))$ occurs less than $(m^2 + m)$ times before i_q or less than $(m^2 + m)$ times after $j_{q+1} + k - 1$. Let $\Delta = \{d(q) | 1 \leq q \leq n\}$ the set of all such points. By hypothesis, there cannot be any $\mathcal{R} \in REQ(\varphi)$ such that $\mathcal{R}$ occurs more than $2 \cdot (m^2 + m)$ times in Δ. Then $|\Delta| \leq 2 \cdot (m^2 + m) \cdot |REQ(\varphi)|$, which is a contradiction. □

Lemma 2 *Let* $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ *be a fulfilling LIS that satisfies* φ. *Suppose that there exists an abundant* k*-sequence of requests* $REQ(\sigma)$ *and let* q *be the index whose existence is guaranteed by Lemma 1. Then, there exists a fulfilling LIS* $\mathbf{L}^* = \langle \mathbb{D}^*, \mathbb{I}(\mathbb{D}^*), \mathcal{L}^* \rangle$ *that satisfies* φ *such that* $D^* = D \setminus \{i_q, \ldots, i_{q+1} - 1\}$.

Proof Let us fix a fulfilling LIS $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ satisfying φ at some $[i, j]$, an abundant k-sequence $REQ(\sigma)$ in $\mathbf{L}$, and the index q identified by Lemma 1. Moreover, let $D^- = \{i_q, \ldots, i_{q+1} - 1\}$ and $D' = D \setminus D^-$.

For the sake of readability, the points in D' will be denoted by the same numbers as in D. We now show how to suitably redefine the evaluation of the intervals in $\mathbb{I}(\mathbb{D}')$ to preserve satisfiability of φ (as a matter of fact, the temporal requests at all points in D' are preserved as well).

First, we consider all points $d < i_q$ and for each of them, for all p such that $0 \leq p \leq k-1$, we put $\mathcal{L}'([d, i_{q+1} + p]) = \mathcal{L}([d, i_q + p])$. Then, for all p, p' such that $0 \leq p \leq p' \leq k-1$, we put $\mathcal{L}'([i_{q+1} + p, i_{q+1} + p']) = \mathcal{L}([i_q + p, i_q + p'])$. In such a way, we guarantee that the intervals whose length has been shortened as an effect of the elimination of the points in D^- have a correct labeling in terms of all length constraints of the forms $\mathsf{len}_{=k'}$ and $\neg\mathsf{len}_{=k'}$. Moreover, since the requests (in both directions) in $\mathbf{L}$ at $i_{q+1} + p$ are equal to the requests at $i_q + p$, this operation is safe with respect to universal and existential requirements. Finally, since the lengths of intervals beginning before i_q and ending after $i_{q+1} + k - 1$ are greater than k both in $\mathbf{L}$ and in $\mathbf{L}'$, there is no need to change their labeling. (Notice that, in D', i_{q+1} turns out to be the immediate successor of $i_q - 1$.)

The structure $\mathbf{L}' = \langle \mathbb{D}', \mathbb{I}(\mathbb{D}'), \mathcal{L}' \rangle$ defined so far is obviously a LIS, but it is not necessarily a fulfilling one. The removal of the points in the set D^- and the relabeling needed to guarantee correctness with respect to the length constraints may generate *defects*, that is, situations in which there exists a point $d < i_q$ (resp., $d \geq i_{q+1} + k$) and a formula of the type $\Diamond_r \psi$ (resp., $\Diamond_l \psi$) belonging to $REQ(d)$ such that ψ was satisfied in $\mathbf{L}$ by some interval $[d, d']$ (resp., $[d', d]$) and it is not satisfied in $\mathbf{L}'$, either because $d' \in D^-$ or because the labeling of $[d, d']$ (resp., $[d', d]$) has changed due to the above relabeling. We have to show how to repair such defects.

First, we collect and order the set of defects (assume that we have r of them). Suppose that the first one concerns the existence of a point $d < i_q$ and a formula $\Diamond_r \psi \in REQ(d)$, which is not satisfied anymore in $\mathbf{L}'$ (the case in which $d \geq i_{q+1} + k$ can be dealt with in a similar way). Since $\mathbf{L}$ is a fulfilling LIS, then there exists an interval $[d, d']$ such that $\psi \in \mathcal{L}([d, d'])$ and either $d' \in D^-$ or $i_{q+1} \leq d' < i_{q+1} + k$ and $\psi \notin \mathcal{L}'([d, d'])$. Moreover, for this to be the case, $\delta(d', d) > k$ in $\mathbf{L}$, and thus the defect necessarily involves a pair of the form $(\Diamond_r \psi, +) \in REQ(d)$. By Lemma 1, there exist at least $n = m^2 + m$ points $\{\bar{d}_1, \ldots, \bar{d}_n\}$ after $i_{q+1}+k-1$ such that $REQ(\bar{d}_i) = REQ(d')$, for $i = 1, \ldots, n$. We will choose one of these points, say $\bar{d}_i$, to satisfy the request. In general, this may require a change in the labeling of the interval $[d, \bar{d}_i]$, and to prevent such a change to make one or more requests in $REQ^L(\bar{d}_i)$ no longer satisfied, we will possibly have to redefine the labeling of more than one interval.

To start with, we take a point $d'' < i_q$ such that $REQ(d'') = REQ(d')$ (the existence of such a point is guaranteed by Lemma 1) and a minimal set of points $P^{d''} \subset D'$ such that, for each $(\Diamond_l \tau, +) \in REQ^L(d'')$, there exists a point $e \in P^{d''}$ such that $\tau \in \mathcal{L}([e, d''])$ and $\delta(e, d'') > k$. Now, for each point $e \in P^{d''}$, let $P_e^{d''}$ be a minimal set of points such that, for each $\Diamond_r \xi \in REQ^R(e)$, there exists a point $f \in P_e^{d''}$ such that $\xi \in \mathcal{L}([e, f])$. Finally, let $Q = \bigcup_{e \in P^{d''}} P_e^{d''}$. By the minimality requirements, we have that $|Q| \leq m^2$, since requests in $REQ^L(d'')$ need at most m points to be satisfied and, for each $e \in P^{d''}$, $REQ^R(e)$ can be satisfied using at most m points.

Consider the set $H = \{\bar{d}_1, \ldots, \bar{d}_n\} \setminus Q$. Since, by construction, $|H| \geq (m^2 + m) - m^2 = m$, there must be some point $\bar{d}_i \in H$ such that in $\mathbf{L}'$ the interval $[d, \bar{d}_i]$ satisfies only those $\Diamond_r$-formulae of $REQ(d)$, if any, that are satisfied over some other interval beginning at d. Then, we create a new LIS $\mathbf{L}'_1$, and we put $\mathcal{L}'_1([d, \bar{d}_i]) = \mathcal{L}([d, d'])$. Since $REQ^R(\bar{d}_i) = REQ^R(d')$, such a change has no impact on the right-neighboring intervals of $[d, \bar{d}_i]$. On the contrary, there may exist one or more $\Diamond_l$-formulae in $REQ^L(\bar{d}_i)$ which, due to the change in the labeling of $[d, \bar{d}_i]$, are not satisfied anymore. In such a case, however, we can recover satisfiability, without introducing any new defect, by putting $\mathcal{L}'_1([e, \bar{d}_i]) = \mathcal{L}([e, d''])$ for all $e \in P^{d''}$. Notice that the intervals $[e, d'']$ cannot be shorter than k by definition of $P^{d''}$, and thus this relabeling is safe with respect to length constraints. The labeling of all other intervals is the one defined by $\mathbf{L}'$.

In this way, we have fixed the first defect without introducing any new defect. If we repeat the above procedure for each of the defects, according to their ordering, we obtain a finite sequence of LISs $\mathbf{L}'_1, \mathbf{L}'_2, \ldots, \mathbf{L}'_r$, where the last one is the LIS $\mathbf{L}^*$ we were looking for.

To conclude the proof, we have to show that $\mathbf{L}^*$ is still a LIS for φ. Let $[d, d']$ be the interval of $\mathbf{L}$ satisfying the formula φ. Since $\Diamond_r \varphi \in CL(\varphi)$, we have that $\Diamond_r \varphi \in REQ(d)$. If d is still present in $\mathbf{L}^*$, then, since the final LIS is fulfilling, we have that there must exist an interval $[d, d'']$ labelled with φ. If d is not a point of $\mathbf{L}^*$ anymore, then Lemma 1 guarantees that there exists another point d'' in $\mathbf{L}^*$ such that $REQ(d'') = REQ(d)$. Again, since $\mathbf{L}^*$ is fulfilling, we have that there must exist an interval $[d'', d''']$ labelled with φ. □

Lemma 2 guarantees that we can eliminate sequences of requests that occur 'sufficiently many' times in a LIS, without 'spoiling' the LIS. This eventually allows us to prove the following small-model theorem for finite satisfiability of MPNL.

Theorem 1 (Small-Model Theorem) *If φ is any finitely satisfiable formula of MPNL, then there exists a fulfilling, finite LIS $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ that satisfies φ such that* $|D| \leq |REQ(\varphi)|^k \cdot (2 \cdot (m^2 + m) \cdot |REQ(\varphi)| + 1) \cdot k + k - 1$.

Proof Let $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ be any finite fulfilling LIS that satisfies φ. If $|D| \leq |REQ(\varphi)|^k \cdot (2 \cdot (m^2 + m)| \cdot REQ(\varphi)| + 1) \cdot k + k - 1$, then we are done. Otherwise, by an application of the pigeonhole principle, for at least one sequence $REQ(\sigma)$ of length k, we have that $REQ(\sigma)$ is abundant. In this case, we apply Lemma 2 sufficiently many times to get the requested maximum length. □

To deal with formulae that are satisfiable only over infinite models, we provide these models with a finite periodic representation, and we bound the lengths of their prefix and period.

Definition 7 A LIS $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ is *ultimately periodic*, with *prefix* L, *period* P, and *threshold* $k < P$ if, for every interval $[i, j]$,

– if $i \geq L$, then $\mathcal{L}([i, j]) = \mathcal{L}([i + P, j + P])$;
– if $j \geq L$ and $\delta(i, j) > k$, then $\mathcal{L}([i, j]) = \mathcal{L}([i, j + P])$.

It is worth noticing that, in every ultimately periodic LIS, $REQ(i) = REQ(i + P)$, for $i \geq L$, and that every ultimately periodic LIS is finitely presentable: it suffices to define its labeling only on the intervals $[i, j]$ such that $j < L + 2 \cdot P + k$; thereafter, it can be uniquely extended by periodicity. It can be easily shown that a finite LIS can be recovered as a special case of ultimately periodic LIS.

Lemma 3 *Let φ be an MPNL formula and $\mathbf{L} = \langle \mathbb{N}, \mathbb{I}(\mathbb{N}), \mathcal{L} \rangle$ be an infinite fulfilling LIS over $\mathbb{N}$ that satisfies φ. Then, there exists an infinite ultimately periodic fulfilling LIS $\mathbf{L}^* = \langle \mathbb{N}, \mathbb{I}(\mathbb{N}), \mathcal{L}^* \rangle$ over $\mathbb{N}$ that satisfies φ.*

Proof First of all, let $[b, e]$ be the interval satisfying φ in $\mathbf{L}$. We define the set $REQ_{inf}(\varphi)$ as the subset of $REQ(\varphi)$ containing all and only the sets of requests that occur infinitely often in $\mathbf{L}$. Let $L, M \in \mathbb{N}$ be such that the following conditions are met: (i) $L \geq e$; (ii) for each point $r \geq L$, $REQ(r) \in REQ_{inf}(\varphi)$; (iii) each set of requests $\mathcal{R} \in REQ_{inf}(\varphi)$ occurs at least $m^2 + m$ times before L and at least $m^2 + m$ times between $L + k$ and M; (iv) for each point $i < L$ and any formula $\Diamond_r \tau \in REQ(i)$, τ is satisfied on some interval $[i, j]$, with $j < M$; and (v) the k-sequences of requests starting at L and at M are the same.

We put $P = M - L$. By condition (iii), $P > k$. We build an infinite ultimately periodic structure $\overline{\mathbf{L}}$ over the domain $\mathbb{N}$ with prefix L, period P, and threshold k. As a first step, for all points $d < M$, we put $\overline{REQ}(d) = REQ(d)$. Then, for all points $M + n$, with $0 \leq n < P$, we put $\overline{REQ}(M + n) = REQ(L + n)$ [by condition (v), this is already the case for $0 \leq n < k$], and, for all points $M + P + n$, with $0 \leq n < k$, we put $\overline{REQ}(M + P + n) = REQ(L + n)$.

The labeling is defined as follows.

For all intervals $[i, j]$ such that $j < M$, we put $\overline{\mathcal{L}}([i, j]) = \mathcal{L}([i, j])$. As for intervals $[i, j]$, with $M \leq j < M + P$, we must distinguish different cases:

(a) if $i \geq M$, we put $\overline{\mathcal{L}}([i, j]) = \mathcal{L}([i - P, j - P])$;
(b) if $i < M$ (and thus $\overline{REQ}(i) = REQ(i)$), we must distinguish three sub-cases:
   (b1) if $\delta(i, j) \leq k$ (and thus, by condition (v), $\overline{REQ}(j) = REQ(j)$), then we put $\overline{\mathcal{L}}([i, j]) = \mathcal{L}([i, j])$;
   (b2) if $k < \delta(i, j) \leq k + P$, we put $\overline{\mathcal{L}}([i, j]) = \mathcal{L}([i, h])$ for some h such that $\overline{REQ}(j) = REQ(h)(= REQ(j - P))$ and $\delta(i, h) > k$ (the existence of such an h is guaranteed by conditions (ii) and (iii); if $M \leq j < M + k$, we can take $h = j$);
   (b3) if $\delta(i, j) > k + P$, we put $\overline{\mathcal{L}}([i, j]) = \mathcal{L}([i, j - P])$.

As for intervals $[i, j]$, with $M + P \leq j < M + P + k$, we must distinguish three cases:

(1) if $i \geq M$, we put $\overline{\mathcal{L}}([i, j]) = \overline{\mathcal{L}}([i - P, j - P])$;
(2) if $i < M$ and $\delta(j, i) > P + k$, then $\overline{\mathcal{L}}([i, j]) = \overline{\mathcal{L}}([i, j - P])$)
(3) if $i < M$ and $\delta(j, i) \leq P + k$, then $\overline{\mathcal{L}}([i, j]) = \overline{\mathcal{L}}([i', j])$, for some i' such that $i' < L$ and $\overline{REQ}(i')(= REQ(i')) = \overline{REQ}(i)$ [the existence of such an i' is guaranteed by condition (iii)].

The above construction labels all subintervals of $[0, M + P + k]$ in a way that is consistent with the definition of LIS, but that is not necessarily fulfilling. The labels of intervals $[i, j]$, with $j < M$, remain unchanged and thus requests of points $i < L$ are not critical, as, by condition (iv), every request of every such point is satisfied on some interval $[i, j]$, with $j < M$. This is not the case with $L \leq i \leq M$. Indeed, it may happen that, for some point $L \leq i \leq M$ and some formula $\Diamond_r \psi \in \overline{REQ}(i)$, there is no interval satisfying ψ in $\overline{\mathbf{L}}$ (the only intervals satisfying it in $\mathbf{L}$ being of the form $[i, j]$, with $j > M + k$). We fix such *defects* as follows. Since $\overline{REQ}(i) = REQ(i)$, there exists a point $j > i$ such that $\psi \in \mathcal{L}([i, j])$ in $\mathbf{L}$. By condition (iii), there exist at least $m^2 + m$ points between $M + k$ and $M + P$ with the same set of requests as j. We proceed exactly as in the proof of Lemma 2: we fix the defect by choosing a point d' in between $M + k$ and $M + P$ and building a new LIS $\mathbf{L}_1$, which is identical to $\overline{\mathbf{L}}$ but for the labeling of the interval $[i, d']$ (we put $\mathcal{L}_1([i, d']) = \mathcal{L}([i, d])$). By applying such a repair procedure to each defect in a systematic manner, e.g., starting from the defect closest to the origin and then moving from left to right, we generate a finite sequence of LISs $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_r$, the last of which is such that every request of every point $i \leq M$ is fulfilled before $M + P$.

The ultimately periodic fulfilling LIS $\mathbf{L}^*$ is obtained from $\mathbf{L}_r$ by completing the specification of the labeling of the intervals in $\mathbb{I}(\mathbb{N})$ in such a way that the conditions of Definition 7 for an ultimately periodic LIS with prefix L, period P, and threshold k are satisfied. Formally, for every $i \geq M + P + k$ we put $REQ^*(i) = REQ^*(i - n \cdot P)$, where n is the least non-negative integer such that $i - n \cdot P < M + P + k$. Then, for every interval $[i, j]$ such that $j \geq M + P + k$, if $\delta(i, j) \leq P + k$, then we put $\mathcal{L}^*([i, j]) = \mathcal{L}^*([i - n \cdot P, j - n \cdot P])$, where n is the least non-negative integers such that $j - n \cdot P < M + P + k$ (notice that, since $\delta(i, j) \leq P + k$, it also holds $i - n \cdot P \geq L$); otherwise, we put $\mathcal{L}^*([i, j]) = \mathcal{L}^*([i - n \cdot P, j - q \cdot P])$, where n and q are respectively the least non-negative integers such that $L \leq i - n \cdot P < M$ and $M + k \leq j - q \cdot P < M + P + k$ (notice that $\delta([i - n \cdot P, j - q \cdot P]) > k$). It is straightforward to check that the labeling $\mathcal{L}^*$ respects all length constraints, and that the resulting structure $\mathbf{L}^* = \langle \mathbb{N}, \mathbb{I}(\mathbb{N}), \mathcal{L}^* \rangle$ is an ultimately periodic fulfilling LIS satisfying φ on $[b, e]$. □

Theorem 2 (Small Periodic Model Theorem) *If φ is any satisfiable formula of MPNL, then there exists a fulfilling, ultimately periodic LIS satisfying φ such that both the length L of the prefix and the length P of the period are less or equal to $|REQ(\varphi)|^k \cdot (2 \cdot (m^2 + m) \cdot |REQ(\varphi)| + 1) \cdot k + k - 1$.*

Proof Existence of an ultimately periodic fulfilling LIS is guaranteed by Lemma 3. The bound on the prefix and of the period can be proved by exploiting Lemma 2. □

Corollary 1 *The satisfiability problem for MPNL, interpreted over $\mathbb{N}$, is decidable.*

The results of this section immediately give a double exponential time non-deterministic procedure for checking the satisfiability of any MPNL-formula φ. Such a procedure non-deterministically checks models whose size is in general $O(2^{k\cdot|\varphi|})$, where $|\varphi|$ is the length of the formula to be checked for satisfiability. It has been shown in [11] that, in the case in which k is represented in binary, the right-neighborhood fragment of MPNL is complete for the class EXPSPACE. This means that, in the general case, the complexity for MPNL is located somewhere in between EXPSPACE and 2NEXPTIME (the exact complexity is still an open problem). It is worth noticing that, whenever k is a constant, it does not influence the complexity class and thus, since we have a NTIME($2^{|\varphi|}$) procedure for satisfiability and a NEXPTIME-hardness result [14], we can conclude that MPNL is NEXPTIME-complete. Similarly, when k is expressed in unary, the value of k increases linearly with the length of the formula and thus NTIME($2^{k\cdot|\varphi|}$) = NTIME($2^{|\varphi|^2}$); therefore, as in the previous case, MPNL is NEXPTIME-complete.

5 MPNL and two-variable fragments of first-order logic for $(\mathbb{N}, <, s)$

5.1 PNL and two-variable fragments of first-order logic

We start with a summary of results from [10], which will then be extended to MPNL. Let us denote by $\mathrm{FO}^2[=]$ the fragment of first-order logic with equality whose language contains only two distinct variables. Moreover, we denote the formulae of $\mathrm{FO}^2[=]$ by $\alpha, \beta, \ldots$ For example, the formula $\forall x(P(x) \rightarrow \forall y \exists x Q(x, y))$ belongs to FO^2, while the formula $\forall x(P(x) \rightarrow \forall y \exists z(Q(z, y) \wedge Q(z, x)))$ does not. We first focus our attention on the extension $\mathrm{FO}^2[=, <]$ of $\mathrm{FO}^2[=]$ over a purely relational vocabulary $\{=, <, P, Q, \ldots\}$ including equality and a distinguished binary relation $<$ interpreted as a linear order. Since atoms in two-variable fragments may involve at most two distinct variables, we can further assume, without loss of generality, that the arity of every relation in the considered vocabulary is exactly 2. Let x and y be the two variables of the language. The formulae of $\mathrm{FO}^2[=, <]$ can be defined recursively as follows:

$$\alpha ::= A_0 \mid A_1 \mid \neg\alpha \mid \alpha \vee \beta \mid \exists x\alpha \mid \exists y\alpha$$
$$A_0 ::= x = x \mid x = y \mid y = x \mid y = y \mid x < y \mid y < x$$
$$A_1 ::= P(x, x) \mid P(x, y) \mid P(y, x) \mid P(y, y),$$

where A_1 deals with (uninterpreted) binary predicates. For technical convenience, we assume that both variables x and y occur as (possibly vacuous) free variables in every formula $\alpha \in \mathrm{FO}^2[=, <]$, that is, $\alpha = \alpha(x, y)$.

Formulae of $\mathrm{FO}^2[=, <]$ are interpreted over *relational models* of the form $\mathbf{M} = \langle \mathbb{D}, V\rangle$, where $\mathbb{D} = \langle D, <\rangle$ is a linearly ordered set and V is a *valuation function* that assigns to every binary relation P a subset of $D \times D$. When we evaluate a formula $\alpha(x, y)$ on a pair of elements a, b, we write $\alpha(a, b)$ for $\alpha[x := a, y := b]$.

The decidability of the satisfiability problem for FO^2 without equality has been proved by Scott [37] by means of satisfiability-preserving reduction of any FO^2-formula to a formula of the form $\forall x \forall y \psi_0 \wedge \bigwedge_{i=1}^{m} \forall x \exists y \psi_i$, which belongs to the Gödel's prefix-defined class of first-order formulae, whose satisfiability problem was shown to be decidable by Gödel [6].

Later on, Mortimer extended this decidability result by including equality in the language [31]. Mortimer's result has been improved by Grädel, Kolaitis, and Vardi who lowered the complexity [23]. Finally, by building on techniques from [23] and taking advantage of an in-depth analysis of the basic 1-types and 2-types in $\mathrm{FO}^2[=, <]$-models, Otto proved the decidability of $\mathrm{FO}^2[=, <]$ over various classes of orders, including $\mathbb{N}$. In [10], Bresolin et al. show that $\mathrm{FO}^2[=, <]$ is expressively complete with respect to PNL^{π}. In the following, we extend this expressive completeness result (in the case of natural numbers) to MPNL.

5.2 Comparing the expressive power of interval and first-order logics

There are various ways to compare the expressive power of different logics. The one we use in this paper is comparing logics with respect to properties they can express. In doing this, we distinguish two different cases: the case in which we compare two interval logics on the same class of models, e.g., different fragments of MPNL, and the case in which we compare an interval logic with a first-order logic, e.g., MPNL and a suitable extension of $\mathrm{FO}^2[=, <]$.

Given two interval logics L and L' interpreted in the same class of models $\mathcal{C}$, we say that L' is *at least as expressive as* L (with respect to $\mathcal{C}$), denoted by L$\preceq_{\mathcal{C}}$L', if there exists an effective translation τ from L to L' (inductively defined on the structure of formulae) such that for every model M in

$\mathcal{C}$, any interval $[i, j]$ in the model, and any formula φ of L, $M, [i, j] \Vdash \varphi$ if and only if $M, [i, j] \Vdash \tau(\varphi)$. Furthermore, we say that L' is *as expressive as* L, denoted by $\text{L'}\equiv_{\mathcal{C}}\text{L}$, if both $\text{L'}\preceq_{\mathcal{C}}\text{L}$ and $\text{L}\preceq_{\mathcal{C}}\text{L'}$, and we say that L' is *strictly more expressive than* L, denoted by $\text{L} \prec_{\mathcal{C}}\text{L'}$, if $\text{L} \preceq_{\mathcal{C}}\text{L'}$ and $\text{L'} \not\preceq_{\mathcal{C}}\text{L}$. Finally, we say that two logics are incomparable if no one of the above cases applies. In the following, we will omit the $\mathcal{C}$ subscript when it will be clear from the context.

When we compare interval logics with first-order logics interpreted in relational models, the above criteria are no longer adequate, since we need to compare logics which are interpreted in different types of model (interval models and relational models). We deal with this complication by following the approach outlined by Venema in [39]. We first define suitable model transformations (from interval models to relational models and vice versa) and then we compare the expressiveness of interval and first-order logics modulo these transformations. In order to define the mapping from interval models to relational models, we associate a binary relation P with every propositional variable $p \in \mathcal{AP}$ of the considered interval logic, as in the following definition.

Definition 8 [10] Let $M = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), V_M \rangle$ be an interval model. The corresponding relational model $\eta(M)$ is a pair of the type $\langle \mathbb{D}, V_{\eta(M)} \rangle$, where for all $p \in \mathcal{AP}$, $V_{\eta(M)}(P) = \{(i, j) \in D \times D : [i, j] \in V_M(p)\}$.

To define the mapping from relational models to interval ones, we have to solve a technical problem: the truth of formulae in interval models is evaluated only on ordered pairs $[i, j]$, with $i \leq j$, while in relational models there is no such constraint. To deal with this problem, we associate two atomic propositions $p^{\leq}$ and $p^{\geq}$ of the interval logic with every binary relation P.

Definition 9 [10] Let $\mathbf{M} = \langle \mathbb{D}, V_{\mathbf{M}} \rangle$ be a relational model. The corresponding interval model $\zeta(\mathbf{M})$ is a triple $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}), V_{\zeta(\mathbf{M})} \rangle$ such that for any binary relation P and any interval $[i, j]$, we have that $[i, j] \in V_{\zeta(\mathbf{M})}(p^{\leq})$ iff $(i, j) \in V_{\mathbf{M}}(P)$ and that $[i, j] \in V_{\zeta(\mathbf{M})}(p^{\geq})$ iff $(j, i) \in V_{\mathbf{M}}(P)$.

Therefore, given an interval logic L_I and a first-order logic L_{FO}, we say that L_{FO} is *at least as expressive as* L_I, denoted by $\text{L}_I \preceq \text{L}_{FO}$, if there exists an effective translation τ from L_I to L_{FO} such that for any interval model M, any interval $[i, j]$, and any formula φ of L_I, $M, [i, j] \Vdash \varphi$ if and only if $\eta(M) \models \tau(\varphi)(i, j)$. Conversely, we say that L_I is *at least as expressive as* L_{FO}, denoted by $\text{L}_{FO} \preceq \text{L}_I$, if there exists an effective translation τ' from L_{FO} to L_I such that for any relational model $\mathbf{M}$, any pair (i, j) of elements, and any formula φ of L_{FO}, $\mathbf{M} \models \varphi(i, j)$ if and only if $\zeta(\mathbf{M}), [i, j] \Vdash \tau'(\varphi)$ if $i \leq j$ or $\zeta(\mathbf{M}), [j, i] \Vdash \tau'(\varphi)$ otherwise. We say that L_I is *as expressive as* L_{FO}, denoted by $\text{L}_I \equiv \text{L}_{FO}$, if $\text{L}_I \preceq \text{L}_{FO}$ and $\text{L}_{FO} \preceq \text{L}_I$. $\text{L}_I \prec \text{L}_{FO}$ and $\text{L}_{FO} \prec \text{L}_I$ are defined as expected.

It should be clear from the context which one of the above notions we use each time: in the rest of this section, we will compare first-order logics with interval ones, while, in Sect. 6, we will compare different interval logics to each other.

5.3 The logic $\text{FO}^2[\mathbb{N}, =, <, s]$

As we already pointed out, the relationships between PNL^{π} and $\text{FO}^2[=, <]$have been investigated by Bresolin et al. [10].

Theorem 3 *$PNL^{\pi} \equiv \text{FO}^2[=, <]$, when interpreted over any class of linearly ordered sets.*

We consider now the extension of $\text{FO}^2[=, <]$over $\mathbb{N}$ with the successor function s, denoted by $\text{FO}^2[\mathbb{N}, =, <, s]$. The terms of the language $\text{FO}^2[\mathbb{N}, =, <, s]$ are of the type $s^k(z)$, where $z \in \{x, y\}$ and $s^k(z)$ denotes z when k=0 and $\underbrace{s(s(\ldots s}_{k}(z)\ldots))$ when $k > 0$. Formulae of $\text{FO}^2[\mathbb{N}, =, <, s]$ can be defined as in the case of the logic $\text{FO}^2[=, <]$, mutatis mutandis. Using 2-pebble games and a standard model-theoretic argument, it is possible to prove that $\text{FO}^2[\mathbb{N}, =, <, s]$ is strictly more expressive than $\text{FO}^2[=, <]$. That result, however, is also a direct consequence of the decidability and expressive completeness results given in [10] and in this paper.

Theorem 4 *The satisfiability problem for* $\text{FO}^2[\mathbb{N}, =, <, s]$ *is undecidable.*

Proof Let $\mathcal{O} = \{(i, j) : i, j \in \mathbb{N} \wedge 0 \leq i \leq j\}$ be the second octant of the integer plane $\mathbb{Z} \times \mathbb{Z}$. The *tiling problem for* $\mathcal{O}$ is the problem of establishing whether a given finite set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$ can tile $\mathcal{O}$. For every tile type $t_i \in \mathcal{T}$, let $right(t_i)$, $left(t_i)$, $up(t_i)$, and $down(t_i)$ be the colors of the corresponding sides of t_i. To solve the problem, one must find a function $f : \mathcal{O} \to \mathcal{T}$ such that

$$right(f(n, m)) = left(f(n + 1, m)), \text{ with } n < m,$$
$$\text{and } up(f(n, m)) = down(f(n, m + 1)).$$

Using König's lemma, one can prove that a tiling system tiles $\mathcal{O}$ if and only if it tiles arbitrarily large squares if and only if it tiles $\mathbb{N} \times \mathbb{N}$ if and only if it tiles $\mathbb{Z} \times \mathbb{Z}$. The undecidability of the first of these tiling problems immediately follows from that of the last one [6].

The reduction from the tiling problem for $\mathcal{O}$ to the satisfiability problem for $\text{FO}^2[\mathbb{N}, =, <, s]$ takes advantage of some special relational symbols, namely, those in the set $Let = \{*, Tile, Id, Id_e, Id_b, Id_d, Corr, T_1, T_2, \ldots, T_k\}$. The reduction consists of three main steps: (i) the encoding of an infinite chain that will be used to represent the tiles, (ii) the encoding of the above-neighbor relation by means of

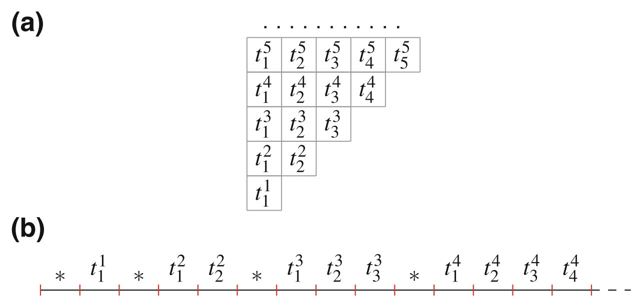

Fig. 1 The encoding of the Octant Tiling Problem: **a** cartesian representation, **b** interval representation

the relation $Corr$, and (iii) the encoding of the right-neighbor relation, which will make use of the successor function. The resulting schema is shown in Fig. 1. Pairs of successive points (*unit intervals*) are used as cells to arrange the tiling, while the relation Id is exploited to represent a row of the octant. Any Id consists of a sequence of unit intervals, each one of which is used either to represent a point of the plane or to separate two Ids. In the former case, it is labeled with the relation $Tile$, while, in the latter case, it is labeled with the relation $*$. Consider now the following formulae:

$$y = s(x) \wedge *(x, y) \quad (1)$$

$$\forall x, y \bigwedge_{P \in Let} (P(x, y) \rightarrow x < y) \quad (2)$$

$$\forall x, y(y = s(x) \leftrightarrow *(x, y) \vee Tile(x, y)) \quad (3)$$

$$\forall x, y(*(x, y) \rightarrow \neg Tile(x, y)) \quad (4)$$

$$\exists x(x = s(y) \wedge Tile(y, x) \wedge *(x, s(x))) \quad (5)$$

The conjunction α_1 of formulae (1), …, (5) guarantees that there exists an infinite sequence of consecutive points $x_0, x_1, x_2 \ldots$. Formula (1) is used to start the chain: it is evaluated over two free variables x and y, that must correspond to two consecutive points, and it forces the predicate $*$ to be true when evaluated over the pair (x, y).

Formula (2) forces the relational symbols in Let to hold only over ordered pairs (x, y) such that $x < y$. Formulae (3) and (4) guarantee that each pair x_i, x_{i+1} is labeled either by $*$ or by $Tile$. Finally, formula (5) states that $*(x_0, x_1)$, $Tile(x_1, x_2)$, and $*(x_2, x_3)$. Now, consider the conjunction α_2 of α_1 and the following formulae:

$$\exists y(y = s^2(x) \wedge Id(x, y)) \quad (6)$$

$$\forall x, y(Id(x, y) \rightarrow *(y, s(y))) \quad (7)$$

$$\forall x, y(Id(x, y) \rightarrow *(x, s(x))) \quad (8)$$

$$\forall x, y(*(x, y) \rightarrow \exists y(s(x) < y \wedge Id(x, y))) \quad (9)$$

$$\forall x, y(Id(x, y) \rightarrow Id_e(s(x), y)) \quad (10)$$

$$\forall x, y(Id_e(x, y) \wedge s(x) < y \rightarrow Id_e(s(x), y)) \quad (11)$$

$$\forall x, y(Id(x, s(y)) \rightarrow Id_b(x, y)) \quad (12)$$

$$\forall x, y(Id_b(x, s(y)) \wedge x < y \rightarrow Id_b(x, y)) \quad (13)$$

$$\forall x, y((Id_e(x, s(y)) \vee Id_d(x, s(y))) \wedge x < y \rightarrow Id_d(x, y)) \quad (14)$$

$$\forall x, y((Id_b(x, y) \vee Id_e(x, y) \vee Id_d(x, y)) \rightarrow \neg Id(x, y)) \quad (15)$$

$$\forall x, y \bigwedge_{\nu, \mu \in \{b, d, e\}, \nu \neq \mu} (Id_\nu(x, y) \rightarrow \neg Id_\mu(x, y)). \quad (16)$$

The formula α_2 builds a chain of Id, in such a way that (i) $Id(x_0, x_2)$ holds, (ii) each Id is followed by another Id, (iii) for each pair $x_i < x_j$, if $Id(x_i, x_j)$, then $*(x_i, x_{i+1})$, (iv) if $Id(x_i, x_j)$ then $\neg Id(x_h, x_k)$, for all $x_i \leq x_h \leq x_k \leq x_j$, with $(x_i, x_j) \neq (x_h, x_k)$, and (v) no pair of points is labeled by both Id_ν and Id_μ, with $\nu, \mu \in \{e, b, d\}$ and $\nu \neq \mu$. For any pair x_i, x_j such that $Id(x_i, x_j)$, the relation Id_e (resp., Id_b, Id_d) holds over all pairs x_k, x_j, with $x_i < x_k < x_j$ (resp., x_i, x_k, with $x_i < x_k < x_j$, x_h, x_k, with $x_i < x_h < x_k < x_j$). Condition (iv) prevents two Ids from holding over two pairs of points x_i, x_j and x_h, x_k such that either $x_i < x_h < x_k = x_j$ ((x_h, x_k) *ends* (x_i, x_j)) or $x_i = x_h < x_k < x_j$ ((x_h, x_k) *begins* (x_i, x_j)) or $x_i < x_h < x_k < x_j$ ((x_h, x_k) *is included in* (x_i, x_j)). Condition (v) excludes the existence of two pairs of points x_i, x_j and x_h, x_k such that $x_h < x_i < x_k < x_j$ ((x_h, x_k) *overlaps* (x_i, x_j)) and both $Id(x_i, x_j)$ and $Id(x_h, x_k)$ hold. Suppose, by contradiction, that there exist two pairs of points x_i, x_j and x_h, x_k such that (x_h, x_k) overlaps (x_i, x_j) and both $Id(x_i, x_j)$ and $Id(x_h, x_k)$ hold. By (10) and (11), we have that $Id_e(x_i, x_k)$ holds. Moreover, by (12) and (13), we have that $Id_b(x_i, x_k)$ holds as well. Thus, formula (16) is not satisfied (contradiction). As a third step, let α_3 be the conjunction of α_2 with the following formulae:

$$\forall x, y(Id(x, y) \rightarrow Corr(s(x), s(y))) \quad (17)$$

$$\forall x, y(Corr(x, y) \rightarrow Tile(x, s(x)) \wedge Tile(y, s(y))) \quad (18)$$

$$\forall x, y(Corr(x, y) \wedge *(s(x), s^2(x)) \rightarrow Tile(y, s(y)) \wedge Tile(s(y), s^2(y)) \wedge *(s^2(y), s^3(y))) \quad (19)$$

$$\forall x, y(Corr(x, y) \wedge \neg *(s(x), s^2(x)) \rightarrow Corr(s(x), s(y))) \quad (20)$$

$$\forall x, y(Id(x, y) \rightarrow \neg Corr(x, y)). \quad (21)$$

Let $Tile(x_i, x_j)$ and $Tile(x_h, x_k)$ hold, and let $x_j < x_h$. We say that the two tiles are *above connected* if and only if $Corr(x_i, x_h)$. From α_3, it follows that the first tile of each Id is above connected to the first tile of the successive Id (formula (17)). Moreover, by taking advantage of the successor function, we extend such a property to the other tiles of any Id, that is, the i-th tile of an Id is above connected to the i-th tile of the successive Id (formula (20)). Finally, formulas (18) and (19) force each Id to have exactly one tile less than the next one. It can be easily shown that if α_3 holds, then the j-th Id provides an encoding of the j-th layer of the octant. Now, let $\alpha_{\mathcal{T}}$ be the conjunction of α_3 and the

Table 1 Translation clauses from $FO^2_r[\mathbb{N}, =, <, s]$ to MPNL

$\tau[x, y](s^k(z) = s^m(z)) = \top$	$(z \in \{x, y\})$,	if $k = m$	$\tau[x, y](\alpha \vee \beta) = \tau[x, y](\alpha) \vee \tau[x, y](\beta)$,	
$= \bot$	$(z \in \{x, y\})$,	if $k \neq m$	$\tau[x, y](\exists x \beta) = \Diamond_r(\tau[y, x](\beta)) \vee \Box_r \Diamond_l(\tau[x, y](\beta))$,	
$\tau[x, y](s^k(z) < s^m(z)) = \bot$	$(z \in \{x, y\})$,	if $k \geq m$	$\tau[x, y](\exists y \beta) = \Diamond_l(\tau[y, x](\beta)) \vee \Box_l \Diamond_r(\tau[x, y](\beta))$,	
$= \top$	$(z \in \{x, y\})$,	if $k < m$	$\tau[x, y](P(s^k(x), s^m(x))) = \Diamond_l \Diamond_r(\mathsf{len}_{=k} \wedge \Diamond_r(\mathsf{len}_{=m-k} \wedge p^{\leq}))$,	if $k < m$
$\tau[x, y](s^k(x) = s^m(y)) = \bot$,		if $k < m$	$= \Diamond_l \Diamond_r(\mathsf{len}_{=k} \wedge \Diamond_r(\mathsf{len}_{=0} \wedge p^{\leq} \wedge p^{\geq}))$,	if $k = m$
$= \mathsf{len}_{=k-m}$,		if $k \geq m$	$= \Diamond_l \Diamond_r(\mathsf{len}_{=m} \wedge \Diamond_r(\mathsf{len}_{=k-m} \wedge p^{\geq}))$,	if $k > m$
$\tau[x, y](s^k(x) < s^m(y)) = \top$,		if $k < m$	$\tau[x, y](P(s^k(y), s^m(y))) = \Diamond_r(\mathsf{len}_{=k} \wedge \Diamond_r(\mathsf{len}_{=m-k} \wedge p^{\leq}))$,	if $k < m$
$= \mathsf{len}_{>k-m}$,		if $k \geq m$	$= \Diamond_r(\mathsf{len}_{=k} \wedge \Diamond_r(\mathsf{len}_{=0} \wedge p^{\leq} \wedge p^{\geq}))$,	if $k = m$
$\tau[x, y](s^m(y) < s^k(x)) = \bot$,		if $k < m$	$= \Diamond_r(\mathsf{len}_{=m} \wedge \Diamond_r(\mathsf{len}_{=k-m} \wedge p^{\geq}))$,	if $k > m$
$= \mathsf{len}_{<k-m}$,		if $k \geq m$	$\tau[x, y](P(x, y)) = p^{\leq}$,	
$\tau[x, y](\neg\alpha) = \neg\tau[x, y](\alpha)$			$\tau[x, y](P(y, x)) = p^{\geq}$	

following formulae:

$$\forall x, y(Tile(x, y) \rightarrow \bigvee_{T \in \mathcal{T}} T(x, y) \wedge \bigwedge_{T, T' \in \mathcal{T},\ T \neq T'} \neg(T(x, y) \wedge T'(x, y)) \quad (22)$$

$$\forall x, y(T(x, y) \wedge Tile(s(x), s(y)) \rightarrow \bigvee_{T' \in \mathcal{T},\ right(T) = left(T')} T'(s(x), s(y))) \quad (23)$$

$$\forall x, y(Corr(x, y) \wedge T(x, s(x))) \rightarrow \bigvee_{T' \in \mathcal{T},\ up(T) = down(T')} T'(y, s(y))). \quad (24)$$

In view of the above steps, it is straightforward to check that, given any set of tile types $\mathcal{T}$, the formula $\alpha_{\mathcal{T}}$ is satisfiable if and only if $\mathcal{T}$ can tile $\mathcal{O}$. Thus, the satisfiability problem of $FO^2[\mathbb{N}, =, <, s]$ is undecidable. □

5.4 Expressive completeness of MPNL for a fragment of $FO^2[\mathbb{N}, =, <, s]$

Let $FO^2_r[\mathbb{N}, =, <, s]$ be the fragment of $FO^2[\mathbb{N}, =, <, s]$ obtained by imposing the following restriction: if both variables x and y occur in the scope of (an occurrence of) a binary relation other than $=$ and $<$, then the successor function s cannot occur in the scope of that occurrence. As an example, each of the formulae $P(s^k(x), s^m(x))$, $P(x, y)$, $s^k(x) = s^m(y)$, and $s^k(x) < s^m(y)$ belongs to $FO^2_r[\mathbb{N}, =, <, s]$, but none of $P(x, s(y))$, $P(s(x), y)$, and, in general, $P(s^n(x), s^m(y))$ and $P(s^n(y), s^m(x))$, with $n + m > 0$, does. It is easy to check that the encoding used to show that $FO^2[\mathbb{N}, =, <, s]$ is undecidable makes an essential use of formulae of the type that we have excluded from the fragment $FO^2_r[\mathbb{N}, =, <, s]$(e.g., see formula 10). By using 2-pebble games and a standard model-theoretic argument, one can show that:

$$FO^2[=, <] \prec FO^2_r[\mathbb{N}, =, <, s] \prec FO^2[\mathbb{N}, =, <, s].$$

To prove that MPNL and $FO^2_r[\mathbb{N}, =, <, s]$are expressively equivalent, we first define the standard translation $ST_{x,y}$ of the former into the latter as

$$ST_{x,y}(\varphi) = x \leq y \wedge ST'_{x,y}(\varphi),$$

where x, y are the two first-order variables in $FO^2_r[\mathbb{N}, =, <, s]$, and

$$ST'_{x,y}(p) = P(x, y);$$
$$ST'_{x,y}(\mathsf{len}_{=k}) - s^k(x) = y;$$
$$ST'_{x,y}(\varphi \vee \psi) - ST'_{x,y}(\varphi) \vee ST'_{x,y}(\psi);$$
$$ST'_{x,y}(\neg\varphi) - \neg ST'_{x,y}(\varphi);$$
$$ST'_{x,y}(\Diamond_l \varphi) - \exists y(y \leq x \wedge ST'_{y,x}(\varphi));$$
$$ST'_{x,y}(\Diamond_r \varphi) - \exists x(y \leq x \wedge ST'_{y,x}(\varphi)).$$

Lemma 4 *For any MPNL-formula φ, any interval model $M = \langle \mathbb{N}, \mathbb{I}(\mathbb{N}), V \rangle$, and any interval $[a, b]$ in M:*

$$M, [a, b] \Vdash \varphi \quad \textit{iff} \quad \eta(M) \models ST_{x,y}(\varphi)[x := a, y := b].$$

Proof Routine structural induction on φ. □

It is worth noticing that, given an MPNL-formula φ, the length of the standard translation $ST_{x,y}(\varphi)$ depends not only on $|\varphi|$, but also on the maximum constant k appearing in length constraints, as atomic propositions of the form $\mathsf{len}_{=k}$ are translated by nesting k-times the successor function s. Hence, the exact complexity of the translation depends on how metric constraints are encoded. When k is constant or encoded in unary, the standard translation is polynomial in the length of $|\varphi|$; when k is encoded in binary, we have that $k = O(2^{|\varphi|})$, and thus the standard translation is exponential in $|\varphi|$.

The inverse translation τ from $FO^2_r[\mathbb{N}, =, <, s]$ to MPNL is given in Table 1. In this case, the choice on the way in which metric constraints are encoded does not affect the complexity: the translation is always exponential in the size of the input formula, due to the clauses for the existential quantifier. The following lemma proves that it is correct.

Lemma 5 *For any formula $\alpha(x, y)$ of* $FO^2_r[\mathbb{N}, =, <, s]$*, any* $FO^2_r[\mathbb{N}, =, <, s]$*-model* $\mathbf{M} = \langle \mathbb{N}, V_{\mathbf{M}} \rangle$ *and any pair $i, j \in \mathbb{N}$, with $i \leq j$:*

(*i*) $\mathbf{M} \models \alpha(i, j)$ *if and only if* $\zeta(\mathbf{M}), [i, j] \Vdash \tau[x, y](\alpha)$, *and*

(*ii*) $\mathbf{M} \models \alpha(j, i)$ *if and only if* $\zeta(\mathbf{M}), [i, j] \Vdash \tau[y, x](\alpha)$.

Proof The proof is by induction on the structural complexity of α (for the sake of simplicity, we only prove claim (i); claim (ii) can be proved similarly):

- $\alpha = (s^k(x) = s^m(x))$. If $k = m$, then both α and its translation $\tau[x, y](\alpha) = \top$ are true, while if $k \neq m$, then α and $\tau[x, y](\alpha) = \bot$ are both false; the same applies when y is used instead of x;
- $\alpha = (s^k(x) < s^m(x))$. If $k \geq m$, then both α and its translation $\tau[x, y](\alpha) = \bot$ are false, while if $k < m$, then α and $\tau[x, y](\alpha) = \top$ are both true; the same applies when y is used instead of x;
- $\alpha = (s^k(x) = s^m(y))$. Let $i < j$. If $k < m$, then $s^k(i) < s^m(j)$, and, since $\mathbf{M} \models \alpha(i, j)$ if and only if $s^k(i) = s^m(j)$, we have that $\mathbf{M} \not\models \alpha(i, j)$. On the other hand, it is immediate to see that $\tau[x, y](\alpha) = \bot$. If $m \leq k$, $s^k(i) = s^m(j)$ if and only if $j - i = k - m$, that is, $\mathbf{M} \models \alpha(i, j)$ if and only if $\zeta(\mathbf{M}), [i, j] \models \mathsf{len}_{=\mathsf{k-m}}$. Likewise for the cases $\alpha = (s^m(y) = s^k(x))$, $\alpha = (s^k(x) < s^m(y))$, and $\alpha = (s^m(y) < s^k(x))$;
- $\alpha = (P(s^k(x), s^m(x)))$. Let $i < j$. If $k < m$, then $s^m(i) - s^k(i) = m - k$ and $s^k(i) - i = k$. Thus, $\mathbf{M} \models \alpha(i, j)$ if and only if P is true over the pair $(s^k(i), s^{m-k}(s^k(i)))$, that is, $\mathbf{M} \models \alpha(i, j)$ if and only if $\zeta(\mathbf{M}), [i, j] \Vdash \Diamond_l \Diamond_r(\mathsf{len}_{=\mathsf{k}} \wedge \Diamond_r(\mathsf{len}_{=\mathsf{m-k}} \wedge p^{\leq}))$. A similar reasoning path can be followed for the case of $m < k$. If $k = m$, then $s^k(i) = s^m(i)$, and thus P must be true over a point-interval, specifically, identified by the pair $(s^k(i), s^k(i))$. Hence, we have that $\mathbf{M} \models \alpha(i, j)$ if and only if $\zeta(\mathbf{M}), [i, j] \Vdash \Diamond_l \Diamond_r(\mathsf{len}_{=\mathsf{k}} \wedge \Diamond_r(\mathsf{len}_{=0} \wedge p^{\leq} \wedge p^{\geq}))$. Likewise, when y substitutes x;
- $\alpha = P(x, y)$ or $\alpha = P(y, x)$. The claim follows from the valuation of $p^{\leq}$ and $p^{\geq}$;
- The Boolean cases are straightforward;
- $\alpha = \exists x \beta$. Suppose that $\mathbf{M} \models \alpha(i, j)$. Then, there is $l \in \mathbf{M}$ such that $\mathbf{M} \models \beta(l, j)$. There are two (non-exclusive) cases: $j \leq l$ and $l \leq j$. If $j \leq l$, by the inductive hypothesis, we have that $\zeta(\mathbf{M}), [j, l] \Vdash \tau[y, x](\beta)$ and thus $\zeta(\mathbf{M}), [i, j] \Vdash \Diamond_r(\tau[y, x](\beta))$. Likewise, if $l \leq j$, by the inductive hypothesis, we have that $\zeta(\mathbf{M}), [l, j] \Vdash \tau[x, y](\beta)$ and thus for every r such that $j \leq r$, $\zeta(\mathbf{M}), [j, r] \Vdash \Diamond_l(\tau[x, y](\beta))$, that is, $\zeta(\mathbf{M}), [i, j] \Vdash \Box_r \Diamond_l(\tau[x, y](\beta))$. Hence we have that $\zeta(\mathbf{M}), [i, j] \Vdash \Diamond_r(\tau[y, x](\beta)) \vee \Box_r \Diamond_l(\tau[x, y](\beta))$, that is, $\zeta(\mathbf{M}), [i, j] \Vdash \tau[x, y](\alpha)$. For the converse direction, it suffices to note that the interval $[i, j]$ has at least one right neighbor, viz. $[j, j]$, and thus the above argument can be reversed;
- $\alpha = \exists y \beta$. Analogous to the previous case. □

Theorem 5 *For any formula* $\alpha(x, y)$ *of* $\mathrm{FO}^2_r[\mathbb{N}, =, <, s]$ *and any* $\mathrm{FO}^2_r[\mathbb{N}, =, <, s]$*-model* $\mathbf{M} = \langle \mathbb{N}, V_{\mathbf{M}} \rangle$, $\mathbf{M} \models \forall x \forall y\, \alpha(x, y)$ *if and only if* $\zeta(\mathbf{M}) \Vdash \tau[x, y](\alpha) \wedge \tau[y, x](\alpha)$. *As a consequence,* $\mathrm{FO}^2_r[\mathbb{N}, =, <, s] \equiv$ MPNL.

From Theorem 5, decidability of $\mathrm{FO}^2_r[\mathbb{N}, =, <, s]$immediately follows. A decision procedure for it can be obtained by first translating the input formula to MPNLand then applying the decision procedure for MPNLdescribed in Sect. 4. Since the length of the translated formula is exponential, no matter how we encode the metric constants in MPNL, the lowest complexity of the procedure is obtained when we choose to use the unary encoding: the satisfiability problem for $\mathrm{FO}^2_r[\mathbb{N}, =, <, s]$ is thus in 2NEXPTIME. A lower bound on the complexity can be given by observing that $\mathrm{FO}^2[=, <]$ is NEXPTIME-hard (the EXPSPACE-hardness result given for MPNLin Sect. 4 cannot be transferred to $\mathrm{FO}^2_r[\mathbb{N}, =, <, s]$, since it relies on the binary encoding).

5.5 Extension of MPNL expressively complete for $\mathrm{FO}^2[\mathbb{N}, =, <, s]$

To cover full $\mathrm{FO}^2[\mathbb{N}, =, <, s]$, MPNL can be extended with additional diamond modalities that shift respectively the beginning, the end, and both endpoints of the current interval to the right by a prescribed distance:

- $M, [i, j] \Vdash \Diamond_e^{+k} \psi$ iff $M, [i, j + k] \Vdash \psi$;
- $M, [i, j] \Vdash \Diamond_b^{+k} \psi$ iff $(i + k \leq j$ and $M, [i + k, j] \Vdash \psi)$ or $(i + k > j$ and $M, [j, i + k] \Vdash \psi)$;
- $M, [i, j] \Vdash \Diamond_{be}^{+k} \psi$ iff $M, [i + k, j + k] \Vdash \psi$.

Let MPNL$^+$ be the resulting language. The standard translation $ST'_{x,y}$ of MPNL-formulae into $\mathrm{FO}^2[\mathbb{N}, =, <, s]$ can be extended to MPNL$^+$ as follows:

$$ST'_{x,y}(\Diamond_e^{+k} \psi) = ST'_{x,y}(\psi)[s^k(y)/y];$$
$$ST'_{x,y}(\Diamond_b^{+k} \psi) = ST'_{x,y}(\psi)[s^k(x)/x];$$
$$ST'_{x,y}(\Diamond_{be}^{+k} \psi) = ST'_{x,y}(\psi)[s^k(x)/x, s^k(y)/y],$$

where $\alpha[t/z]$ denotes the result of the simultaneous substitution of the term t for all free occurrences of z in α.

It is immediate to see that if $ST'_{x,y}(\psi) \in \mathrm{FO}^2[\mathbb{N}, =, <, s]$, then $ST'_{x,y}(\psi)[s^k(x)/x, s^m(y)/y] \in \mathrm{FO}^2[\mathbb{N}, =, <, s]$ for any k, $m \in \mathbb{N}$, and thus the translation of all formulae of MPNL$^+$ remains within $\mathrm{FO}^2[\mathbb{N}, =, <, s]$. Conversely, we can extend the translation τ from $\mathrm{FO}^2_r[\mathbb{N}, =, <, s]$ to MPNL to a translation from $\mathrm{FO}^2[\mathbb{N}, =, <, s]$ to MPNL$^+$ by adding the clauses for the atomic formulae in Table 2. The extensions of the expressive completeness results are routine.

To conclude this subsection, we recall that Venema [39] has shown in a similar way that the interval temporal logic

Table 2 The translation from $\mathrm{FO}^2[\mathbb{N},=,<,s]$ to MPNL^+: the additional clause for $\tau[x,y](P(s^k(x),s^m(y)))$

$\tau[x,y](P(s^k(x),s^m(y)))=$	
$\langle be\rangle^{+k}\langle e\rangle^{+(m-k)}p^{\leq}$,	if $k<m$
$(\mathsf{len}_{>0}\wedge\langle be\rangle^{+k}p^{\leq})\vee(\mathsf{len}_{=0}\wedge\langle be\rangle^{+k}(p^{\leq}\wedge p^{\geq}))$,	if $k=m$
$(\mathsf{len}_{>\mathsf{k-m}}\wedge\langle be\rangle^{+m}\langle b\rangle^{+(k-m)}p^{\leq})\vee$	
$(\mathsf{len}_{=\mathsf{k-m}}\wedge\langle be\rangle^{+m}\langle b\rangle^{+(k-m)}(p^{\leq}\wedge p^{\geq}))\vee$	
$(\mathsf{len}_{<\mathsf{k-m}}\wedge\langle be\rangle^{+m}\langle b\rangle^{+(k-m)}p^{\geq})$,	if $k>m$

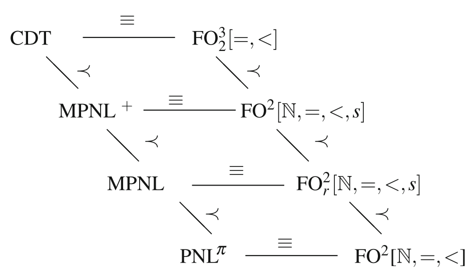

Fig. 2 Expressive completeness results for interval logics

CDT, involving binary modalities based on the ternary interval relation 'chop' and its residuals (denoted respectively by C, D, and T) is expressively complete for the fragment of first-order logic with equality with three variables of which at most two are free, denoted by $\mathrm{FO}^3_2[=,<]$. Note that, when interpreted in $\mathbb{N}$, the successor function is definable in this fragment, which therefore strictly extends $\mathrm{FO}^2[\mathbb{N},=,<,s]$. The resulting hierarchy of expressive completeness results is depicted in Fig. 2. Notice also that all the proposed translations from the first-order languages into interval ones are exponential in the size of the input formula.[2]

6 Classifying the expressive power of MPNL

In the previous sections, we have studied the expressiveness and the computational properties of MPNL. A natural question is whether there exist other interesting variants of PNL that deserve to be analyzed. In this section, we define a family of metric languages, and we compare their expressive power. As it will be proved in the following, MPNL is able to encode all the languages in the family, thus being the most expressive metric extension of PNL.

Let $\sim\in\{<,\leq,=,\geq,>\}$, $k\in\mathbb{N}$, and $k'\in\mathbb{N}\cup\{\infty\}$. We consider a set of *metric modalities* of the form $\Diamond_r^{\sim k}$, $\Diamond_r^{[k,k']}$, $\Diamond_r^{(k,k')}$, $\Diamond_r^{[k,k')}$, and $\Diamond_r^{(k,k']}$, as well as their inverses $\Diamond_l^{\sim k}$, $\Diamond_l^{[k,k']}$, $\Diamond_l^{(k,k')}$, $\Diamond_l^{[k,k')}$, and $\Diamond_l^{(k,k']}$, with the following semantics:

- $M,[i,j]\Vdash\Diamond_r^{\sim k}\psi$ iff there exists $m\geq j$ such that $\delta(j,m)\sim k$ and $M,[j,m]\Vdash\psi$;
- $M,[i,j]\Vdash\Diamond_r^{[k,k']}\psi$ iff there exists $m\geq j$ such that $k\leq\delta(j,m)\leq k'$ and $M,[j,m]\Vdash\psi$;
- $M,[i,j]\Vdash\Diamond_r^{(k,k')}\psi$ iff there exists $m\geq j$ such that $k<\delta(j,m)<k'$ and $M,[j,m]\Vdash\psi$.

The semantic clauses for $\Diamond_r^{[k,k')}$ and $\Diamond_r^{(k,k']}$, as well as those for the inverse modalities, are defined likewise. It is easy to show that all metric modalities are definable by exploiting the length constraints, e.g.:

$$\Diamond_r^{\sim k}\psi := \Diamond_r(\psi\wedge\mathsf{len}_{\sim\mathsf{k}}),$$

and thus that all languages in the family are fragments of MPNL. Let $\kappa\in\{<k,\leq k,=k,\geq k,>k,[k,k'],(k,k'),[k,k'),(k,k']\}$, and let $\Diamond_o^{\kappa}$ be any of the two modal operators $\Diamond_l^{\kappa}$ and $\Diamond_r^{\kappa}$. The dual modalities are defined as usual, that is, $\Box_o^{\kappa}\psi=\neg\Diamond_o^{\kappa}\neg\psi$. Let ϵ be a special symbol such that $\Diamond_r^{\epsilon k}=\Diamond_r$ and $\Diamond_l^{\epsilon k}=\Diamond_l$, for any k, and let $S\subseteq\{\epsilon,<,\leq,=,\geq,>,[],(),[),(]\}$. We will denote by MPNL^S the language that features:

(i) the modal operators $\Diamond_l^{\sim k}$ and $\Diamond_r^{\sim k}$ for each $k\in\mathbb{N}$ and $\sim\in S\cap\{\epsilon,<,\leq,=,\geq,>\}$;

(ii) the modal operators $\Diamond_l^{[k,k']}$ and $\Diamond_r^{[k,k']}$ (resp., $\Diamond_l^{(k,k')}$ and $\Diamond_r^{(k,k')}$, $\Diamond_l^{[k,k')}$ and $\Diamond_r^{[k,k')}$, $\Diamond_l^{(k,k']}$ and $\Diamond_r^{(k,k']}$), for each $k\in\mathbb{N}$, $k'\in\mathbb{N}\cup\{\infty\}$, if $[]\in S$ (resp., $()\in S$, $[)\in S$, $(]\in S$).

We will denote by MPNL_l^S the extension of MPNL^S with the length constraints (this means that $\mathrm{MPNL}_l^{\epsilon}$ is exactly the language MPNL of the previous sections). For the sake of simplicity, we will omit the curly brackets in the superscript; for example, if $S=\{<,>\}$, we will write simply $\mathrm{MPNL}^{<,>}$ instead of $\mathrm{MPNL}^{\{<,>\}}$. Thus, we have that $\mathrm{MPNL}^{\epsilon}\equiv\mathrm{PNL}$ and $\mathrm{MPNL}_l^{\epsilon}\equiv\mathrm{MPNL}$. Moreover, by the following lemma, we can reduce the number of interesting fragments:

Lemma 6 *If $o\in\{r,l\}$, whenever $\Diamond_o^{<k}$ (resp., $\Diamond_o^{[k,k']}$, $\Diamond_o^{(k,k']}$) is included in the language, then $\Diamond_o^{\leq k}$ (resp., $\Diamond_o^{[k,k')}$, $\Diamond_o^{(k,k')}$) can be defined, and the other way around.*

Proof See Table 3, left column. □

Thus, without loss of generality, from now on we can focus our attention on languages characterized by subsets of the set $\{\epsilon,<,=,>,\geq,[],()\}$. As we will see, some languages will be expressive enough to embed non-metric PNL, while some others will not. We will use the expression *weak Metric Propositional Neighborhood Logics* (wMPNL) to denote the latter.

In order to compare the expressive power of interval languages, we use two standard techniques in modal logic, based on bisimulation [4] and bisimulation games [22].

[2] At present, we do not know whether a polynomial translation for any of these cases exists.

Table 3 Equivalences between metric operators, where $o \in \{r, l\}$

$\Diamond_o^{<k}\psi \Leftrightarrow$	$\bot$ $(k = 0)$	$\Diamond_o\psi \Leftrightarrow$	$\Diamond_o^{\geq 0}\psi$
	$\Diamond_o^{\leq k-1}\psi$ $(k > 0)$		$\Diamond_o^{[0,\infty]}\psi$
$\Diamond_o^{[k,k']}\psi \Leftrightarrow$	$\Diamond_o^{[k,k')}\psi$ $(k' = \infty)$	$\Diamond_o^{<k}\psi \Leftrightarrow$	$\Diamond_o^{=0}\psi \vee \ldots \vee \Diamond_o^{=k-1}\psi$
	$\Diamond_o^{[k,k'+1)}\psi$ $(k' \neq \infty)$	$\Diamond_o^{=k}\psi \Leftrightarrow$	$\Diamond_o^{[k,k]}\psi$
$\Diamond_o^{[k,k')}\psi \Leftrightarrow$	$\bot$ $(k' = 0)$	$\Diamond_o^{>k}\psi \Leftrightarrow$	$\Diamond_o^{\geq k+1}\psi$
	$\Diamond_o^{[k,k'-1]}$ $(k' > 0)$		$\Diamond_o^{(k,\infty)}\psi$
	$\Diamond_o^{[k,k']}\psi$ $(k' = \infty)$	$\Diamond_o^{\geq k}\psi \Leftrightarrow$	$\Diamond_o^{[k,\infty]}\psi$
$\Diamond_o^{(k,k']}\psi \Leftrightarrow$	$\Diamond_o^{(k,k'+1)}\psi$ $(k' \neq \infty)$	$\Diamond_o^{(k,k')}\psi \Leftrightarrow$	$\bot$ $(k' \leq 1)$
	$\Diamond_o^{(k,k')}\psi$ $(k' = \infty)$		$\Diamond_o^{[k+1,k'-1]}\psi$ $(k' > 1)$
$\Diamond_o^{(k,k')}\psi \Leftrightarrow$	$\bot$ $(k' \leq 1)$		$\Diamond_o^{[k+1,k']}\psi$ $(k' = \infty)$
	$\Diamond_o^{(k,k'-1]}$ $(k' > 1)$		
	$\Diamond_o^{(k,k']}\psi$ $(k' = \infty)$		

Given an interval logic L, for each modality $\Diamond$ in the language of L, we denote by $R_\Diamond$ the (interval) relation on which $\Diamond$ is based. Now, given a pair of L-models M, M', with $M = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), V \rangle$ and $M' = \langle \mathbb{D}', \mathbb{I}(\mathbb{D}'), V' \rangle$, we say that a relation $Z \subseteq \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$ is a *bisimulation* if $([a, b], [a', b']) \in Z$ implies that (i) $[a, b]$ and $[a', b']$ satisfy the same atomic propositions, (ii) for every relation $R_\Diamond$ and every interval $[c, d]$ such that $[a, b]\ R_\Diamond\ [c, d]$, there exists an interval $[c', d']$ such that $[a', b']\ R_\Diamond\ [c', d']$ and $[c, d], [c', d'] \in Z$, and (iii) for every relation $R_\Diamond$ and every interval $[c', d']$ such that $[a', b']\ R_\Diamond\ [c', d']$, there exists an interval $[c, d]$ such that $[a, b]\ R_\Diamond\ [c, d]$ and $[c, d], [c', d'] \in Z$. Interval logics are invariant under bisimulation, as it is the case with modal logic [4].

Proposition 1 *Let L be a language for interval logics, M and M′ two L-models, and $Z \subseteq \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$ be a bisimulation. Then, every pair $([a, b], [a', b']) \in Z$ is such that $[a, b]$ and $[a', b']$ satisfy the same L-formulae.*

The above proposition can be proved by induction on the structural complexity of formulae.

The notion of bisimulation game can be viewed as a generalization of the notion of bisimulation. In the context of interval logics, we define the notion of a *N-moves bisimulation game* (for the interval logic L) to be played by two players, Player I and Player II, on a pair of L-models M, M', with $M = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), V \rangle$ and $M' = \langle \mathbb{D}', \mathbb{I}(\mathbb{D}'), V' \rangle$. The game starts from a given *initial configuration*, where a *configuration* is a pair of intervals $([a, b], [a', b'])$, with $[a, b] \in \mathbb{I}(\mathbb{D})$ and $[a', b'] \in \mathbb{I}(\mathbb{D}')$. A configuration $([a, b], [a', b'])$ is *matching* if $[a, b]$ and $[a', b']$ satisfy the same atomic propositions in their respective models. The *moves* of the game depend on the modal operators of L: for each modality $\Diamond$ in the language of L, Player I can play the corresponding move: choose M (resp., M'), and an interval $[c, d]$ (resp., $[c', d']$) such that $[a, b]\ R_\Diamond\ [c, d]$ (resp., $[a', b']\ R_\Diamond\ [c', d']$). Player II must reply by choosing an interval $[c', d']$ (resp., $[c, d]$) in M' (resp., M), which leads to the new configuration $([c, d], [c', d'])$. If after any given round the current configuration is not matching, Player I wins the game; otherwise, after N rounds, Player II wins the game. Intuitively, Player II has a *winning strategy* in the N-moves bisimulation game on the models M and M' with a given initial configuration if she can win regardless of the moves played by Player I; otherwise, Player I has a winning strategy. A formal definition of winning strategy can be found in [22]. The following key property of N-move bisimulation games can be proved routinely, in analogy with similar results about bisimulation games in modal logic [22].[3]

Proposition 2 *Let L be a language for interval logics with finitely many atomic propositions. For all $N \geq 0$, Player II has a winning strategy in the N-move L-bisimulation game on M and M′ with initial configuration $([a, b], [a', b'])$ if and only if $[a, b]$ and $[a', b']$ satisfy the same L-formulae with modal depth at most N.*

In order to prove that a modal operator $\bigcirc$ is not definable in L, it suffices to construct a pair of interval models M and M' and a bisimulation (resp., a bisimulation game such that Player II has a winning strategy) between them, relating a pair of intervals $[a, b] \in M$ and $[a', b'] \in M'$, such that $M, [a, b] \Vdash \bigcirc p$, but $M', [a', b'] \nVdash \bigcirc p$.

[3] In Proposition 2, we make use of the notion of modal depth of an L-formula φ. Let us denote the modal depth of φ by $mdepth(\varphi)$. As usual, $mdepth(\varphi)$ can be inductively defined as follows: (i) $mdepth(p) = 0$, for each $p \in \mathcal{AP}$; (ii) $mdepth(\neg\varphi) = mdepth(\varphi)$, $mdepth(\varphi \vee \psi) = max\{mdepth(\varphi), mdepth(\psi)\}$, $mdepth(\Diamond\varphi) = mdepth(\varphi) + 1$, for each modality $\Diamond$ of the language.

6.1 The class of wMPNL

Here, we analyze the set of languages in wMPNL. Formally, wMPNL is the subset of MPNL defined as follows:

wMPNL = {L | L ∈ MPNL and PNL $\not\preceq$ L}.

The following lemma states some basic results that we will use to classify languages in wMPNL.

Lemma 7 *If $o \in \{r, l\}$, whenever any of the modalities in $\{\Diamond_o^{\geq k}, \Diamond_o^{[k,k']}\}$ (resp., $\{\Diamond_o^{=k}, \Diamond_o^{[k,k']}\}$, $\{\Diamond_o^{\geq k}, \Diamond_o^{(k,k')}, \Diamond_o^{[k,k']}\}$) is included in the language, then $\Diamond_o$ (resp., $\Diamond_o^{<k}$, $\Diamond_o^{>k}$) can be defined. Similarly, whenever $\Diamond_o^{[k,k']}$ is included, then $\Diamond_o^{=k}$, $\Diamond_o^{\geq k}$, and $\Diamond_o^{(k,k')}$ can be defined.*

Proof See Table 3, right column. □

Theorem 6 *Let $\mathcal{S}_w = \{\{<\}, \{>\}, \{=\}, \{()\}, \{<, =\}, \{>, ()\}\}$. We have that wMPNL = {MPNL^S, MPNL$_l^S$ | $S \in \mathcal{S}_w$}.*

Proof As a preliminary step, notice that, by Lemma 7, it immediately follows MPNL$^=$ ≡ MPNL$^{<,=}$ and MPNL$^{()}$ ≡ MPNL$^{>,()}$. Thus, we can disregard the logics MPNL$^{<,=}$ and MPNL$^{>,()}$. Next, we show that both MPNL and MPNL^S. First, we show that both MPNL^S and MPNL$_\ell^S$ belong to wMPNL for each $S \in \mathcal{S}_w$. To this end, we prove that PNL $\not\preceq$ MPNL$_l^S$, for each $S \in \mathcal{S}_w$. From this, it immediately follows that, for each $S \in \mathcal{S}_w$, PNL $\not\preceq$ MPNL^S as well. Moreover, by Lemma 7, we have that MPNL$_l^<$ $\preceq$ MPNL$_l^=$ and MPNL$_l^>$ $\preceq$ MPNL$_l^{()}$, and thus it suffices to show that PNL $\not\preceq$ MPNL$_l^=$ and PNL $\not\preceq$ MPNL$_l^{()}$.

PNL $\not\preceq$ MPNL$_l^=$. It is easy to show that classical, non-metric modal operators of PNL can be expressed using formulae of MPNL$_l^=$ of infinite length. For example, it is possible to express the formula $\Diamond_r p$ of PNL by means the infinite formulae $\Diamond_r^{=0} p \vee \Diamond_r^{=1} p \vee \ldots \Diamond_r^{=i} p \vee \ldots$ Nevertheless, suppose, by contradiction, that there exists a finite formula $\varphi \in$ MPNL$_l^=$ such that $\varphi \equiv \Diamond_r p$. This means that φ contains a finite number of modal operators. Let $t \in \mathbb{N}$ be the largest number such that $\Diamond_r^{=t}$ or $\Diamond_l^{=t}$ occurs in φ, and, for any $t \in \mathbb{N}$, define ^{t}MPNL$_l^=$ as the restriction of MPNL$_l^=$ to the set of modalities $\{\Diamond_r^{=k}, \Diamond_l^{=k} \mid 0 \leq k \leq t\}$. Now, let $M = \langle \mathbb{D} = \mathbb{N}, \mathbb{I}(\mathbb{D}), V\rangle$ and $M' = \langle \mathbb{D}' = \mathbb{N}, \mathbb{I}(\mathbb{D}'), V'\rangle$, $\mathcal{AP} = \{p\}$, $V(p) = \{[1, t+2]\}$, $V'(p) = \emptyset$, and $Z \subset \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$ defined as $Z = \{([i, j], [i', j']) \mid \delta(i, j) \leq t\}$. It is possible to show that Z is a bisimulation for ^{t}MPNL$_l^=$. Since $M, [1, 1] \Vdash \Diamond_r p$, $M', [1', 1'] \nVdash \Diamond_r p$, and $[1, 1]$ is Z-related with $[1', 1']$, we have a contradiction.

PNL $\not\preceq$ MPNL$_l^{()}$. Again, suppose that for some $\varphi \in$ MPNL$_l^{()}$ it is the case that $\varphi \equiv \Diamond_r p$. Consider $M = \langle \mathbb{D} = \mathbb{N}, \mathbb{I}(\mathbb{D}), V\rangle$, $M' = \langle \mathbb{D}' = \mathbb{N}, \mathbb{I}(\mathbb{D}'), V'\rangle$, $\mathcal{AP} = \{p\}$, $V(p) = \{[1, 1]\}$, $V'(p) = \emptyset$, and $Z \subset \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$ defined as $Z = \{([i, j], [i', j']) \mid i \neq j\}$. As before, Z is a bisimulation for MPNL$_l^{()}$. Since $M, [0, 1] \Vdash \Diamond_r p$, $M', [0', 1'] \nVdash \Diamond_r p$, and $[0, 1]$ is Z-related with $[0', 1']$, we have a contradiction.

To complete the proof, we show that no other language belongs to wMPNL, that is, neither MPNL^S nor MPNL$_l^S$ belongs to wMPNL for any $S \notin \mathcal{S}_w$. Let $S \subseteq \{\epsilon, <, =, >, \geq, [], ()\}$ such that $S \notin \mathcal{S}_w$. We must show that PNL $\preceq$ MPNL^S and PNL $\preceq$ MPNL$_l^S$. Since MPNL^S $\preceq$ MPNL$_l^S$, it suffices to show that PNL $\preceq$ MPNL^S. If $\epsilon \in S$, then clearly PNL $\preceq$ MPNL^S, since PNL ≡ MPNL$^\epsilon$. If $\geq \in S$ or $[] \in S$, then the result immediately follows from Lemma 7. If $\{<, >\} \subseteq S$, then the thesis immediately follows by the fact that $\Diamond_o \psi$ is defined by $\Diamond_o^{<1} \psi \vee \Diamond_o^{>0} \psi$ for each $o \in \{r, l\}$. The rest of the cases are consequences of the considered ones and of previous lemmas. □

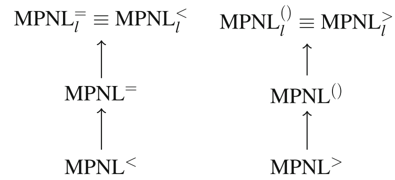

Fig. 3 Relative expressive power of the metric languages belonging to wMPNL. An *arrow* going from L to L′ denotes that L′ is strictly more expressive than L. Logics which are not connected through any path are incomparable

We now establish how the various languages of wMPNL relate to each other in terms of expressive power.

Theorem 7 *The relative expressive power of the languages of the class wMPNL is as depicted in Fig. 3, where each arrow means that the language at the top is strictly more expressive than the one at the bottom.*

Proof By Lemma 7, we already know that MPNL$^<$ $\preceq$ MPNL$^=$, MPNL$_l^<$ $\preceq$ MPNL$_l^=$, MPNL$^>$ $\preceq$ MPNL$^{()}$, and that MPNL$_l^>$ $\preceq$ MPNL$_l^{()}$. To complete the proof, it remains to show that MPNL$^=$ $\not\preceq$ MPNL$^<$, MPNL$_l^=$ $\preceq$ MPNL$_l^<$, MPNL$^{()}$ $\not\preceq$ MPNL$^>$, and MPNL$_l^{()}$ $\preceq$ MPNL$_l^>$. **MPNL$^=$ $\not\preceq$ MPNL$^<$.** It suffices to show that $\Diamond_r^{=k}$ cannot be defined in MPNL$^<$. Suppose the contrary, and let $M = \langle \mathbb{D} = \mathbb{N}, \mathbb{I}(\mathbb{D}), V\rangle$, $M' = \langle \mathbb{D}' = \{0'\}, \mathbb{I}(\mathbb{D}'), V'\rangle$, $\mathcal{AP} = \{p\}$, $V(p) = \mathbb{I}(\mathbb{D})$, $V'(p) = \mathbb{I}(\mathbb{D}') = \{[0', 0']\}$, and $Z = \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$. It is possible to show that Z is a bisimulation for MPNL$^<$. Since it holds that $M, [0, 0] \Vdash \Diamond_r^{=1} p$, $M', [0', 0'] \nVdash \Diamond_r^{=1} p$, and $[0, 0]$ is Z-related to $[0', 0']$, we have a contradiction.

MPNL$^{()}$ $\not\preceq$ MPNL$^>$. For any $t \in \mathbb{N}$, consider the language ^{t}MPNL$^>$, that is, as before, the restriction of MPNL$^>$ to the set of modalities $\{\Diamond_r^{>k}, \Diamond_l^{>k} \mid 0 \leq k \leq t\}$. Let $N \in \mathbb{N}$. Moreover, let $M = \langle \mathbb{D} = \mathbb{N}, \mathbb{I}(\mathbb{D}), V\rangle$, $M' = \langle \mathbb{D}' = \mathbb{N}, \mathbb{I}(\mathbb{D}'), V'\rangle$, $\mathcal{AP} = \{p\}$, $V(p) = \{[i, j] \mid \delta(i, j)$ is odd and $\delta(i, j) \leq t + 1\}$, $V'(p) = \{[i', j'] \mid \delta(i, j)$ is odd,

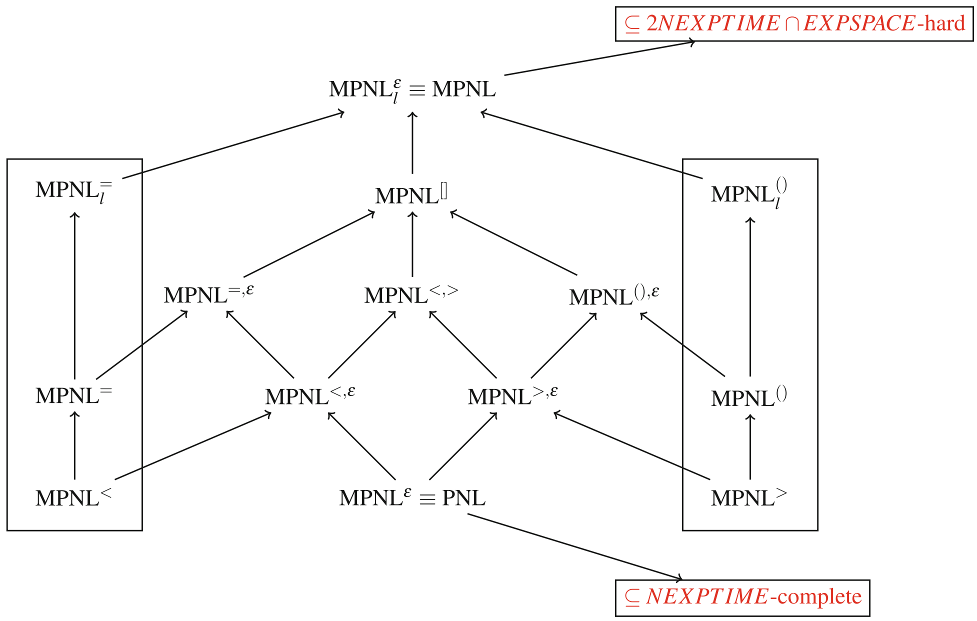

Fig. 4 Relative expressive power of the metric languages belonging to MPNL. *Fragments* inside the *boxes* belong to wMPNL (see Fig. 3)

$\delta(i,j) \le t+1$, and $[i',j'] \ne [(a-1)',a']\}$, where $a = (N-1)\cdot(t+1)+3$, and consider the relation $Z = \{([i,j],[k',l']) \mid \delta(i,j) = \delta(k,l) \le t+1$ and $[k',l'] \ne [(a-1)',a']\} \cup \{([i,j],[i',k']) \mid \delta(i,j) > t+N+1$ and $\delta(i,k) > t+1\} \cup \{([a-1,a],[(a-3)',a']), ([a-1,a],[(a-1)',(a+2)'])\} \cup \{([i,j],[(a-1)',a']) \mid \delta(i,j) = 2\}$. It is possible to show that Z represents a winning strategy for Player II with initial configuration $([a,b],[a',b'])$ (for any b) in the N-moves bisimulation game for ${}^t\mathrm{MPNL}^{>}$. However, we have that $M,[a,b] \Vdash \Diamond_l^{(0,2)} p$ and $M',[a',b'] \nVdash \Diamond_l^{(0,2)} p$, which means that the formula $\Diamond_l^{(0,2)} p$ cannot be expressed in the language ${}^t\mathrm{MPNL}^{>}$ for any $t, N \in \mathbb{N}$. Thus, we have the result.

$\mathrm{MPNL}_l^{=}$ $\preceq \mathrm{MPNL}_l^{<}$, $\mathrm{MPNL}_l^{()} \preceq \mathrm{MPNL}_l^{>}$. This is immediate by observing that, for each $o \in \{r,l\}$, $\Diamond_o^{=k}\psi$ is defined by $\Diamond_o^{<k+1}(\mathsf{len}_{=k} \wedge \psi)$, and that $\Diamond_o^{(k,k')}\psi$ is defined by $\Diamond_o^{>k}(\mathsf{len}_{<k'} \wedge \psi)$ (if $k' \ne \infty$) or by $\Diamond_o^{>k}\psi$ (if $k' = \infty$).

From the above results, we have that $\mathrm{MPNL}^{<} \prec \mathrm{MPNL}^{=}$, $\mathrm{MPNL}_l^{<} \equiv \mathrm{MPNL}_l^{=}$, $\mathrm{MPNL}^{>} \prec \mathrm{MPNL}^{()}$, and $\mathrm{MPNL}_l^{>} \equiv \mathrm{MPNL}_l^{()}$. We show that each language in the set $\{\mathrm{MPNL}^{<}, \mathrm{MPNL}^{=}, \mathrm{MPNL}_l^{=}\}$ is incomparable with any language in the set $\{\mathrm{MPNL}^{>}, \mathrm{MPNL}^{()}, \mathrm{MPNL}_l^{()}\}$. To this end, it suffices to show that $\mathrm{MPNL}^{<} \not\preceq \mathrm{MPNL}_l^{()}$ and $\mathrm{MPNL}^{>} \not\preceq \mathrm{MPNL}_l^{=}$, which can be done as in Theorem 6. Finally, we must show that $\mathrm{MPNL}^{=} \prec \mathrm{MPNL}_l^{=}$ and $\mathrm{MPNL}^{()} \prec \mathrm{MPNL}_l^{()}$. It is easy to see that $\mathrm{MPNL}^{=} \preceq \mathrm{MPNL}_l^{=}$ and $\mathrm{MPNL}^{()} \preceq \mathrm{MPNL}_l^{()}$. To show that $\mathrm{MPNL}_l^{=} \not\preceq \mathrm{MPNL}^{=}$, for any $t \in \mathbb{N}$, consider the language ${}^t\mathrm{MPNL}^{=}$, defined as before. Let $N \in \mathbb{N}$. Moreover, let $M = \langle \mathbb{D} = \mathbb{N}, \mathbb{I}(\mathbb{D}), V \rangle$, $M' = \langle \mathbb{D}' = \mathbb{N}, \mathbb{I}(\mathbb{D}'), V' \rangle$, $\mathcal{AP} = \emptyset$, $V(p) = V'(p) = \emptyset$, and let Z be the relation $\{([i,j],[i',j']) \mid i,j \in \mathbb{N}\} \cup \{([a,a+1],[a',(a+2)'])\} \cup \{([i,j],[(i+1)',(j+1)']) \mid i,j \in \mathbb{N}\}$, where $a = (N-1)\cdot(t+1)$. It is possible to show that Z represents a winning strategy for Player II with initial configuration $([a,a+1],[a',(a+2)'])$ in the N-moves bisimulation game for ${}^t\mathrm{MPNL}^{=}$. However, $M,[a,a+1] \Vdash \mathsf{len}_{=1}$ and $M',[a',(a+2)'] \nVdash \mathsf{len}_{=1}$, which means that the formula $\mathsf{len}_{=1}$ cannot be expressed in the language ${}^t\mathrm{MPNL}^{=}$ for any $t, N \in \mathbb{N}$. Thus, we have the result. By exploiting a very similar argument, it is possible to show that $\mathrm{MPNL}_l^{()} \not\preceq \mathrm{MPNL}^{()}$. □

6.2 Expressive power of languages in the class MPNL

In this section, we deal with the problem of classifying all the fragments of the class MPNL with respect to their relative expressive power. Figure 4 shows how the various languages are related to each other.

Lemma 8 *The following equivalences hold:*

1. $\mathrm{MPNL}^{<,>} \equiv \mathrm{MPNL}^{<,\ge}$;
2. $\mathrm{MPNL}^{<,()} \equiv \mathrm{MPNL}^{=,()} \equiv \mathrm{MPNL}^{=,>} \equiv \mathrm{MPNL}^{=,\ge} \equiv \mathrm{MPNL}^{[]}$;
3. $\mathrm{MPNL}^{>,\epsilon} \equiv \mathrm{MPNL}^{\ge}$;
4. $\mathrm{MPNL}^{\ge,()} \equiv \mathrm{MPNL}^{(),\epsilon}$.

Proof It suffices to use Lemma 7 and the equivalences in Table 4 (left column). □

Corollary 2 *If $S = \{\epsilon, <, =, >, \ge, (), []\}$, then we have that $MPNL^S \equiv MPNL^{[]}$ and $MPNL_l^S \equiv MPNL_l^{[]}$.*

Theorem 8 *The relative expressive power of the languages in the class MPNL is as depicted in Fig. 4, where each arrow*

Table 4 Additional equivalences between metric operators, with $o \in \{r, l\}$

$\Diamond_o^{\geq k}\psi \Leftrightarrow$	$\Diamond_o^{<1}\psi \vee \Diamond_o^{>0}\psi$	$k = 0$	$\Diamond_o^{<k}\psi \Leftrightarrow$	$\Diamond_o^{[0,k-1]}\psi$	$k > 0$
	$\Diamond_o^{>k-1}\psi$	$k > 0$		$\perp$	$k = 0$
$\Diamond_o^{(k,k')}\psi \Leftrightarrow$	$\Diamond_o^{=k+1}\psi \vee \ldots \vee \Diamond_o^{=k'-1}\psi \vee \perp$	$k' \neq \infty$	$\Diamond_o^{>k}\psi \Leftrightarrow$	$\Diamond_o^{[k+1,\infty]}\psi$	
	$\Diamond_o^{>k}\psi$	$k' = \infty$	$\Diamond_o^{[k,k']}\psi \Leftrightarrow$	$\Diamond_o(\mathsf{len}_{\geq \mathsf{k}} \wedge \mathsf{len}_{\leq \mathsf{k'}} \wedge \psi)$	$k' \neq \infty$
$\Diamond_o^{[k,k']}\psi \Leftrightarrow$	$\Diamond_o^{(k-1,k'+1)}\psi$	$k > 0, k' \neq \infty$		$\Diamond_o(\mathsf{len}_{\geq \mathsf{k}} \wedge \psi)$	$k' = \infty$
	$\Diamond_o^{<k'+1}\psi$	$k = 0, k' \neq \infty$	$\Diamond_o^{=k}\psi \Leftrightarrow$	$\Diamond_o(\mathsf{len}_{=\mathsf{k}} \wedge \psi)$	
	$\Diamond_o^{(k-1,k')}\psi$	$k > 0, k' = \infty$	$\Diamond_o^{(k,k')}\psi \Leftrightarrow$	$\Diamond_o(\mathsf{len}_{>\mathsf{k}} \wedge \mathsf{len}_{<\mathsf{k'}} \wedge \psi)$	$k' \neq \infty$
	$\Diamond_o^{(k,k')}\psi \vee \Diamond_o^{<1}\psi$	$k = 0, k' = \infty$		$\Diamond_o(\mathsf{len}_{>\mathsf{k}} \wedge \psi)$	$k' = \infty$
$\Diamond_o^{\geq k}\psi \Leftrightarrow$	$\Diamond_o\psi$	$k = 0$			
	$\Diamond_o^{>k-1}\psi$	$k > 0$			

means that the language at the top is strictly more expressive than the one at the bottom.

Proof To prove this result, one can exploit bisimulations (and bisimulation games), as in the previous theorems, plus the equivalences in Table 4, right column, and all the above results. For this reason, we only detail the proof of one case, namely, $\mathrm{MPNL}^{<} \not\preceq \mathrm{MPNL}^{(),\epsilon}$ (the proofs of the other cases are very similar).

For any $t \in \mathbb{N}$, let us define the language ${}^t\mathrm{MPNL}^{(),\epsilon}$ the same way we did before. Let $N \in \mathbb{N}$. Moreover, let $M = \langle \mathbb{D} = \mathbb{N}, \mathbb{I}(\mathbb{D}), V \rangle$, $M' = \langle \mathbb{D}' = \mathbb{N}, \mathbb{I}(\mathbb{D}'), V' \rangle$, $\mathcal{AP} = \{p\}$, $V(p) = \{[i, i], [i, i+1] \mid i \in \mathbb{N}\}$, $V'(p) = \{[i', i'], [i', (i+1)'] \mid i \in \mathbb{N}\} \setminus \{[a', a']\}$, where $a = (N-1) \cdot (t+1) + 2$, and consider the relation $Z = \{([i, j], [k', l']) \mid \delta(i, j) = \delta(k, l) \text{ and } [k', l'] \neq [a', a']\} \cup \{([a, a], [a', (a+1)']), ([a, a], [(a-1)', a'])\} \cup \{([i, i+2], [a', a']) \mid i \in \mathbb{N}\}$. It is possible to show that Z represents a winning strategy for Player II with initial configuration $([a, b], [a', b'])$ (for any b) in the N-moves bisimulation game for ${}^t\mathrm{MPNL}^{(),\epsilon}$. However, we have that $M, [a, b] \Vdash \Diamond_l^{<1} p$ and $M', [a', b'] \nVdash \Diamond_l^{<1} p$, which means that the formula $\Diamond_l^{<1} p$ cannot be expressed in ${}^t\mathrm{MPNL}^{(),\epsilon}$ for any $t, N \in \mathbb{N}$. Thus, we have the result. □

7 Concluding remarks

In this paper, we have proposed and studied metric extensions of Propositional Neighborhood Logic over the interval structure of natural numbers $\mathbb{N}$. We have demonstrated that these are expressive and natural languages to reason about that structure by proving the complexity and expressive completeness results summarized in Table 5. First, we have considered a very expressive language in this class, called MPNL, and shown the decidability of its satisfiability problem. Then, we have identified an appropriate fragment called $\mathrm{FO}^2_r[\mathbb{N}, =, <, s]$ of $\mathrm{FO}^2[\mathbb{N}, =, <, s]$ (the two-variable fragment of first-order logic with equality, order, successor, and any family of binary relations, interpreted on the structure of natural numbers), denoted by $\mathrm{FO}^2_r[\mathbb{N}, =, <, s]$, and we have proved that MPNL is expressively complete for such a fragment. Decidability of $\mathrm{FO}^2_r[\mathbb{N}, =, <, s]$ immediately follows. Then, we have shown how to extend MPNL in order to obtain an interval logic that is expressively complete for full $\mathrm{FO}^2[\mathbb{N}, =, <, s]$, which we have proved to be undecidable. Finally, we have discussed the variety of fragments of MPNL and studied their expressiveness.

Table 5 Complexity and expressive completeness results

PNL^{π}	NEXPTIME complete	$\mathrm{FO}^2[=, <]$ [10]	NEXPTIME complete [33]
MPNL	2NEXPTIME, EXPSPACE hard	$\mathrm{FO}^2_r[\mathbb{N}, =, <, s]$	2NEXPTIME, NEXPTIME hard
MPNL^{+}	Undecidable	$\mathrm{FO}^2[\mathbb{N}, =, <, s]$	Undecidable

The results obtained here are amenable to some fairly straightforward generalizations, e.g., from $\mathbb{N}$ to $\mathbb{Z}$. An important open problem is to find the exact complexity of the satisfiability problem for MPNL, when constraints are represented in binary, and the identification of the fragment(s) of MPNL where the complexity jumps occur. Another interesting open problem is to determine more precisely the (un)decidability border in the family of metric propositional neighborhood logics by identifying maximal decidable extensions of MPNL.

From a more practical point of view, we plan to implement the decision procedure for MPNL presented in this paper, and to study the application of the logic in the modeling and verification of reactive systems.

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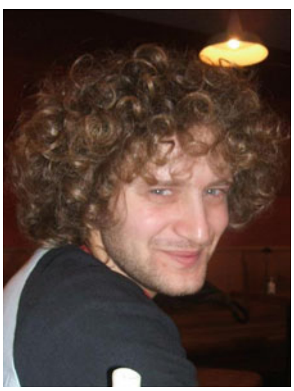

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