Statistical Methods & Applications (2023) 32:211–235
https://doi.org/10.1007/s10260-022-00652-3

ORIGINAL PAPER

Trend resistant balanced bipartite block designs

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Accepted: 11 June 2022 / Published online: 18 August 2022

Abstract

Balanced Bipartite Block (BBPB) designs resistant against the trend are used when the interest of the experimenter is in making comparisons between two sets of treatments that are disjoint, and there is the presence of systematic trend within a block. This paper deals with the bipartite block model incorporating trend component. The general methodology has been described related to BBPB designs incorporating trend effect. The conditions for a BBPB design to be trend resistant are also obtained. Further, methods of constructing trend resistant BBPB designs are discussed. The designs so obtained are trend resistant and are more efficient for estimating the contrasts pertaining to two treatments from different sets.

Keywords Block design · Balanced bipartite · Trend resistant design

1 Introduction

Experiments are generally undertaken to compare effects of several conditions/treatments/processes on some phenomena. Designing an experiment is a systematic method to determine the relationship between factors affecting a process and the output of that process taking into account all the sources of variability in the experiment. Accordingly the response is modeled considering the sources of variability in the experiment. Usually the interest is in making all pair-wise comparisons among the treatments with equal precision. There are many experimental situations where it is

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desired to compare several test treatments to a standard treatment called control. The main interest here lies in making test treatment–control comparisons with as much precision as possible, and comparisons within the test treatments are of less interest. For example, in agricultural experiments, the experimenter aims to test a set of new varieties of a crop with an already existing variety and determine which of the varieties perform better in comparison to the existing variety. The designs that are efficient for all pair-wise comparisons may not be efficient for this subset of comparisons. While conducting large scale experiments with crop varieties for identification of varieties for release, when there are a number of popular and prevalent varieties under cultivation, the experimenters would like to include more than one variety as control with which the comparisons are to be made. This experimental situation gives rise to designs for comparing treatments belonging to two disjoint sets of treatments. The two sets are disjoint in the sense that there are no common treatments between the two. The balanced block design so obtained to compare two disjoint sets of treatments is called Balanced Bipartite Block (BBPB) design. The interest here is to estimate the contrasts of the type (τ_i–τ_j) with as high precision as possible, where τ_i and τ_j belong to 1st and 2nd set of treatments, respectively.

A lot of work has been done on different aspects of BBPB designs and its optimality under various experimental setup (Majumdar 1986; Hedayat et al. 1988; Kageyama and Sinha 1988; Kageyama and Sinha 1991; Jacroux 1990; Jaggi et al. 1996; Jaggi 1996; Parsad et al. 1996; Jaggi and Gupta 1997; Jaggi et al. 1999; Jacroux 2003; Gupta et al. 2002; Altan 2005; Iqbal et al. 2011; Pravender and Patel 2016; Mandal et al. 2016; Abeyanayake et al. 2011). Besides, Gupta and Parsad (2001) provided a detailed review on block designs for comparing test treatments with control treatments.

In another case of experiments under block design set up, experiments may be carried out using plots occurring in long, narrow rows wherein spatial fertility trends may occur. In such situations, the response may also depend on the spatial position of the experimental unit within a block. One way to overcome such situations is the suitable arrangement of treatments over plots within a block such that the arranged design is capable of completely eliminating the effects of defined components of a common trend. Trend-free designs are quite useful in the experimental situations that may occur in Green house experiments where the source of heat is located on sides of the house and the experimental units (pots) are kept in lines; in poultry experiments where the source of heat is at the centre of the shed and chicks of early age are in the cages; in animal experiments where littermates (animals born in the same litter) are experimental units within a block i.e. litters are blocks. Other such experiments are orchard and vineyard experiments on undulating topography, experiments in which response variable of interest is affected by slowly migrating insects entering the area from one side, laboratory experiments where the responses to the experimental units may be affected within time periods by instrument drift or analyst fatigue, etc.

1.1 Experimental situation (Cox 1958)

Consider an experiment to investigate the effect on the textile process of changing the relative humidity. Suppose that three levels of relative humidity 50, 60 and 70% are to be used. To obtain uniform experimental units a suitable quantity of raw material was taken and thoroughly mixed and then divided into say, nine experimental units. The first batch was processed at one relative humidity in the first period, the second batch at different relative humidity in the second period, and so on. Superimposed on any treatment effects and on random variations remaining, it is likely to be a smooth trend due to the aging of the material. It would be of interest to estimate this trend explicitly and set up the experiment so that the trend has little or no influence on the estimates of treatment effects. Here, there are three humidity levels as three treatments T_{50}, T_{60} and T_{70}. Since the experimental material is having trend over time, the treatments are to be allocated in the following order to eliminate the effect of linear trend:

Treatments	T_{60}	T_{50}	T_{70}	T_{70}	T_{60}	T_{50}	T_{50}	T_{70}	T_{60}
Plot position	1	2	3	4	5	6	7	8	9
Coefficient of orthogonal polynomial (linear)	−4	−3	−2	−1	0	1	2	3	4

To examine whether the allotment of treatments are orthogonal to trend, test whether the sum of the coefficients of orthogonal polynomials is zero for each treatment separately. For treatment number T_{50}, sum of the coefficients of polynomials $(-3+1+2)$ is zero. Similarly for T_{60} and T_{70} sum of the coefficients of the polynomials is also zero. Hence, treatments allocated are orthogonal to a linear trend. Thus, the linear trend does not affect any contrast among these treatments. Such designs have been called Trend Free Block (TFB) designs (Bradley and Yeh 1980). These designs are constructed in such a manner that treatment effects and trend effects are orthogonal. A necessary condition for a design to be linear trend free was derived by Yeh and Bradley (1983). Yeh et al. (1985) highlighted concepts and properties of Nearly TFB designs with linear and quadratic trends over plots within blocks. Jacroux (1990, 1993) constructed trend-resistant designs for comparing a set of test treatments with a set of controls. Jacroux et al. (1995, 1997) developed some methods for identifying efficient designs when different blocks may have linear trend effect of different slopes. Atkinson and Donev (1996) developed an algorithm to construct a series of exact optimum designs resistant to linear and quadratic time trend. Majumdar and Martin (2002) gave a method for determining optimal designs for comparing treatments in blocks when units within blocks are linearly ordered, and a polynomial trend influences the response. Carrano et al. (2006) developed an integer programming approach for the construction of trend-free split-plot designs.

Bhowmik et al. (2014) obtained trend free designs which are balanced in the presence of uni-directional and bi-directional neighbour effects from immediate neighbouring units. Sarkar et al. (2017) studied BBPB designs in the presence of systematic trend and developed some classes of trend free bipartite block designs which are balanced for neighbour effects.

This article deals with Trend Resistant Balanced Bipartite Block (TR-BBPB) designs with two disjoint sets of treatments {one set (tests), and others may be controls}. The interest here is to estimate the contrasts between test treatments and control treatments with higher precision. Series of TR-BBPB designs for comparing a treatment from set 1 to treatment from set 2, with more precision, have been developed.

2 Bipartite block designs incorporating trend effect

Consider the following model in block design set-up for v treatments ($v=v_1+v_2$; v_1 treatments in first set and v_2 treatments in second set) and b blocks of size k each incorporating trend component (within-block trend effects are represented by orthogonal polynomials of pth degree, $p<k$):

$$\mathbf{Y} = \mu\mathbf{1} + \boldsymbol{\Delta}'\boldsymbol{\tau} + \mathbf{D}'\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\rho} + \mathbf{e} \tag{1}$$

where $\mathbf{Y}$ is a $n\times 1$ vector of observations, $\mathbf{1}$ is a $n\times 1$ vector of unity, $\boldsymbol{\Delta}'$ is a $n\times(v_1+v_2)$ matrix of observations versus treatments, $\boldsymbol{\tau}$ is a $(v_1+v_2)\times 1$ vector of treatment effects, $\mathbf{D}'$ is a $n\times b$ incidence matrix of observations versus blocks, $\boldsymbol{\beta}$ is a $b\times 1$ vector of block effects, $\mathbf{Z}\ \boldsymbol{\rho}$ represents the trend effects. The matrix $\mathbf{Z}$, of order $n\times p$, is the matrix of coefficients given by $\mathbf{Z} = \mathbf{1}_b \otimes \mathbf{F}$ where $\mathbf{F}$ is a $k\times p$ matrix with columns representing the (normalized) orthogonal polynomials and $\mathbf{e}$ is a $n\times 1$ vector of errors following a normal distribution with zero mean and constant variance. Further, $\mathbf{1}'\mathbf{F} = \mathbf{0}$, $\mathbf{F}'\mathbf{F} = \mathbf{I}_p$ and hence $\mathbf{Z}'\mathbf{Z} = b\mathbf{I}_p$.

Let $\mathbf{N}$ be a $(v_1+v_2)\times b$ incidence matrix, which is partitioned as

$$\boldsymbol{\Delta}\mathbf{D}' = \mathbf{N} = \begin{bmatrix}\mathbf{N}_1\\ \mathbf{N}_2\end{bmatrix},$$

where $\mathbf{N}_1$ is a $v_1\times b$ incidence matrix pertaining to v_1 treatments and $\mathbf{N}_2$ is a $v_2\times b$ incidence matrix pertaining to v_2 treatments.

The model (1) can be written as

$$\mathbf{Y} = \mathbf{X}_1\boldsymbol{\theta}_1 + \mathbf{X}_2\boldsymbol{\theta}_2 + \mathbf{e} \tag{2}$$

where $\mathbf{X}_1 = [\boldsymbol{\Delta}'] = \begin{bmatrix}\boldsymbol{\Delta}'_1 & \boldsymbol{\Delta}'_2\end{bmatrix}$, $\mathbf{X}_2 = \begin{bmatrix}\mathbf{1} & \mathbf{D}' & \mathbf{Z}\end{bmatrix}$, $\boldsymbol{\theta}_1 = \boldsymbol{\tau}$ and $\boldsymbol{\theta}_2 = \begin{bmatrix}\mu & \boldsymbol{\beta}' & \boldsymbol{\rho}'\end{bmatrix}'$.

$\mathbf{X}_1$ is the matrix of effects of interest and $\mathbf{X}_2$ is the matrix of nuisance effects. $\boldsymbol{\theta}_1$ is the vector of parameters of interest and $\boldsymbol{\theta}_2$ is the vector of nuisance parameters.

Therefore,

$$\mathbf{X}'_1\mathbf{X}_1 = \boldsymbol{\Delta}\boldsymbol{\Delta}' = \mathbf{R} = \begin{bmatrix}\boldsymbol{\Delta}_1\boldsymbol{\Delta}'_1 & \boldsymbol{\Delta}_1\boldsymbol{\Delta}'_2\\ \boldsymbol{\Delta}_2\boldsymbol{\Delta}'_1 & \boldsymbol{\Delta}_2\boldsymbol{\Delta}'_2\end{bmatrix} = \begin{bmatrix}r_1\mathbf{I}_{v_1} & \mathbf{0}\\ \mathbf{0} & r_2\mathbf{I}_{v_2}\end{bmatrix},$$

$$\mathbf{X}'_1\mathbf{X}_2 = \begin{bmatrix}\boldsymbol{\Delta}_1\mathbf{1} & \boldsymbol{\Delta}_1\mathbf{D}' & \boldsymbol{\Delta}_1\mathbf{Z}\\ \boldsymbol{\Delta}_2\mathbf{1} & \boldsymbol{\Delta}_2\mathbf{D}' & \boldsymbol{\Delta}_2\mathbf{Z}\end{bmatrix} = \begin{bmatrix}r_1\mathbf{1}_{v_1} & \mathbf{N}_1 & \boldsymbol{\Delta}_1\mathbf{Z}\\ r_2\mathbf{1}_{v_2} & \mathbf{N}_2 & \boldsymbol{\Delta}_2\mathbf{Z}\end{bmatrix},$$

$$\text{and } \mathbf{X}'_2\mathbf{X}_2 = \begin{bmatrix}\mathbf{1}'\mathbf{1} & \mathbf{1}'\mathbf{D}' & \mathbf{1}'\mathbf{Z}\\ \mathbf{D}\mathbf{1} & \mathbf{D}\mathbf{D}' & \mathbf{D}\mathbf{Z}\\ \mathbf{Z}'\mathbf{1} & \mathbf{Z}'\mathbf{D}' & \mathbf{Z}'\mathbf{Z}\end{bmatrix} = \begin{bmatrix}n & k\mathbf{1}' & \mathbf{0}\\ k\mathbf{1} & k\mathbf{I}_b & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & b\mathbf{I}_p\end{bmatrix}$$

Using the procedure of matrix inversion (Searle 1971), the joint information matrix (the matrix that provides information on the estimability of various parameters of interest) for estimating different treatment effects from both the sets is obtained as:

$$\mathbf{C}=\mathbf{X}'_1\mathbf{X}_1-\mathbf{X}'_1\mathbf{X}_2(\mathbf{X}'_2\mathbf{X}_2)^{-}\mathbf{X}'_2\mathbf{X}_1$$

$$\mathbf{C}=\begin{bmatrix} r_1\mathbf{I}_{v_1}-\frac{1}{k}\mathbf{N}_1\mathbf{N}'_1-\frac{1}{b}\mathbf{\Delta}_1\mathbf{Z}\mathbf{Z}'\mathbf{\Delta}'_1 & -\frac{1}{k}\mathbf{N}_1\mathbf{N}'_2-\frac{1}{b}\mathbf{\Delta}_1\mathbf{Z}\mathbf{Z}'\mathbf{\Delta}'_2 \\ -\frac{1}{k}\mathbf{N}_2\mathbf{N}'_1-\frac{1}{b}\mathbf{\Delta}_2\mathbf{Z}\mathbf{Z}'\mathbf{\Delta}'_1 & r_2\mathbf{I}_{v_2}-\frac{1}{k}\mathbf{N}_2\mathbf{N}'_2-\frac{1}{b}\mathbf{\Delta}_2\mathbf{Z}\mathbf{Z}'\mathbf{\Delta}'_2 \end{bmatrix}$$

r_1 and r_2 are the replications of the first and second set of treatments, respectively.

The $(v_1+v_2)\times(v_1+v_2)$ matrix $\mathbf{C}$ is symmetric, non-negative definite with zero row and column sums and Rank($\mathbf{C}$) $\leq$ (v_1+v_2)-1.

Definition 1 A block design is said to be linear trend free if the adjusted treatment sum of squares of block model with linear trend component is the same as the adjusted treatment sum of squares of block model without linear trend component.

Definition 2 A bipartite block design is said to be balanced with respect to set 1 versus set 2 if each treatment from a set appears together with every other treatment of the same set constant number of times (say, λ^*_{ii}, i=1,2) and each treatment from a set appears together with every other treatment of a different set a constant number of times (say, λ^*_{12}).

Definition 3 A BBPB design is said to be Trend Resistant Balanced Bipartite Block (TR-BBPB) design if adjusted treatment sum of squares of block model with trend is same as adjusted treatment sum of squares of block model without trend.

Definition 4 A bipartite block design is said to be Group divisible partially balanced bipartite block design if adjusted treatment sum of squares of block model with trend is same as adjusted treatment sum of squares of block model without trend and treatments following a GD association scheme.

Theorem 1 *A block design with two disjoint sets of treatments incorporating trend component is said to be trend free iff* $\mathbf{\Delta}_1\mathbf{Z}=\mathbf{0}$ *and* $\mathbf{\Delta}_2\mathbf{Z}=\mathbf{0}$, *where the symbols have their usual meaning as defined earlier.*

Proof As defined in (2), $\mathbf{X}_2=[\mathbf{1}\quad \mathbf{D}'\quad \mathbf{Z}]$. Let $\mathbf{X}_3=[\mathbf{1}\quad \mathbf{D}']$.

Define,

$$\mathbf{A}_u=\mathbf{I}_n-\mathbf{X}_u(\mathbf{X}'_u\mathbf{X}_u)^{-}\mathbf{X}'_u,\quad (u=2,3)$$

$$\mathbf{Q}_{u\tau}=\begin{bmatrix}\mathbf{\Delta}_1\\ \mathbf{\Delta}_2\end{bmatrix}\mathbf{A}_u\begin{bmatrix}\mathbf{\Delta}'_1 & \mathbf{\Delta}'_2\end{bmatrix}=\begin{bmatrix}\mathbf{\Delta}_1\mathbf{A}_u\mathbf{\Delta}'_1 & \mathbf{\Delta}_1\mathbf{A}_u\mathbf{\Delta}'_2\\ \mathbf{\Delta}_2\mathbf{A}_u\mathbf{\Delta}'_1 & \mathbf{\Delta}_2\mathbf{A}_u\mathbf{\Delta}'_2\end{bmatrix}$$

Thus, $\mathbf{X}'_2\mathbf{X}_2=\begin{bmatrix} n & k\mathbf{1}' & \mathbf{0}\\ k\mathbf{1} & k\mathbf{I}_b & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & b\mathbf{I}_p\end{bmatrix}$ and $\mathbf{X}'_3\mathbf{X}_3=\begin{bmatrix} n & k\mathbf{1}'\\ k\mathbf{1} & k\mathbf{I}_b\end{bmatrix}$.

A g-inverse of $\mathbf{X'}_2\mathbf{X}_2$ and $\mathbf{X'}_3\mathbf{X}_3$ is given, respectively, by.

$$\left(\mathbf{X}_2'\mathbf{X}_2\right)^{-}= \begin{bmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{k}\mathbf{I}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{b}\mathbf{I}_p \end{bmatrix} \text{ and } \mathbf{X}_3'\mathbf{X}_3 = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \frac{1}{k}\mathbf{I}_b \end{bmatrix}$$

$$\text{Hence, } \mathbf{A}_2 = \mathbf{I}_n - \mathbf{X}_2(\mathbf{X'}_2\mathbf{X}_2)^{-}\mathbf{X'}_2 = \mathbf{I}_n - \frac{1}{k}\mathbf{D'D} - \frac{1}{b}\mathbf{ZZ'} \tag{3}$$

$$\text{and } \mathbf{A}_3 = \mathbf{I}_n - \mathbf{X}_3(\mathbf{X'}_3\mathbf{X}_3)^{-}\mathbf{X'}_3 = \mathbf{I}_n - \frac{1}{k}\mathbf{D'D} \tag{4}$$

Now adjusted treatment sum of squares matrix ($\mathbf{T}_t$) of the model (1) with respect to two sets of treatments is

$$\mathbf{T}_t = \begin{bmatrix} \mathbf{Y'A}_2\mathbf{\Delta}_1'\mathbf{Q}_{2\tau}^{-}\mathbf{\Delta}_1\mathbf{A}_2\mathbf{Y} & \mathbf{Y'A}_2\mathbf{\Delta}_1'\mathbf{Q}_{2\tau}^{-}\mathbf{\Delta}_2\mathbf{A}_2\mathbf{Y} \\ \mathbf{Y'A}_2\mathbf{\Delta}_2'\mathbf{Q}_{2\tau}^{-}\mathbf{\Delta}_1\mathbf{A}_2\mathbf{Y} & \mathbf{Y'A}_2\mathbf{\Delta}_2'\mathbf{Q}_{2\tau}^{-}\mathbf{\Delta}_2\mathbf{A}_2\mathbf{Y} \end{bmatrix}$$

Adjusted treatment sum of squares matrix ($\mathbf{T}_0$) under general block design model i.e. $\mathbf{Y} = \mu\,\mathbf{1} + \mathbf{\Delta}'\,\tau + \mathbf{D}'\,\boldsymbol{\beta} + \mathbf{e}$ is

$$\mathbf{T}_0 = \begin{bmatrix} \mathbf{Y'A}_3\mathbf{\Delta'}_1\mathbf{Q}_{3\tau}^{-}\mathbf{\Delta}_1\mathbf{A}_3\mathbf{Y} & \mathbf{Y'A}_3\mathbf{\Delta'}_1\mathbf{Q}_{3\tau}^{-}\mathbf{\Delta}_2\mathbf{A}_3\mathbf{Y} \\ \mathbf{Y'A}_3\mathbf{\Delta'}_2\mathbf{Q}_{3\tau}^{-}\mathbf{\Delta}_1\mathbf{A}_3\mathbf{Y} & \mathbf{Y'A}_3\mathbf{\Delta'}_2\mathbf{Q}_{3\tau}^{-}\mathbf{\Delta}_2\mathbf{A}_3\mathbf{Y} \end{bmatrix}$$

According to the definition, the design will be trend free if $\mathbf{T}_t = \mathbf{T}_0$, i.e.

$$\begin{aligned} &\mathbf{Y'A}_2\mathbf{\Delta'}_1\mathbf{Q}_{2\tau}^{-}\mathbf{\Delta}_1\mathbf{A}_2\mathbf{Y} = \mathbf{Y'A}_3\mathbf{\Delta'}_1\mathbf{Q}_{3\tau}^{-}\mathbf{\Delta}_1\mathbf{A}_3\mathbf{Y} \\ &\mathbf{\Delta}_1\mathbf{A}_2\mathbf{\Delta'}_1 = \mathbf{\Delta}_1\mathbf{A}_3\mathbf{\Delta'}_1 \\ &\mathbf{\Delta}_1(\mathbf{A}_3 - \mathbf{A}_2)\mathbf{\Delta'}_1 = \mathbf{0} \end{aligned} \tag{5}$$

$$\begin{aligned} &\mathbf{Y'A}_2\mathbf{\Delta'}_1\mathbf{Q}_{2\tau}^{-}\mathbf{\Delta}_2\mathbf{A}_2\mathbf{Y} = \mathbf{Y'A}_3\mathbf{\Delta'}_1\mathbf{Q}_{3\tau}^{-}\mathbf{\Delta}_2\mathbf{A}_3\mathbf{Y} \\ &\mathbf{\Delta}_1\mathbf{A}_2\mathbf{\Delta'}_1 = \mathbf{\Delta}_1\mathbf{A}_3\mathbf{\Delta'}_2 \\ &\mathbf{\Delta}_1(\mathbf{A}_3 - \mathbf{A}_2)\mathbf{\Delta'}_2 = \mathbf{0} \end{aligned} \tag{6}$$

$$\begin{aligned} &\mathbf{Y'A}_2\mathbf{\Delta'}_2\mathbf{Q}_{2\tau}^{-}\mathbf{\Delta}_1\mathbf{A}_2\mathbf{Y} = \mathbf{Y'A}_3\mathbf{\Delta'}_2\mathbf{Q}_{3\tau}^{-}\mathbf{\Delta}_1\mathbf{A}_3\mathbf{Y} \\ &\mathbf{\Delta}_2\mathbf{A}_2\mathbf{\Delta'}_1 = \mathbf{\Delta}_2\mathbf{A}_3\mathbf{\Delta'}_1 \\ &\mathbf{\Delta}_2(\mathbf{A}_3 - \mathbf{A}_2)\mathbf{\Delta'}_1 = \mathbf{0} \end{aligned} \tag{7}$$

And

$$\begin{aligned} &\mathbf{Y'A}_2\mathbf{\Delta'}_2\mathbf{Q}_{2\tau}^{-}\mathbf{\Delta}_2\mathbf{A}_2\mathbf{Y} = \mathbf{Y'A}_3\mathbf{\Delta'}_2\mathbf{Q}_{3\tau}^{-}\mathbf{\Delta}_2\mathbf{A}_3\mathbf{Y} \\ &\mathbf{\Delta}_2\mathbf{A}_2\mathbf{\Delta'}_2 = \mathbf{\Delta}_2\mathbf{A}_3\mathbf{\Delta'}_2 \\ &\mathbf{\Delta}_2(\mathbf{A}_3 - \mathbf{A}_2)\mathbf{\Delta'}_2 = \mathbf{0} \end{aligned} \tag{8}$$

Substituting the value of $\mathbf{A}_2$ and $\mathbf{A}_3$ into Eqs. (5–8) and solving the corresponding equation we get

$$\begin{bmatrix} \boldsymbol{\Delta}_1\mathbf{ZZ}'\boldsymbol{\Delta}'_1 & \boldsymbol{\Delta}_1\mathbf{ZZ}'\boldsymbol{\Delta}'_2 \\ \boldsymbol{\Delta}_2\mathbf{ZZ}'\boldsymbol{\Delta}'_1 & \boldsymbol{\Delta}_2\mathbf{ZZ}'\boldsymbol{\Delta}'_2 \end{bmatrix} = \mathbf{0} \tag{9}$$

From Eqs. (9), it is seen that for the design to be trend free, the condition $\boldsymbol{\Delta}_1\mathbf{Z} = \mathbf{0}$ and $\boldsymbol{\Delta}_2\mathbf{Z} = \mathbf{0}$ must satisfy.

To prove the sufficiency, we assume that the condition given in the above theorem is true i.e., $\boldsymbol{\Delta}_1\mathbf{Z} = \mathbf{0}$ and $\boldsymbol{\Delta}_2\mathbf{Z} = \mathbf{0}$. Pre-multiplying and post-multiplying both sides of Eqs. (3) and (4) by Δ and Δ' respectively, we get:

$$\begin{aligned}\mathbf{Q}_{2\tau} &= \begin{bmatrix} \boldsymbol{\Delta}_1 \\ \boldsymbol{\Delta}_2 \end{bmatrix} \left[\mathbf{I}_n - \frac{1}{k}\mathbf{D}'\mathbf{D} - \frac{1}{b}\mathbf{ZZ}'\right] \begin{bmatrix} \boldsymbol{\Delta}'_1 & \boldsymbol{\Delta}'_2 \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{\Delta}_1\left(\mathbf{I}_n - \frac{1}{k}\mathbf{D}'\mathbf{D} - \frac{1}{b}\mathbf{ZZ}'\right)\boldsymbol{\Delta}'_1 & \boldsymbol{\Delta}_1\left(\mathbf{I}_n - \frac{1}{k}\mathbf{D}'\mathbf{D} - \frac{1}{b}\mathbf{ZZ}'\right)\boldsymbol{\Delta}'_2 \\ \boldsymbol{\Delta}_2\left(\mathbf{I}_n - \frac{1}{k}\mathbf{D}'\mathbf{D} - \frac{1}{b}\mathbf{ZZ}'\right)\boldsymbol{\Delta}'_1 & \boldsymbol{\Delta}_2\left(\mathbf{I}_n - \frac{1}{k}\mathbf{D}'\mathbf{D} - \frac{1}{b}\mathbf{ZZ}'\right)\boldsymbol{\Delta}'_2 \end{bmatrix}\end{aligned}$$

Substitution of $\boldsymbol{\Delta}_1\mathbf{Z} = \mathbf{0}$ and $\boldsymbol{\Delta}_2\mathbf{Z} = \mathbf{0}$ in $\mathbf{Q}_{2\tau}$ results

$$\mathbf{Q}_{2\tau} = \begin{bmatrix} \boldsymbol{\Delta}_1\left(\mathbf{I}_n - \frac{1}{k}\mathbf{D}'\mathbf{D}\right)\boldsymbol{\Delta}'_1 & \boldsymbol{\Delta}_1\left(\mathbf{I}_n - \frac{1}{k}\mathbf{D}'\mathbf{D}\right)\boldsymbol{\Delta}'_2 \\ \boldsymbol{\Delta}_2\left(\mathbf{I}_n - \frac{1}{k}\mathbf{D}'\mathbf{D}\right)\boldsymbol{\Delta}'_1 & \boldsymbol{\Delta}_2\left(\mathbf{I}_n - \frac{1}{k}\mathbf{D}'\mathbf{D}\right)\boldsymbol{\Delta}'_2 \end{bmatrix}$$

and

$$\begin{aligned}\mathbf{Q}_{3\tau} &= \begin{bmatrix} \boldsymbol{\Delta}_1 \\ \boldsymbol{\Delta}_2 \end{bmatrix} \left[\mathbf{I}_n - \frac{1}{k}\mathbf{D}'\mathbf{D}\right] \begin{bmatrix} \boldsymbol{\Delta}'_1 & \boldsymbol{\Delta}'_2 \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{\Delta}_1\left(\mathbf{I}_n - \frac{1}{k}\mathbf{D}'\mathbf{D}\right)\boldsymbol{\Delta}'_1 & \boldsymbol{\Delta}_1\left(\mathbf{I}_n - \frac{1}{k}\mathbf{D}'\mathbf{D}\right)\boldsymbol{\Delta}'_2 \\ \boldsymbol{\Delta}_2\left(\mathbf{I}_n - \frac{1}{k}\mathbf{D}'\mathbf{D}\right)\boldsymbol{\Delta}'_1 & \boldsymbol{\Delta}_2\left(\mathbf{I}_n - \frac{1}{k}\mathbf{D}'\mathbf{D}\right)\boldsymbol{\Delta}'_2 \end{bmatrix}\end{aligned}$$

As, $\mathbf{Q}_{2\tau}$=$\mathbf{Q}_{3\tau}$, thus it is clear that $\mathbf{T}_t$=$\mathbf{T}_0$. Hence the condition given in the above theorem is both necessary and sufficient.

3 Construction of trend resistant balanced bipartite block (TR-BBPB) Designs

Method 3.1 Let $v_1 = mn, b_1, r_1, k_1, \lambda_{11}\,\text{and}\,\lambda_{12} = \lambda_{11} + u\,(u > 0)$ be the parameters of a group divisible Partially Balanced Incomplete Block (PBIB) design (Dey 2010), where the first associates appear together λ_{11} times and second associates λ_{12} times. Another design is obtained from (m, n) group divisible association scheme by treating m blocks each of size n with parameters $v_1 = mn, b_2 = m, r_2 = 1, k_2 = n, \lambda_{21} = 1,$ and $\lambda_{22} = 0$, where in this design, the first associates appear together λ_{21} times and second associates λ_{22} times. We consider $k_1 - k_2 = v_2(\neq 0)$. Augment v_2

treatments to all the blocks of the design obtained from the association scheme. Obtain b_2 more blocks by applying the fold-over procedure to all blocks of the association scheme i.e. appending the vertical mirror image of the blocks below the original set of blocks. Considering all the b_1+2b_2 blocks together would result in a TR-BBPB design with parameters $v_1, v_2, b = b_1 + 2b_2, \mathbf{r}' = \left[(r_1+2)\mathbf{1}'_{v_1} \quad 2b_2\mathbf{1}'_{v_2} \right], k = k_1, \lambda^*_{11} = \lambda_{12}, \lambda^*_{12} = 2$ and $\lambda^*_{22} = 2b_2$.

The general form of the information matrix for these designs is obtained as

$$\mathbf{C} = \frac{1}{k}\begin{bmatrix} \{(r_1+2)(k-1)+\lambda_{12}\}\mathbf{I}_{v_1} - \lambda_{12}\mathbf{1}_{v_1}\mathbf{1}'_{v_1} & -2\mathbf{1}_{v_1}\mathbf{1}'_{v_2} \\ -2\mathbf{1}_{v_2}\mathbf{1}'_{v_1} & 2b_2k\mathbf{I}_{v_2} - 2b_2\mathbf{1}_{v_2}\mathbf{1}'_{v_2} \end{bmatrix}$$

Example 1 Consider semi-regular group divisible design (SR-36, Clatworthy 1973) with parameters $v_1=8$, $b_1=8$, $r_1=4$, $k_1=4$, $\lambda_{11}=0$ and $\lambda_{12}=2$.

1	2	3	4
5	6	7	8
2	7	8	1
6	3	4	5
3	8	1	6
7	4	5	2
4	1	6	7
8	5	2	3

The design obtained by taking the (4, 2) group divisible association scheme with parameters $v_1 = 8,\ b_2 = 4,\ r_2 = 1,\ k_2 = 2,\ \lambda_{21} = 1$ and $\lambda_{22} = 0$ is as follows:

1,	5
2,	6
3,	7
4,	8

Augmenting $k_1-k_2=v_2=2$ new treatments in all the blocks of the association scheme, taking its fold and combining all the blocks results in the following TR-BBPB design with $v_1 = 8, v_2 = 2, b = 16, \mathbf{r}' = \left[6\mathbf{1}'_{14} \quad 8\mathbf{1}'_2 \right],\ k = 4, \lambda^*_{11} = 2$, $\lambda^*_{12} = 2, \lambda^*_{22} = 8$:

−3	−1	1	3
1	2	3	4
5	6	7	8
2	7	8	1
6	3	4	5
3	8	1	6
7	4	5	2
4	1	6	7
8	5	2	3
1	5	9	10
2	6	9	10
3	7	9	10
4	8	9	10
10	9	5	1
10	9	6	2
10	9	7	3
10	9	8	4

Orthogonal trend component of degree one without normalization (Fisher and Yates 1963) is given in the upper row and

$$\mathbf{F} = \begin{bmatrix} \frac{-3}{\sqrt{20}} & \frac{-1}{\sqrt{20}} & \frac{1}{\sqrt{20}} & \frac{3}{\sqrt{10}} \end{bmatrix}' = [\,-0.67 \quad -0.22 \quad 0.22 \quad 0.67\,]'$$

The information matrix for treatment effects is obtained as

$$\mathbf{C} = \frac{1}{4}\begin{bmatrix} 20\mathbf{I}_{14} - 2\mathbf{1}_{14}\mathbf{1}'_{14} & -2\mathbf{1}_{14}\mathbf{1}'_{2} \\ -2\mathbf{1}_{2}\mathbf{1}'_{14} & 32\mathbf{I}_{2} - 8\mathbf{1}_{2}\mathbf{1}'_{2} \end{bmatrix}$$

The variance of any estimated elementary contrast among the treatments belonging to first set is $V_{11}=0.40\sigma^2$ and the variance of any estimated elementary contrast between the treatments belonging to the first and second set is $V_{12}=0.362\sigma^2$.

Remark 1 For m=n, the design obtained is a trend resistant partially balanced bipartite block (TR-PBBPB) design concerning the first v_1 set of treatments following a GD association scheme. The general form of the information matrix for these designs is obtained as.

$$\mathbf{C}=\frac{1}{k}\begin{bmatrix}\{(r_1+2)(k-1)+\lambda_{12}\}\mathbf{I}_m-\lambda_{12}\mathbf{1}_m\mathbf{1}'_m & -\mathbf{I}_m-\lambda_{12}\mathbf{1}_m\mathbf{1}'_m & -\mathbf{I}_m-\lambda_{12}\mathbf{1}_m\mathbf{1}'_m & -2\mathbf{1}_m\mathbf{1}'_m \\ -\mathbf{I}_m-\lambda_{12}\mathbf{1}_m\mathbf{1}'_m & \{(r_1+2)(k-1)+\lambda_{12}\}\mathbf{I}_m-\lambda_{12}\mathbf{1}_m\mathbf{1}'_m & -\mathbf{I}_m-\lambda_{12}\mathbf{1}_m\mathbf{1}'_m & -2\mathbf{1}_m\mathbf{1}'_m \\ -\mathbf{I}_m-\lambda_{12}\mathbf{1}_m\mathbf{1}'_m & -\mathbf{I}_m-\lambda_{12}\mathbf{1}_m\mathbf{1}'_m & \{(r_1+2)(k-1)+\lambda_{12}\}\mathbf{I}_m-\lambda_{12}\mathbf{1}_m\mathbf{1}'_m & -2\mathbf{1}_m\mathbf{1}'_m \\ -2\mathbf{1}_m\mathbf{1}'_m & -2\mathbf{1}_m\mathbf{1}'_m & -2\mathbf{1}_m\mathbf{1}'_m & 2b_2k\mathbf{I}_{v_2}-2b_2\mathbf{1}_{v_2}\mathbf{1}'_{v_2}\end{bmatrix}$$

Example 2 Consider a semi-regular group divisible design (SR 65, Clatworthy 1973) with parameters $v_1 = 9$, $b_1 = 9$, $r_1 = 6$, $k_1 = 6$, $\lambda_{11} = 3$ and $\lambda_{12} = 4$. The (3, 3) group divisible association scheme is as follows:

1,	4,	7
2,	5,	8
3,	6,	9

Following design is a TR-PBBPB design with parameters $v_1 = 9$, $v_2 = 3$, $b = 15$, $\mathbf{r}' = \left[8\mathbf{1}'_{14} \quad 6\mathbf{1}'_2\right]$, $k = 6$, $\lambda^*_{11} = 4$, $\lambda^*_{12} = 2$ and $\lambda^*_{22} = 6$ following a GD association scheme:

−5	−3	−1	1	3	5
1	2	3	5	6	4
8	1	2	3	9	7
4	9	7	8	5	6
6	7	1	9	2	5
3	4	5	1	8	9
2	6	4	7	3	8
9	8	6	2	4	1
7	5	8	6	1	3
5	3	9	4	7	2
1	4	7	10	11	12
2	5	8	10	11	12
3	6	9	10	11	12
12	11	10	7	4	1
12	11	10	8	5	2
12	11	10	9	6	3

Orthogonal trend component of degree one without normalization (Fisher and Yates 1963) is given in the upper row and

$$\mathbf{F} = \left[\frac{-5}{\sqrt{70}} \quad \frac{-3}{\sqrt{70}} \quad \frac{-1}{\sqrt{70}} \quad \frac{1}{\sqrt{70}} \quad \frac{3}{\sqrt{70}} \quad \frac{5}{\sqrt{70}} \right]'$$
$$= [-0.60 \quad -0.36 \quad -0.12 \quad 0.12 \quad 0.36 \quad 0.60]'$$

The information matrix for this design is derived as

$$\mathbf{C} = \frac{1}{6}\begin{bmatrix} 44\mathbf{I}_3 - 4\mathbf{1}_3\mathbf{1}'_3 & -\mathbf{I}_3 - 4\mathbf{1}_3\mathbf{1}'_3 & -\mathbf{I}_3 - 4\mathbf{1}_3\mathbf{1}'_3 & -2\mathbf{1}_3\mathbf{1}'_3 \\ -\mathbf{I}_3 - 4\mathbf{1}_3\mathbf{1}'_3 & 44\mathbf{I}_3 - 4\mathbf{1}_3\mathbf{1}'_3 & -\mathbf{I}_3 - 4\mathbf{1}_3\mathbf{1}'_3 & -2\mathbf{1}_3\mathbf{1}'_3 \\ -\mathbf{I}_3 - 4\mathbf{1}_3\mathbf{1}'_3 & -\mathbf{I}_3 - 4\mathbf{1}_3\mathbf{1}'_3 & 44\mathbf{I}_3 - 4\mathbf{1}_3\mathbf{1}'_3 & -2\mathbf{1}_3\mathbf{1}'_3 \\ -2\mathbf{1}_3\mathbf{1}'_3 & -2\mathbf{1}_3\mathbf{1}'_3 & -2\mathbf{1}_3\mathbf{1}'_3 & 36\mathbf{I}_3 - 6\mathbf{1}_3\mathbf{1}'_3 \end{bmatrix}$$

Method 3.2 Consider a symmetric Balanced Incomplete Block (BIB) design constructed through the initial block (method of symmetrically repeated differences) with parameters $v_1^* = b_1^*,\ r_1^* = k_1^*$ and λ_1^*. Consider another unreduced symmetric BIB design having the same block size with parameters $v_2^* = b_2^*,\ r_2^* = k_2^*\ \left(= k_1^*\right)$ and λ_2^*. Taking the fold-over of the second design and combining all the blocks would result in a TR-BBPB design with parameters $v_1 = v_1^* - v_2^*,\ v_2 = v_2^*,\ b = b_1^* + 2b_2^*,\ \mathbf{r}' = \left[r_1^*\mathbf{1}'_{v_1} \quad \left(r_1^* + 2r_2^*\right)\mathbf{1}'_{v_2}\right], k = k_1^* = k_2^*, \lambda_{11}^* = \lambda_1^*,\ \lambda_{12}^* = \lambda_1^*$ and $\lambda_{22}^* = \lambda_1^* + 2\lambda_2^*$.

The general information matrix for this class of design is obtained as

$$\mathbf{C} = \frac{1}{k}\begin{bmatrix} v\mathbf{I}_{v_1} - \mathbf{1}_{v_1}\mathbf{1}'_{v_1} & -\mathbf{1}_{v_1}\mathbf{1}'_{v_2} \\ -\mathbf{1}_{v_2}\mathbf{1}'_{v_1} & (kr_1 - v_1)\mathbf{I}_{v_2} - \{r_1 - (k+1)\}\mathbf{1}_{v_2}\mathbf{1}'_{v_2} \end{bmatrix}$$

Example 3 Consider a symmetric BIB design with parameters $v_1^* = b_1^* = 13,\ r_1^* = k_1^* = 4$ and $\lambda_1^* = 1$ and an unreduced BIB design with parameters $v_2^* = b_2^* = 5,\ r_2^* = k_2^* = 4$ and $\lambda_2^* = 3$. Following TR-BBPB design is obtained with $v_1 = 8,\ v_2 = 5,\ b = 23,\ \mathbf{r}' = \left[4\mathbf{1}'_8 \quad 14\mathbf{1}'_5\right],\quad k = k_1^* = k_2^* = 4,\ \lambda_{11}^* = 1,\ \lambda_{12}^* = 1$ and $\lambda_{22}^* = 7$ (Here treatment numbers 1–5 are treatments of the second set):

−3	−1	1	3
1	2	4	10
2	3	5	11
3	4	6	12
4	5	7	13
5	6	8	1
6	7	9	2
7	8	10	3
8	9	11	4
9	10	12	5

−3	−1	1	3
10	11	13	6
11	12	1	7
12	13	2	8
13	1	3	9
1	2	3	4
1	2	3	5
1	2	4	5
1	3	4	5
2	3	4	5
4	3	2	1
5	3	2	1
5	4	2	1
5	4	3	1
5	4	3	2

Orthogonal trend component of degree one without normalization [Fisher and Yates (1963)] is given in the upper row and

$$\mathbf{F} = \begin{bmatrix} \frac{-3}{\sqrt{20}} & \frac{-1}{\sqrt{20}} & \frac{1}{\sqrt{20}} & \frac{3}{\sqrt{10}} \end{bmatrix}' = [-0.67 \quad -0.22 \quad 0.22 \quad 0.67]'$$

The information matrix for two sets of treatment effects is obtained as

$$\mathbf{C} = \frac{1}{4}\begin{bmatrix} 13\mathbf{I}_8 - \mathbf{1}_8\mathbf{1}'_8 & -\mathbf{1}_8\mathbf{1}'_5 \\ -\mathbf{1}_5\mathbf{1}'_8 & 43\mathbf{I}_5 - 7\mathbf{1}_5\mathbf{1}'_5 \end{bmatrix}$$

The variance of any estimated elementary contrast among the treatments belonging to the first set is $V_{11}=0.615\sigma^2$ and variance of any estimated elementary contrast between the treatments belonging to the first and second set is $V_{12}=0.444\sigma^2$.

Method 3.3 Let D_1 be any symmetric BIB design with parameters $v_1^* = b_1^*,\ r_1^* = k_1^*$ and λ_1^* and D_2 be an unreduced symmetric BIB design with parameters $v_2^* = b_2^*,\ r_2^* = k_2^*\ (= k_1^*)$ and λ_2^*. From each block of design D_2, develop $(k_2^* - 1)$ blocks by rotating the treatments clockwise, resulting in $k_2^* b_2^*$ blocks. Juxtapose design D_1, and the new form of D_2, and the resultant design is a TR-BBPB design with parameters $v_1 = v_1^* - v_2^*,\ v_2 = v_2^*,\ b = b_1^* + k_2^* b_2^*,\ \mathbf{r}' = [r_1^* \mathbf{1}'_{v_2} \quad (r_1^* + k_2^* r_2^*)\mathbf{1}'_{v_1}],\ k = k_1^* = k_2^*,\ \lambda_{11}^* = \lambda_1^*,\ \lambda_{12}^* = \lambda_1^*$ and $\lambda_{22}^* = \lambda_1^* + (k_2^* - 1)\lambda_2^*$.

The general information matrix for this design is obtained as

$$\mathbf{C} = \frac{1}{k}\begin{bmatrix} v\lambda_1^*\mathbf{I}_{v_1} - \lambda_1^*\mathbf{1}_{v_1}\mathbf{1}'_{v_1} & -\lambda_1^*\mathbf{1}_{v_1}\mathbf{1}'_{v_2} \\ -\lambda_1^*\mathbf{1}_{v_2}\mathbf{1}'_{v_1} & \{r_1(k-1) + v\lambda_1^*\}\mathbf{I}_{v_2} - v\lambda_1^*\mathbf{1}_{v_2}\mathbf{1}'_{v_2} \end{bmatrix}$$

where $v = v_1^* + v_2^*$

Example 4 Let D_1 be asymmetric BIB design with parameters $v_1^* = b_1^* = 7$, $r_1^* = k_1^* = 3$ and $\lambda_1^* = 1$ and D_2 be an unreduced BIB design with parameters $v_2^* = b_2^* = 4$, $r_2^* = k_2^* = 3$ and $\lambda_2^* = 2$. Following is a TR-BBPB design with $v_1 = 3$, $v_2 = 4$, $b = 19$, $\mathbf{r}' = \left[3\mathbf{1}'_3 \quad 12\mathbf{1}'_4\right]$, $k = k_1^* = k_2^* = 3$, $\lambda_{11}^* = 5$, $\lambda_{12}^* = 1$ and $\lambda_{22}^* = 1$ (1, 2, 3 and 4 are the treatments of the second set):

−1	0	1
1	2	4
2	3	5
3	4	6
4	5	7
5	6	1
6	7	2
7	1	3
1	2	3
2	3	1
3	1	2
1	2	4
2	4	1
4	1	2
1	3	4
3	4	1
4	1	3
2	3	4
3	4	2
4	2	3

Here,

$$\mathbf{F} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}' = [-0.707 \quad 0 \quad 0.707]'$$

The incidence matrix for this design is given as

$$\mathbf{C} = \frac{1}{3}\begin{bmatrix} 7\mathbf{I}_3 - \mathbf{1}_3\mathbf{1}'_3 & -\mathbf{1}_3\mathbf{1}'_4 \\ -\mathbf{1}_4\mathbf{1}'_3 & 31\mathbf{I}_4 - 7\mathbf{1}_4\mathbf{1}'_4 \end{bmatrix}$$

The variance of any estimated elementary contrast among the treatments belonging to the first set is $V_{11}=0.608\sigma^2$. The variance of any estimated elementary contrast between the treatments belonging to the first and second set is $V_{12}=0.193\sigma^2$.

Method 3.4 Consider a BIB design with parameters $v^*=sm+1$ (prime or prime power), $b^*=sv^*$, $r^*=sm$, $k^*=m$ and $\lambda^*=m$ obtained by developing following initial block(s) modulo v:

$$x^w,\ x^{w+s},\ x^{w+s},\ldots x^{w+(m-1)s} \text{ for } w = 0,1,\ldots,s-1$$

where x is the primitive element of GF (v). Substitute the last u set of treatments of the design with the last treatment of the second set, second last u set of treatment with second last treatment of the second set, likewise v^*-3 number of treatments can be replaced by p number of treatment of second set and the resulting design is a TR-BBPB design with parameters $v_1 = (v^* - pu), v_2 = p,\ b = sv^*,\ r_1 = sm,\ r_2 = usm,\ k = m,\ \lambda_{11}^* = \lambda^*,\ \lambda_{12}^* = 2\lambda^*$ and $\lambda_{22}^* = 4\lambda^*$.

The joint information matrix for this design is given as

$$\mathbf{C} = \frac{(k-1)}{k}\begin{bmatrix} \left(v^*\mathbf{I}_{v_1} - \mathbf{1}_{v_1}\mathbf{1}'_{v_1}\right) & -u\mathbf{1}_{v_1}\mathbf{1}'_{v_2} \\ -u\mathbf{1}_{v_2}\mathbf{1}'_{v_1} & u\left(v^*\mathbf{I}_{v_2} - u\mathbf{1}_{v_2}\mathbf{1}'_{v_2}\right) \end{bmatrix}$$

The variance of any estimated elementary contrast among the treatments belonging to the first set is $V_{11}=\frac{uk}{v^*(k-1)}\sigma^2$ and variance of any estimated elementary contrast between the treatments belonging to first and second set is $V_{12}=\frac{k(u+1)}{uv^*(k-1)}\sigma^2$.

Example 5 Let m=6, s=1, then the following initial block modulo 7 is obtained:

1	3	2	6	4	5

Developing this initial block, the following design with parameters v*=7, b*=7, r*=6, k*=6 and l*=5 is obtained:

1	3	2	6	4	5
2	4	3	0	5	6
3	5	4	1	6	0
4	6	5	2	0	1
5	0	6	3	1	2
6	1	0	4	2	3
0	2	1	5	3	4

By substituting (4, 5) by 4 and (0, 6) by 5, the following TR-BBPB design is obtained with parameters v_1=3, v_2=2, b=7, r_1=6, r_2=12, k=6 $\lambda^*_{11} = 5,\ \lambda^*_{12} = 10,$ and $\lambda^*_{22} = 20$:

$$\mathbf{F} = \begin{bmatrix} \frac{-5}{\sqrt{70}} & \frac{-3}{\sqrt{70}} & \frac{-1}{\sqrt{70}} & \frac{1}{\sqrt{70}} & \frac{3}{\sqrt{70}} & \frac{5}{\sqrt{70}} \end{bmatrix}'$$
$$= [-0.60 \quad -0.36 \quad -0.12 \quad 0.12 \quad 0.36 \quad 0.60]'$$

−5	−3	−1	1	3	5
1	3	2	5	4	4
2	4	3	5	4	5
3	4	4	1	5	5
4	5	4	2	5	1
4	5	5	3	1	2
5	1	5	4	2	3
5	2	1	4	3	4

The information matrix for the above design is

$$\mathbf{C} = \frac{5}{6}\begin{bmatrix} 7\mathbf{I}_3 - \mathbf{1}_3\mathbf{1}_3 & -2\mathbf{1}_3\mathbf{1}'_2 \\ -2\mathbf{1}_2\mathbf{1}'_3 & 14I_2 - 4\mathbf{1}_2\mathbf{1}'_2 \end{bmatrix}$$

The variance of any estimated elementary contrast among the treatments belonging to the first set is V_{11}=0.343σ^2 and the variance of any estimated elementary contrast between the treatments belonging to the first and second set is V_{12}=0.257σ^2.

Method 3.5 Consider a BIB design with parameters v^*, b^*, r^*, k^* and λ^*. In each block of this design, augment v_2 number of treatments. Juxtapose the design with its fold-over form. A TR-BBPB design is obtained with parameters $v_1 = v^*,\ v_2,\ b = 2b^*,\ \mathbf{r}' = [2\,r^*\mathbf{1}'_{v_1} \quad 2b^*\mathbf{1}'_{v_2}],\ k = k^* + v_2,\ \lambda^*_{11} = 2\lambda^*,\ \lambda^*_{12} = 2r^*$ and $\lambda^*_{22} = 2b^*$.

The general information matrix of this design so obtained is

$$\mathbf{C} = \frac{1}{k}\begin{bmatrix} \{2r^*(k-1) + 2\lambda^*\}\mathbf{I}_{v_1} - 2\lambda^*\mathbf{1}_{v_1}\mathbf{1}'_{v_1} & -2r^*\mathbf{1}_{v_1}\mathbf{1}'_{v_2} \\ -2r^*\mathbf{1}_{v_2}\mathbf{1}'_{v_1} & 2b^*k\mathbf{I}_{v_2} - 2b^*\mathbf{1}_{v_2}\mathbf{1}'_{v_2} \end{bmatrix}$$

Example 6 Let $v^* = 7,\ b^* = 7,\ r^* = 3,\ k^* = 3$ and $\lambda^* = 1$ be the parameters of a BIB design. Augment three treatments (8, 9, and10) in each block of this design and fold-over the whole design. By appending both the form one after another, the following TR-BBPB design is obtained with parameters $v_1 = 7,\ v_2 = 3, b = 14, \mathbf{r}' = \left[6\mathbf{1}'_9 \quad 14\mathbf{1}'_2 \right]$, k=6, $\lambda^*_{11} = 2,\ \lambda^*_{12} = 6$ and $\lambda^*_{22} = 14$:

−5	−3	−1	1	3	5
1	2	4	8	9	10
2	3	5	8	9	10
3	4	6	8	9	10
4	5	7	8	9	10
5	6	1	8	9	10
6	7	2	8	9	10
7	1	3	8	9	10
10	9	8	4	2	1
10	9	8	5	3	2
10	9	8	6	4	3
10	9	8	7	5	4
10	9	8	1	6	5
10	9	8	2	7	6
10	9	8	3	1	7

The information matrix of this design is

$$\mathbf{C} = \frac{1}{6}\begin{bmatrix} 32\mathbf{I}_7 - 2\mathbf{1}_7\mathbf{1}'_7 & -6\mathbf{1}_7\mathbf{1}'_3 \\ -6\mathbf{1}_3\mathbf{1}'_7 & 84\mathbf{I}_3 - 14\mathbf{1}_3\mathbf{1}'_3 \end{bmatrix}$$

The variance of any estimated elementary contrast among the treatments belonging to the first set is V_{11}=0.375σ^2 and the variance of any estimated elementary contrast between the treatments belonging to the first and second set is V_{12}=0.256σ^2.

SAS code has been developed for obtaining the information matrix and also the variances of the TR-BBPB and is given in the Appendix. A list of TR-BBPB designs for v_1<50 and v_2<10, along with the two types of variances V_{11} and $V_{12,}$ is given in Table 1.

4 Illustration

We used the design from Example 5 as an example, where three newly developed feeds (numbered 1–3) were to be compared to two existing feed types (treatments 4 and 5, for which more replications are possible) and a breed of cow from seven age groups (represented by A, B, C, D, E, F, and G) were available for the trial. Let's state the data is milk yield (in Kg) during a specific time period (say, a week) during lactation. Because milk yield of any breed diminishes as lactation advances beyond

Table 1 List of TR-BBPB Designs for v1 < 50 and v2 < 10

v_1	v_2	b	r_1	r_2	k	λ^*_{11}	λ^*_{12}	λ^*_{22}	V_{11}	V_{12}
3	4	11	3	6	3	1	2	4	0.6857	0.4000
3	4	19	3	12	3	5	1	1	0.6083	0.1935
4	1	12	6	12	3	2	1	1	0.4286	0.2857
4	2	12	6	12	4	2	6	12	0.4000	0.2750
4	2	8	6	8	5	1	2	2	0.3571	0.3006
4	3	8	6	8	6	1	3	6	0.3529	0.2990
5	6	17	5	10	5	2	2	6	0.3557	0.2174
5	6	41	5	30	5	2	2	10	0.2945	0.0704
5	1	10	8	10	5	3	8	10	0.2632	0.2303
5	2	10	8	10	6	2	12	20	0.2609	0.2293
5	3	10	8	10	7	2	8	10	0.2593	0.2287
5	3	20	8	20	5	3	8	20	0.2941	0.1926
5	2	20	12	20	5	1	3	6	0.1852	0.1407
5	3	20	12	20	6	8	12	20	0.1818	0.1394
7	6	43	4	24	4	2	2	4	0.6154	0.3982
7	1	14	6	14	4	2	6	14	0.4000	0.2667
7	2	14	6	14	5	2	6	14	0.3846	0.2601
7	3	14	6	14	6	2	6	14	0.3750	0.2560
7	1	14	8	14	5	1	3	6	0.2778	0.2083
7	2	14	8	14	6	2	20	40	0.2727	0.2062
8	2	16	6	8	4	2	2	8	0.4000	0.3625
8	5	23	4	14	4	1	1	7	0.6150	0.4440
8	5	33	4	20	4	1	1	3	0.6154	0.4131
8	2	28	14	28	6	3	3	12	0.1579	0.1137
8	3	28	14	28	7	3	14	28	0.1556	0.1127
9	2	24	8	24	5	2	8	24	0.2941	0.1863
9	3	15	8	24	6	4	2	6	0.2857	0.1825
9	4	24	8	24	7	2	8	24	0.2800	0.1800
11	2	22	10	22	7	2	30	40	0.2188	0.1540
11	3	22	10	22	8	2	10	22	0.2162	0.1528
12	3	24	7	12	5	3	2	12	0.3333	0.2778
12	5	24	9	12	7	7	7	17	0.2414	0.2356
14	2	42	10	14	4	6	2	4	0.2500	0.2232
15	6	33	5	15	5	9	9	27	0.4762	0.3382
48	9	75	8	24	8	27	27	81	0.2807	0.1948

V_{11}: The variance factor for any estimated elementary contrast among the treatments belonging to the first set. V_{12}: The variance factor for any estimated elementary contrast between the treatments belonging to the first and second set the second or third week, there is evidence of a systematic trend component in this experiment, which may alter the precision of the results. Following is the layout of the experiment along with hypothetical data set::

Age group	Lactation period (in weeks)					
	5 [0.60]	**3** [0.36]	**1** [0.12]	(**−1**) [−0.12]	(**−3**) [−0.36]	(**−5**) [−0.60]
A	134.86 (1)	146.53 (3)	164.07 (2)	88.87 (5)	126.39 (4)	82.49 (4)
B	204.77 (2)	157.05 (4)	149.28 (3)	106.74 (5)	123.60 (4)	131.66 (5)
C	259.38 (3)	151.05 (4)	174.66 (4)	199.78 (1)	127.28 (5)	168.54 (5)
D	142.22 (4)	127.94 (5)	113.25 (4)	123.33 (2)	96.35 (5)	119.76 (1)
E	159.16 (4)	147.86 (5)	143.05 (5)	135.46 (3)	139.79 (1)	110.76 (2)
F	156.19 (5)	170.09 (1)	125.04 (5)	143.58 (4)	144.01 (2)	177.13 (3)
G	192.90 (5)	169.76 (2)	169.60 (1)	147.76 (4)	132.22 (3)	137.51 (4)

Distinct letters (A, B, C, D, E, F, and G) denote a breed of cows belonging to various age groups. The numbers inside parenthesis indicate the treatments (different types of feeds: 3 newly developed ones and 2 existing feed types), and the values are given in each column corresponding to each row represent the milk yield (in Kg) of a cow belonging to a specific age group. The numbers in the first row (in bold phase) denote an orthogonal trend component of degree one of size six that has not been normalised. Trend effects have been measured using these coefficients. The numbers in [] represent the normalised orthogonal trend component of size six, degree one. Therefore based on the model, one can choose $\mathbf{F}$ as

$$\begin{aligned}\mathbf{F} &= \left[\frac{5}{\sqrt{70}} \quad \frac{3}{\sqrt{70}} \quad \frac{1}{\sqrt{70}} \quad \frac{-1}{\sqrt{70}} \quad \frac{-3}{\sqrt{70}} \quad \frac{-5}{\sqrt{70}}\right]' \\ &= [0.60 \quad 0.36 \quad 0.12 \quad -0.12 \quad -0.36 \quad -0.60]'\end{aligned}$$

Since the above design is a trend-free design, the adjusted treatment sum of squares arising from the effects of treatments under the model (1) with trend component is the same as the adjusted treatment sum of squares under the usual model without trend component.

4.1 Analytical procedure with and without systematic trend

When there is evidence of systematic trend components in the experimental material, they should be included in the model for proper model specification because these remote effects may affect the response and thus have a direct impact on the precision of experiments and interpretation of the results. As a result, if these impacts are not taken into account, the results may be incorrect. With respect to design involving systematic trend component of degree u and n number of experimental units, the sources of variation and degrees of freedom in the ANOVA table can be split up with respect to design involving systematic trend component of degree u and n number of experimental units [u will be equal to 1 in case of linear trend as described in the experimental situation].

Initially, the above data set was analyzed by using the usual two-way Analysis of Variance (ANOVA) using '*lm*' function of R with two known sources of variation as feeds and age group without considering the trend information. It has been observed (Table 2) that the treatment effects came out to be non-significant.

The same data set was then evaluated again, taking into account the trend impact and including a linear trend component in the model. Interestingly, the major effect of interest, i.e. the effects of feed, was shown to be significant at the 5% level of significance (Table 3), i.e. when the trend component was taken into account, the effects of feeds became significantly different from each other at 5% level of significance. Further, it is also evident from the ANOVA tables (Tables 2 and 3) that incorporation of trend effect resulted in a huge reduction of residual mean square.

As a result, trend effects from the experimental material can have a considerable impact on the experiment's precision. When evidence of trend effects is found, one must incorporate these effects into the model and interpret the data correctly in order to draw valid conclusions from the experiment. The importance of these types of remote but considerable impacts on experimental precision and interpretation of results is highlighted by the significance of the effects owing to the addition of systematic trend component. The existence or absence of a trend component has little effect on the sum of squares of both sources of variations, according to ANOVA of the above data, even though the interpretation has changed dramatically. This is because the experiment layout is based on trend-free designs with a one-way blocking structure. The sum of squares of all three separate sources of variability remains the same in both the presence and absence of a systematic trend component since the experimenter used a trend-free design.

The standard errors for pair-wise comparison between treatments belonging to two different sets, i.e. say Feed 1 versus Feed 4 and also for comparing the treatments

Table 2 ANOVA without considering trend effects

Source	DF	Sum Sq	Mean Sq	F value	Pr>F
Age-group	6	15,368	2561.38	3.51	0.0009
Feeds	4	5842	1460.58	1.99	0.119
Residuals	31	22,653	730.74		

Table 3 ANOVA incorporating trend effects

Source	DF	Sum Sq	Mean Sq	F value	Pr>F
Age-group	6	15,368	2561.38	6.21	0.0003
Feeds	4	5842	1460.58	3.54	0.0176
Trend	1	10,272	10,272.02	24.89	<.0001
Residuals	30	12,381	412.69		

Table 4 Result of contrast analysis

Contrast	Estimate	SE	DF	t.ratio	*p*.value
Feed 1 versus feed 4	17.8	10.3	30	1.731	0.0937
Feed 1 versus feed 2	−2.99	11.9	30	−0.251	0.8035
Set 1 feeds versus set 2 feeds	141	38.5	30	3.671	0.0009

belonging to same set (within test treatments), i.e. say Feed 1 versus Feed 2 have been obtained and given in Table 4. It is seen that the precision of the pair-wise comparison between Feed 1 versus Feed 4 is more.

The next step in the analysis is to compare the average effect of two categories of feed. Contrast analysis is used for comparing the existing feeds (4 and 5) vs new feeds (1, 2 and 3). So the hypothesis to be tested here is H_0: (Feed1 + Feed2 + Feed3)/3 = (Feed4 + Feed5)/2. The analysis result has been given in Table 4.

It indicates that both groups of feeds (existing versus new ones) resulted in significantly different milk yield. All the codes used for the analysis are developed in R and provided in Appendix 1.

5 Conclusion

TR-BBPB designs are highly beneficial in experimental circumstances where two sets of treatments are to be compared and where there may be indications of a systematic trend other than the source of variability being addressed. Although the effect of trend is minor, it can significantly impact response and should be included in the model for proper model specification. TR-BBPB designs have been developed that are suited for circumstances where plots are located in long, narrow rows and fertility trends may arise. For estimating the differences between two treatments from separate sets, the developed trend resistant designs are more efficient. The TR-BBPB designs will eliminate the effects of trend effects, resulting in a gain in precision in test treatment versus control comparisons.

Appendix

Appendix 1

R codes for the analysis of data generated from linear trend free block design

```
trend_free_design <- read_csv("trend free design.csv") # Importing data file for analysis
#fix(trend_free_design)  #To view the file
attach(trend_free_design)
Age_Group<-factor(Age_Group)
Feeds<-factor(Feeds)

# ANOVA without trend effects
output<-lm(Milk_Yield~Age_Group+Feeds)

library(car)
Anova(output, type="III") # For type III sum of squares



# ANOVA without trend effects
output1<-lm(Milk_Yield~Age_Group+Feeds+Trend)
Anova(output1, type="III")

# Contrast Analysis
library(lsmeans)
lsm<-lsmeans(output1,"Feeds")
#Contrast for comparing trt1 with ctrl1
contrast(lsm, list(TestVSControl=c(1,0,0,-1,0)))
#Contrast for comparing trt1 with trt2
contrast(lsm, list(TestVSTest=c(1,-1,0,0,0)))
# Contrast for comparing the average effect of Set 1 Feeds vs Set 2 Feeds
contrast(lsm, list(Set1VSSet2=c(2,2,2,-3,-3)))
```

Appendix 2

SAS code for obtaining the c-matrix and variance of estimate of elementary treatment contrast for comparing treatments within first set and first set versus second set for TR-BBPB design

```
%let v1=5; /*number of treatments in first set*/
%let v2=2; /*number of treatments in second set*/
PROC IML;
/*enter design blockwise (separate each block by comma)*/
a ={1   2   3   4   6   7   ,
1   2   3   5   6   7   ,
1   2   4   5   6   7   ,
1   3   4   5   6   7   ,
2   3   4   5   6   7   ,
7   6   4   3   2   1   ,
7   6   5   3   2   1   ,
7   6   5   4   2   1   ,
7   6   5   4   3   1   ,
7   6   5   4   3   2
};
N = j(max(a),nrow(a),0);
        do i = 1 to nrow(a);
        do j = 1 to ncol(a);
N [a[i,j],i]= N [a[i,j],i]+1;
end;
end;

*print n;

RT = N[,+];
R = j(max(a),max(a),0);
        do i = 1 to max(a);
R[i,i] = RT[i];
end;
print r;

KT = N[+,];
K = j(nrow(a),nrow(a),0);
        do i = 1 to nrow(a);
K[i,i] = KT[i];
end;

C = R-N*inv(K)* N`;
ginv=ginv(c);

Print c;
*print ginv;

i1=i(&t);
i2=i(&con);
j1=j(&t, 1,1);
j2=j(&con, 1,1);
p1=j2@i1;
p2=-i2@j1;
p=p1||p2;
```

```
n_t=comb(&t,2);
p_t=j(n_t,&t+&con,0);
k=1;
        do i=1 to &t;
        do j=i+1 to &t;
p_t[k,i]=1;
p_t[k,j]=-1;
k=k+1;
end;
end;
*print p_t;

n_con=comb(&con,2);
p_con=j(n_con,&t+&con,0);
k=1;
        do i=&t+1 to &t+&con;
        do j=i+1 to &t+&con;
p_con[k,i]=1;
p_con[k,j]=-1;
k=k+1;
end;
end;

*print p_con;
*print p;
var_tc=p*ginv*p`;
var_t=p_t*ginv*p_t`;
var_con=p_con*ginv*p_con`;
print var_tc;
print var_t;
*print var_con;

quit;
```

Acknowledgements We are grateful to the editor and anonymous reviewers for the constructive comments that have led to considerable improvement in the article. The financial assistance received by the first author from PG School, IARI, New Delhi is duly acknowledged. We also acknowledge the facilities provided by ICAR-IASRI, New Delhi, P.G. School, IARI, New Delhi, Visva-Bharati, Sriniketan, West Bengal and ICAR-CMFRI, Kochi for conducting the research.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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