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ORIGINAL ARTICLE

A test of concordance based on Gini's mean difference

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Abstract A new rank correlation index, which can be used to measure the extent of concordance or discordance between two rankings, is proposed. This index is based on Gini's mean difference computed on the totals ranks corresponding to each unit and it turns out to be a special case of a more general measure of the agreement of m rankings. The proposed index can be used in a test for the independence of two criteria used to rank the units of a sample, against their concordance/discordance. It can then be regarded as a competitor of other classical methods, such as Kendall's tau. The exact distribution of the proposed test-statistic under the null hypothesis of independence is studied and its expectation and variance are determined; moreover, the asymptotic distribution of the test-statistic is derived. Finally, the implementation of the proposed test and its performance are discussed.

Keywords Nonparametric tests · Rank correlation methods · Gini's mean difference · Distributive compensation

1 Introduction

Suppose that m judges are asked to rank n objects or, more generally, that a sample of n units are ranked according to m different criteria. Assuming that

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there are no ties, we will denote by $R_{1j}, R_{2j}, \ldots, R_{nj}$ the sequences of ranks corresponding to the jth judge ($j = 1, \ldots, m$). The well-known *problem of m rankings* consists of measuring the extent of agreement of the m judges' rankings. Of course, the above problem simplifies when $m = 2$; in this case, indeed, two opposite extreme situations can be observed, each corresponding to the complete agreement or disagreement of the two judges. More specifically, when the two judges perfectly agree, $R_{i1} = R_{i2}$ ($i = 1, \ldots, n$), whereas $R_{i1} = n+1-R_{i2}$ ($i = 1, \ldots, n$), if the two judges completely disagree. The measures of agreement or disagreement of the two rankings are commonly known as *rank correlation indexes* (see Kendall and Gibbons 1990). Among them the most famous are Spearman's rho,

$$\rho = \frac{12}{n^3 - n} \sum_{i=1}^{n} \left(R_{i1} - \frac{n+1}{2}\right)\left(R_{i2} - \frac{n+1}{2}\right) = 1 - \frac{6\sum_{i=1}^{n}(R_{i1} - R_{i2})^2}{n^3 - n}, \quad (1)$$

and Kendall's tau,

$$\tau = 1 - \frac{4P}{n^2 - n}, \quad (2)$$

where P denotes the number of discordant pairs of ranks, i.e. the number of pairs (R_{i1}, R_{i2}) and (R_{l1}, R_{l2}) such that $R_{i1} < R_{i2}$ and $R_{l1} > R_{l2}$ or $R_{i1} > R_{i2}$ and $R_{l1} < R_{l2}$ ($i, l = 1, \ldots, n$). Another measure of rank correlation is Gini's cograduation index:

$$G = \frac{2}{g} \sum_{i=1}^{n} \{|R_{i1} + R_{i2} - n - 1| - |R_{i1} - R_{i2}|\}$$

(Gini 1954; see also Cifarelli et al. 1996), where the normalization constant g equals n^2 when n is even and $n^2 - 1$ when n is odd.

When $m > 2$, the judges may all agree about the n objects but they cannot completely disagree. Indeed if, say, the first judge disagrees with the second, the third judge must agree either with the first or with the second, and so on. In spite of this conclusion, the agreement among m judges can be measured on the basis of a suitable mean of the rank correlation indexes computed on the possible $\binom{m}{2}$ couples of judges. If Kendall's tau is used as a measure of rank correlation, one gets

$$\tau_{av} = 1 - \frac{2}{\binom{m}{2}\binom{n}{2}} \sum_{j=1}^{n} \sum_{i=1}^{j-1} r_{ij}(m - r_{ij}),$$

where r_{ij} denotes the number of judges who agree about ordering the ith object higher that the jth object (see Hehrenberg 1952 and Hays 1960). It can be easily shown that, when $m = 2$, the double summation in the above formula reduces to P in (2). If Spearman's rho is used in the average of the rank correlation indexes, the following measure can be defined:

$$\rho_{av} = \frac{24}{m(m-1)(n^3-n)} \sum_{j=1}^{m}\sum_{k=1}^{j-1}\sum_{i=1}^{n}\left(R_{ij} - \frac{n+1}{2}\right)\left(R_{ik} - \frac{n+1}{2}\right).$$

Instead of averaging the rank correlation indexes of every couple of judges, a different logic can be followed to measure their agreement. It is sufficient to note that, if the m judges completely agree on the ranking of the n objects, the total ranks $T_i = \sum_{j=1}^{m} R_{ij}$ $(i = 1, \ldots, n)$ will be all multiples of m (the worst object for every judges will totally score m, the second object will score $2m$, and so on); on the other hand, the more the m judges disagree, the more the total ranks will be close to each other (an object ranked first for a judge may be last for another one, and so on). Hence, if an index of *variability* of the total ranks is computed, it will measure the extent of agreement of the m judges. Kendall proposes to compute the variance of the total ranks, $S = \frac{1}{n}\sum_{i=1}^{n}\left(T_i - \frac{m(n+1)}{2}\right)^2$ and defines the well-known *coefficient of concordance*:

$$W = \frac{12\,S}{m(n^3-n)}.$$

Note that W is obtained as the ratio of S over its maximum value in the case of complete agreement and hence takes values in $[0, 1]$ (see Kendall and Gibbons 1990). An interesting issue is that W turns out to be a linear function of ρ_{av}:

$$\rho_{\mathrm{av}} = \frac{mW-1}{m-1}.$$

Hence a test of concordance of m rankings can be equivalently based on W and ρ_{av}; moreover, W reduces to ρ when $m = 2$.

In this paper we propose to measure the agreement of the m rankings by a different measure of variability computed on the total ranks $T_1, \ldots, T_n$, i.e. *Gini's mean difference* (Gini 1912; see also David 1968; Kotz and Johnson 1982, pp. 436–437). For N observations $x_1, \ldots, x_N$, relating to a quantitative variable X, Gini's mean difference (without repetition) can be defined as

$$\Delta(X) = \frac{1}{N^2-N}\sum_{i=1}^{N}\sum_{l=1}^{N}|x_i - x_l|,$$

that is as the mean of the $N(N-1)$ absolute differences between every couple of different observations. If this measure of variability is applied to the total ranks $T_1, \ldots, T_n$, the following measure of agreement of m rankings is obtained:

$$D^{(m)} = \frac{\Delta(T)}{\max \Delta(T)} = \frac{3\,\Delta(T)}{m(n+1)}, \tag{3}$$

where $\Delta(T) = \frac{1}{n^2-n} \sum_{i=1}^{n} \sum_{l=1}^{n} |T_i - T_l|$. Note that the maximum value max $\Delta(T) = \frac{m(n+1)}{3}$ is attained when the total ranks $T_1, \ldots, T_n$, are all multiple of m; hence, as for W, $D^{(m)}$ ranges in $[0, 1]$. Moreover, note that $D^{(m)}$ cannot be regarded as a linear function of a suitable mean of $\binom{m}{2}$ known rank correlation indexes, differently from W.

The minimum value $D^{(m)} = 0$, corresponding to a minimum agreement among the m judges, is closely related to the concept of *compensation* developed in Zenga (2003). Suppose that k quantitative variates $X_1, \ldots, X_k$ are observed on N units and let $\Delta(X_1), \ldots, \Delta(X_k)$ denote the Gini's mean differences computed on these variates; moreover, let $Y = X_1 + \cdots + X_k$ and $\Delta(Y)$ be the total of the k variates and its mean difference. It can be shown that $\Delta(Y) = \sum_{i=1}^{N} \Delta(X_i)$ if and only if the ranking of the N units does not change when the values of each of the k variate are used as a criterion for sorting (i.e. if the k variate are *cograduated*); conversely, when the rankings according to the k variates differ, $\Delta(Y) < \sum_{i=1}^{N} \Delta(X_i)$. Moreover, the minimum value $\Delta(Y) = 0$ is reached when the sum Y takes the same value for each observed unit, which means that, if some variates assume high values on a specific unit, the remaining variates must show lower values. This last situation is termed in Zenga (2003) as *distributive compensation* among the k variates; in addition, the ratio $1 - \frac{\Delta(Y)}{\sum_{i=1}^{N} \Delta(X_i)}$ can be proposed as a measure of departure of data from this situation. In our settings, the situation of distributive compensation is equivalent to a “lack" of agreement among the m judges. Hence, in a sense, the concept of compensation can be regarded as the opposite of concordance even when $m > 2$, that is when the concept of discordance cannot be applied.

This paper examines the properties of $D^{(m)}$ in the particular case $m = 2$, that is when it can be regarded as a rank correlation index as well as Kendall's tau, Spearman's rho or Gini's cograduation index. Indeed it can be noted that, differently from W, under the condition $m = 2$, $D^{(m)}$ does not produce a known index. In the following, the proposed index will then be simply denoted as

$$D = \frac{3}{2(n^3 - n)} \sum_{i=1}^{n} \sum_{j=1}^{n} \left| R_{i1} + R_{i2} - R_{j1} - R_{j2} \right|. \tag{4}$$

Consider a population described by the bivariate r.v. (X, Y) with joint distribution function $H(x, y)$ (which will be supposed to be continuous from now on) with marginal distribution functions $F(x) = H(x, \infty)$ and $G(y) = H(\infty, y)$. The proposed measure D can be regarded as the natural estimator, based on a bivariate random sample $(X_1, Y_1), \ldots, (X_n, Y_n)$ drawn from (X, Y), of the functional

$$\mathcal{D}(H) = \frac{3}{2} \int_{\Re^4} |F(x_1) + G(y_1) - F(x_2) - G(y_2)| \; \mathrm{d}H(x_1, y_1) \, \mathrm{d}H(x_2, y_2).$$

Hence, the use of D as a rank correlation measure can be further justified by analyzing some properties of the functional $\mathcal{D}(H)$ as a measure of monotone dependence. First of all it can be easily proved that $\mathcal{D}(H^-) = 0$, where $H^-(x, y) = \max\{0, F(x) + G(y) - 1\}$, that is in case of perfect negative dependence; conversely, $\mathcal{D}(H^+) = 1$, where $H^+(x, y) = \min\{F(x), G(y)\}$, that is in case of perfect positive dependence. Moreover, following Tchen (1980), if H_2 is more concordant than H_1 (that is if $H_1(x, y) \le H_2(x, y)$ for all $(x, y) \in \Re^2$), a natural requirement of the functional would be to agree with such an ordering, i.e. to be such that $\mathcal{D}(H_1) \le \mathcal{D}(H_2)$. This fact can be proved by using Tchen (1980, Corollary 3, p. 822). In the notation of the same corollary, let $n = q = 2$ and consider the product measure $\hat{H} = H \times H$. Now write $\mathcal{D}(H) = \int_{\Re^4} |r_1(x_1, x_2) + r_2(y_1, y_2)| \, \mathrm{d}\hat{H}(x_1, y_1, x_2, y_2)$ where $r_1(x_1, x_2) = F(x_1) - F(x_2)$ and $r_2(y_1, y_2) = G(y_1) - G(y_2)$. The results follow by noticing that, for $p = 1, 2$, the number of functions r_1 and r_2 decreasing in their pth coordinate is always an even number and that the function $|r_1 + r_2|$ is super-additive. Hence the functional $\mathcal{D}(H)$ agrees with the so-called ordering of monotone dependence. Some further characteristics of the functional $\mathcal{D}(H)$ will be discussed in the next section.

Rather than as a (sample) measure of concordance, the interest of this paper is mainly in the use of D as a test-statistic for independence (of two judges or two criteria) against concordance/discordance. To implement this test, the distribution of the test-statistic D under the null hypothesis of independence is considered in the next section. The expectation and the variance of the test-statistic are derived; moreover, the asymptotic distribution of D is determined and some conclusions about the quality of its approximation are drawn. Section 3 discusses the implementation of a test of independence based on D and gives indications about the performance of the test, compared to its classical nonparametric competitors. Finally, some conclusions and directions for future research are discussed in Sect. 4.

2 Distribution of D under independence

Every rank correlation measure described in Sect. 1 can be used in a test for the independence of two judges or two criteria of ranking. In particular, values of D close to 0 or to 1 can lead to the rejection of the null hypothesis of independence toward the alternative of discordance or concordance.[1] To determine the critical values of the test, a deeper analysis of the distribution of D under the null hypothesis is hence needed. Under H_0, every possible joint realization of the two sequences $R_{11}, \ldots, R_{n1}$ and $R_{12}, \ldots, R_{n2}$ have the same probability $1/(n!)^2$. The distribution of the test-statistic D can then be computed by enumerating all the $(n!)^2$ points of the sample space and by recording the frequency $f(d)$ of

[1] Being derived as a special case of $D^{(m)}$, D ranges in the interval $[0, 1]$. To build an index ranging in the usual interval $[-1, +1]$ (like ρ, τ and G), it is sufficient to consider the transformation $2D - 1$; of course this transformation is useless when D is a test-statistic.

Table 1 Distribution of D under H_0 when $n = 6$

0.000	**0.143**	**0.229**	**0.257**	**0.286**	**0.314**	**0.343**	**0.371**
0.001	0.007	0.008	0.013	0.006	0.017	0.011	0.022
0.400	**0.429**	**0.457**	**0.486**	**0.514**	**0.543**	**0.571**	**0.600**
0.031	0.018	0.028	0.047	0.032	0.044	0.056	0.053
0.629	**0.657**	**0.686**	**0.714**	**0.743**	**0.771**	**0.800**	**0.829**
0.064	0.063	0.060	0.065	0.058	0.060	0.057	0.051
0.857	**0.886**	**0.914**	**0.943**	**0.971**	**1.000**		
0.044	0.031	0.026	0.019	0.007	0.001		

occurrence of each different value d taken by D; the probability of $D = d$ will then be given by the ratio $f(d)/(n!)^2$.

Note that the value taken by D does not change if the summation in (4) is taken by rearranging the terms so that, say, the first sequence of ranks is in ascending order. Hence the distribution of D can be more easily computed by setting $R_{i1} = i$ $(i = 1, \dots, n)$ and by enumerating all the $n!$ possible realizations of the ranks $R_{12}, \dots, R_{n2}$. In this way, every point of the sample space corresponds to a different permutation of the set $\{1, 2, \dots, n\}$. This procedure is quite simple if n is not large. Table 1 reports the distribution of D under H_0 when $n = 6$.

Table 1 shows that the distribution of D under H_0 is not symmetric. This fact looks strange as the null distribution of other common rank correlation measures, such as τ, ρ and G, is symmetric around zero. One can try to explain such an asymmetry by further analyzing the properties of the above-mentioned functional $\mathcal{D}(H)$. Indeed it can be easily shown that $\mathcal{D}(H^0) = 0.7$, where $H^0(x, y) = F(x)\,G(y)$, i.e. when the marginal components are independent;[2] moreover, differently from other common measures of monotone dependence, $\mathcal{D}(H^0) \neq \mathcal{D}((H^- + H^+)/2)$, that is the functional does not consider symmetrically positive and negative dependence. Following Cifarelli and Regazzini (1990), a measure of monotone dependence should be defined so that it vanishes in case of *indifference*, that is, roughly speaking, for such situations characterized by a compensation of the departure from H^+ and the departure from H^-; special cases of indifference are then independence and $H = (H^- + H^+)/2$ [see Cifarelli and Regazzini (1990) and Cifarelli et al. (1996) for further details]. The measure $\mathcal{D}(H)$ may of course be modified to vanish in case of independence but it cannot take the same value for each situation termed as indifference. This fact happens because D was originally proposed as a measure of the concordance of m rankings, even if this paper is mainly concerned with the case $m = 2$; hence the functional $\mathcal{D}(H)$ mainly measures concordance or equivalently departure from perfect negative dependence. To get a monotone-dependence measure, as suggested in Cifarelli and Regazzini (1990), one could subtract to $\mathcal{D}(H)$ an analogous functional of

[2] Note that 0.7 is also the asymptotic expected value of D under independence, as it will be proved later in this Section.

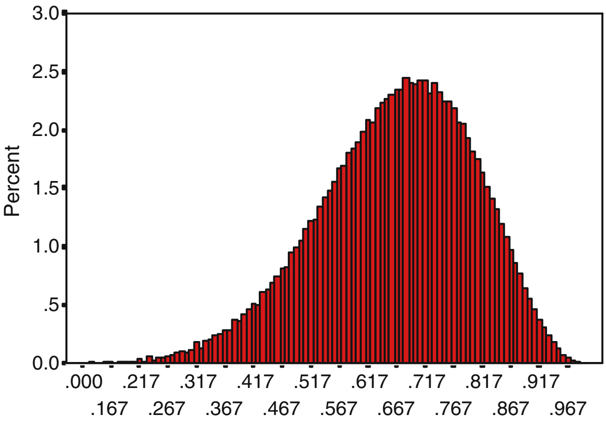

Fig. 1 Graph of the distribution of D under H_0 when $n = 9$

discordance. For instance, if the new functional $\tilde{\mathcal{D}}(H) = \mathcal{D}(H) - \bar{\mathcal{D}}(H)$ is defined (where $\bar{\mathcal{D}}(H) = \frac{3}{2}\int_{\Re^4} |F(x_1) - G(y_1) - F(x_2) + G(y_2)|\ \mathrm{d}H(x_1, y_1)\,\mathrm{d}H(x_2, y_2)$), it can be easily shown that $\tilde{\mathcal{D}}(H^-) = -1$, $\tilde{\mathcal{D}}(H^+) = +1$ and that $\tilde{\mathcal{D}}(H^0) = \tilde{\mathcal{D}}((H^- + H^+)/2) = 0$. This definition would of course imply to modify even the expression of D by subtracting an analogous index based on the counter-ranks of one of the two sequences; such a modification is an interesting hint but, as above explained, is out of the purposes of this paper.

Of course, the discussed asymmetry of $\mathcal{D}(H)$ and of its natural estimator D is not without consequences. First of all, as it will be discussed in Sect. 3, the determination of the critical values of a two-sided test based on D can be troublesome. Moreover, the power function of the same test is likely to be asymmetric, as it will be pointed out later; this fact implies that the test may result powerful just for some kind of alternatives. Fortunately, at least concerning the null distribution of D, the detected asymmetry rapidly vanishes as n increases: Fig. 1 reports the graph of the distribution for $n = 9$ and shows that the null distribution seems to converge to normality (this conjecture will be proved later in this section). Some anomalies connected with the asymmetry of D are hence likely to be overcome when n is large (see for instance the discussion about the power function in Sect. 3.1).

The distribution of D cannot be determined numerically when n is large, as the number of permutations to be enumerated becomes unpracticable. However, some characteristics of the distribution of D can be derived analytically. The next theorem provides the expected value and the variance of D under H_0. The proof of this theorem requires some tedious computations to determine the exact distribution of the summands in the formula (4); for the sake of brevity, by following a suggestion of the anonymous referee, the proof is hence omitted (a whole detailed proof can be found in Borroni (2006) available at the web site www.dimequant.unimib.it).

Theorem 1 *Under the null hypothesis of independence,*

$$\mathrm{E}(D|\mathrm{H}_0) = \frac{7n^2 - 10n + 2}{10n(n-1)} \tag{5}$$

and

$$\begin{aligned} &\mathrm{Var}(D|\mathrm{H}_0) \\ &\quad = \frac{1751n^6 - 3090n^5 + 233n^4 + 1428n^3 - 2173n^2 - 9822n + 4725}{12600n^3(n-2)(n-1)^2(n+1)} && \textit{when } n \textit{ is odd} \\ &\quad = \frac{1751n^6 - 3090n^5 - 5020n^4 + 192n^3 - 5344n^2 - 20448n - 10008}{12600n^2(n-3)(n-1)^2(n+1)^2} && \textit{when } n \textit{ is even.} \end{aligned} \tag{6}$$

The asymptotic expected value and variance of D under H_0 can be easily derived from their exact formulas above. The whole asymptotic distribution can be obtained by following a technique similar to the one used in Cifarelli and Regazzini (1977) for Gini's cograduation index (see also Cifarelli and Regazzini 1974). The following theorem gives such a result:

Theorem 2 *Under the null hypothesis of independence, $\sqrt{n}\,D$ is asymptotically normally distributed with mean $\frac{7}{10}$ and variance $\frac{1751}{12600}$.*

Proof Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a random sample whose elements are iid according to the absolutely continuous bivariate cdf $H(x, y)$. Moreover let $F(x) = H(x, \infty)$ and $G(y) = H(\infty, y)$. Now consider the U-statistic

$$U_n = \binom{n}{3}^{-1} \sum_{(n,3)} \Psi_0(X_{i_1}, Y_{i_1}; X_{i_2}, Y_{i_2}; X_{i_3}, Y_{i_3})$$

where the sum $\sum_{(n,3)}$ is taken over the $\binom{n}{3}$ subsets $1 \leq i_1 < i_2 < i_3 \leq n$ of $\{1, \ldots, n\}$; the three-degree symmetric kernel of U_n is defined as

$$\begin{aligned} &\Psi_0(X_1, Y_1; X_2, Y_2; X_3, Y_3) \\ &\quad = \frac{1}{2}\,[\Psi(X_1, Y_1; X_2, Y_2; X_3, Y_3) + \Psi(X_2, Y_2; X_1, Y_1; X_3, Y_3) \\ &\qquad + \Psi(X_3, Y_3; X_2, Y_2; X_1, Y_1)]\,, \end{aligned}$$

where

$$\begin{aligned} &\Psi(X_1, Y_1; X_2, Y_2; X_3, Y_3) \\ &\quad = \left[2\,S\,(F(X_2) + G(Y_2) - F(X_3) - G(Y_3)) - 1\right] \\ &\qquad \cdot [S(X_2 - X_1) + S(Y_2 - Y_1) - S(X_3 - X_1) - S(Y_3 - Y_1)] \end{aligned}$$

($S(a)$ being 1 if $a \geq 0$ and 0 elsewhere).

It can be first proved that $\sqrt{n}U_n$ and $\sqrt{n}D$ are asymptotically equally distributed. Note that

$$\sum_{(n,3)} \Psi_0(X_{i_1}, Y_{i_1}; X_{i_2}, Y_{i_2}; X_{i_3}, Y_{i_3}) = \frac{1}{4} \sum_{i \neq j \neq l} \Psi(X_l, Y_l; X_i, Y_i; X_j, Y_j);$$

moreover, recall that, for each real couple (a, b), $|a-b| = \left[2\, S(a-b) - 1\right](a-b)$. As n tends to infinity, it can then be written

$$(D-U_n) \simeq \frac{3}{2\,n^2} \sum_{i \neq j} \left\{ \left| \frac{R_{i1}}{n} + \frac{R_{i2}}{n} - \frac{R_{j1}}{n} - \frac{R_{j2}}{n} \right| - \left| F(X_i) + G(Y_i) - F(X_j) - G(Y_j) \right| \right\};$$

it has then to be proved that $\sqrt{n}(D - U_n)$ tends to zero in probability, that is $\lim_{n\to\infty} \Pr\{\sqrt{n}|D - U_n| < \epsilon\} = 1$ for every $\epsilon > 0$. Note that

$$\begin{aligned}
&\sqrt{n}|D - U_n| \\
&\quad \leq \tfrac{3}{2n^{3/2}} \sum_{i \neq j} \left| \left| \tfrac{R_{i1}}{n} + \tfrac{R_{i2}}{n} - \tfrac{R_{j1}}{n} - \tfrac{R_{j2}}{n} \right| - \left| F(X_i) + G(Y_i) - F(X_j) - G(Y_j) \right| \right| \\
&\quad \leq \tfrac{3}{2n^{3/2}} \sum_{i \neq j} \left| \tfrac{R_{i1}}{n} + \tfrac{R_{i2}}{n} - \tfrac{R_{j1}}{n} - \tfrac{R_{j2}}{n} - F(X_i) - G(Y_i) + F(X_j) + G(Y_j) \right| \\
&\quad \leq \tfrac{3}{2n^{3/2}} \sum_{i \neq j} \left\{ \left| \tfrac{R_{i1}}{n} - F(X_i) \right| + \left| \tfrac{R_{i2}}{n} - G(Y_i) \right| + \left| \tfrac{R_{j1}}{n} - F(X_j) \right| + \left| \tfrac{R_{j2}}{n} - G(Y_j) \right| \right\} \\
&\quad \simeq \tfrac{3}{\sqrt{n}} \sum_{i} \left| \tfrac{R_{i1}}{n} - F(X_i) \right| + \tfrac{3}{\sqrt{n}} \sum_{i} \left| \tfrac{R_{i2}}{n} - G(Y_i) \right|
\end{aligned}$$

Hence

$$\begin{aligned}
&\Pr\left\{\sqrt{n}|D - U_n| < \epsilon\right\} \\
&\quad \geq \Pr\left\{ \left(\tfrac{3}{\sqrt{n}} \sum_i \left| \tfrac{R_{i1}}{n} - F(X_i) \right| < \tfrac{\epsilon}{2} \right) \cap \left(\tfrac{3}{\sqrt{n}} \sum_i \left| \tfrac{R_{i2}}{n} - G(Y_i) \right| < \tfrac{\epsilon}{2} \right) \right\} \\
&\quad \geq \Pr\left\{ \tfrac{3}{\sqrt{n}} \sum_i \left| \tfrac{R_{i1}}{n} - F(X_i) \right| < \tfrac{\epsilon}{2} \right\} + \Pr\left\{ \tfrac{3}{\sqrt{n}} \sum_i \left| \tfrac{R_{i2}}{n} - G(Y_i) \right| < \tfrac{\epsilon}{2} \right\} - 1 \\
&\quad \geq 1 - \tfrac{6}{\epsilon\sqrt{n}} \sum_i \mathrm{E}\left\{ \left| \tfrac{R_{i1}}{n} - F(X_i) \right| + \left| \tfrac{R_{i2}}{n} - G(Y_i) \right| \right\}
\end{aligned}$$

The result follows now by noticing that, for $i = 1, \ldots, n$, there always exists $k > 0$ so that $\mathrm{E}\left\{\left|\frac{R_{i1}}{n} - F(X_i)\right|\right\} < \frac{k}{n}$ and $\mathrm{E}\left\{\left|\frac{R_{i2}}{n} - G(Y_i)\right|\right\} < \frac{k}{n}$; hence $\lim_{n\to\infty} \Pr\{\sqrt{n}|D - U_n| < \epsilon\} \geq 1$

To get the asymptotic distribution of U_n, one has then to compute the function $g_1(x_1, y_1) = \mathrm{E}\{\Psi_0(x_1, y_1; X_2, Y_2; X_3, Y_3)\}$; the general theory of U-statistics states that U_n is asymptotically normally distributed with mean θ and variance $9\sigma_1^2$, where $\theta = \mathrm{E}\{g_1(X_1, Y_1)\}$ and $\sigma_1^2 = \mathrm{Var}\{g_1(X_1, Y_1)\}$. Under H_0, that is under the hypothesis $H(x, y) = F(x)\,G(y)$ (for every (x, y)), some calculations show that

$$\begin{aligned} & g_1(x_1, y_1) \\ &= \frac{1}{3}F^4(x_1) - F^2(x_1) - \frac{2}{3}F(x_1) + \frac{1}{3}G^4(y_1) - G^2(y_1) - \frac{2}{3}G(y_1) \\ &\quad +2 + 2F^2(x_1)G(y_1) + 2F(x_1)G^2(y_1) \qquad\qquad (\text{if } 0 \le F(x_1) + G(y_1) \le 1) \\ &= \frac{1}{3}F^4(x_1) - \frac{4}{3}F^3(x_1) + 3F^2(x_1) - \frac{14}{3}F(x_1) + \frac{1}{3}G^4(y_1) - \frac{4}{3}G^3(y_1) \\ &\quad +3G^2(y_1) - \frac{14}{3}G(y_1) + \frac{10}{3} + 8F(x_1)G(y_1) - 2 + F^2(x_1)G(y_1) \\ &\quad -2F(x_1)G^2(y_1) \qquad\qquad (\text{if } 1 < F(x_1) + G(y_1) \le 2) \end{aligned}$$

and that $\theta = \frac{7}{10}$, $\sigma_1^2 = \frac{1751}{113400}$, which complete the proof. □

From Fig. 1 above, the normal curve seems to be a quite accurate approximation of the distribution of D even when n is small. To investigate further, Table 2 compares the exact values of the cumulative probabilities of D with their

Table 2 Comparison of the exact cdf of D and its asymptotic approximation ($n = 8$)

value	0.000	0.083	0.143	0.155	0.167	0.179	0.190	0.202	0.214
exact	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.002	0.003
appr.	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001
0.226	0.238	0.250	0.262	0.274	0.286	0.298	0.310	0.321	0.333
0.004	0.005	0.006	0.007	0.008	0.011	0.013	0.016	0.018	0.022
0.001	0.002	0.002	0.003	0.004	0.005	0.006	0.008	0.010	0.012
0.345	0.357	0.369	0.381	0.393	0.405	0.417	0.429	0.440	0.452
0.027	0.031	0.036	0.043	0.049	0.056	0.065	0.074	0.084	0.095
0.015	0.018	0.022	0.027	0.032	0.039	0.046	0.055	0.064	0.075
0.464	0.476	0.488	0.500	0.512	0.524	0.536	0.548	0.560	0.571
0.107	0.121	0.135	0.151	0.168	0.187	0.205	0.227	0.248	0.272
0.088	0.102	0.117	0.134	0.152	0.173	0.194	0.218	0.243	0.269
0.583	0.595	0.607	0.619	0.631	0.643	0.655	0.667	0.679	0.690
0.297	0.323	0.349	0.378	0.407	0.436	0.466	0.498	0.529	0.562
0.297	0.326	0.356	0.387	0.419	0.451	0.484	0.516	0.549	0.581
0.702	0.714	0.726	0.738	0.750	0.762	0.774	0.786	0.798	0.810
0.594	0.626	0.657	0.689	0.719	0.749	0.777	0.805	0.831	0.855
0.613	0.644	0.674	0.703	0.731	0.757	0.782	0.806	0.827	0.848
0.821	0.833	0.845	0.857	0.869	0.881	0.893	0.905	0.917	0.929
0.877	0.898	0.917	0.933	0.948	0.960	0.971	0.979	0.986	0.992
0.866	0.883	0.898	0.912	0.925	0.936	0.945	0.954	0.961	0.968
0.940	0.952	0.964	0.976	0.988	1.000				
0.995	0.998	0.999	1.000	1.000	1.000				
0.973	0.978	0.982	0.985	0.988	0.990				

normal asymptotic approximations, for $n = 8$; note that the difference between the exact cumulative distribution function and its approximations never exceeds 0.026, this value occurring when the cdf is evaluated for D around 0.9. This result is quite good but it has to be pointed out that, when D is used as a test-statistic, the approximation of the tails of its distribution should be very accurate, especially if low significance levels are applied. Fortunately, the quality of the asymptotic approximation improves fast as the sample size increases: further comparisons show that, not only the difference between the exact and the approximated cdf reduces, but even that the highest differences are observed in the central part of the distribution; for instance, when $n = 12$, the highest difference of 0.023 is observed when the cdf is evaluated at around 0.69.

3 A test of independence based on D

D can be used as a test-statistic in a two-sided or a one-sided test of independence; indeed, the null hypothesis of independence can be tested against the two-sided alternative of lack of independence (H_1) or against one of the one-sided alternative of concordance (H_1^+) or discordance (H_1^-). When n is small, the determination of the critical values can be accomplished numerically on the basis of the left and right tails of the exact distribution of D under H_0; when n is large, the above-discussed asymptotic approximation has to be applied. Tables 3 and 4 report, for $n = 6, \ldots, 17$ and for some selected values of α, the values l_α (Table 3) and r_α (Table 4) such that, under H_0, $\Pr\{D \leq l_\alpha\} \leq \alpha$ and $\Pr\{D \geq r_\alpha\} \leq \alpha$. Up to $n = 12$, these critical values were determined by computing the exact null distribution of D (recall that, according to what reported in Sect. 2, this task requires to enumerate all the permutations of the set $\{1, \ldots, n\}$). Due to the known computational limits, the critical values for $n = 13, \ldots, 17$ were instead determined by *simulating* the null distribution of D, i.e. by randomly drawing 5 millions permutations of the set $\{1, \ldots, n\}$; to

Table 3 Critical values of test based on D (left tail)

$n \backslash \alpha$	0.005	0.01	0.025	0.05	0.1	0.15
6	0.0000	0.1429	0.2287	0.2857	0.3714	0.4287
7	0.1786	0.2143	0.2857	0.3571	0.4286	0.4643
8	0.2381	0.2738	0.3333	0.3929	0.4524	0.4881
9	0.2750	0.3083	0.3667	0.4167	0.4750	0.5167
10	0.3030	0.3394	0.3879	0.4364	0.4909	0.5273
11	0.3273	0.3591	0.4091	0.4546	0.5046	0.5409
12	0.3461	0.3776	0.4266	0.4685	0.5175	0.5489
13*	0.3654	0.3956	0.4423	0.4808	0.5275	0.5577
14*	0.3802	0.4088	0.4527	0.4923	0.5363	0.5648
15*	0.3929	0.4214	0.4643	0.5018	0.5429	0.5714
16*	0.4044	0.4323	0.4735	0.5088	0.5500	0.5765
17*	0.4154	0.4424	0.4828	0.5172	0.5564	0.5821

*Simulated

Table 4 Critical values of test based on D (right tail)

$n\backslash\alpha$	0.15	0.1	0.05	0.025	0.01	0.005
6	0.8571	0.8857	0.9429	0.9714	0.9714	1.0000
7	0.8393	0.8750	0.9107	0.9286	0.9643	0.9822
8	0.8214	0.8571	0.8929	0.9167	0.9405	0.9524
9	0.8167	0.8417	0.8750	0.9000	0.9250	0.9417
10	0.8061	0.8303	0.8667	0.8909	0.9212	0.9323
11	0.8000	0.8273	0.8591	0.8818	0.9091	0.9227
12	0.7972	0.8182	0.8496	0.8741	0.9021	0.9161
13*	0.7940	0.8159	0.8461	0.8681	0.8956	0.9093
14*	0.7890	0.8110	0.8396	0.8637	0.8879	0.9033
15*	0.7857	0.8071	0.8357	0.8589	0.8821	0.8982
16*	0.7838	0.8044	0.8323	0.8544	0.8779	0.8926
17*	0.7806	0.8002	0.8284	0.8993	0.8738	0.8885

*Simulated

test the reliability of this procedure, the distribution of ρ was simulated as well and the corresponding critical values were compared with the ones reported in the literature (notice that the same techniques was applied in Rizzi (1971) for Gini's cograduation index).

Concerning the two-sided test, the null hypothesis should be rejected whenever $D \leq d_1$ or $D \geq d_2$; some specifications are now needed. First of all, as noted in Sect. 2, when n is small the distribution of D under H_0 is not symmetric. Hence, splitting the significance level α exactly into two parts for the two tails of the distribution may not be optimal. In addition, as the distribution of D is discrete, the usual rule of determining the critical values d_1 and d_2 so that $\Pr\{D \leq d_1\} \leq \alpha/2$ and $\Pr\{D \geq d_2\} \leq \alpha/2$ may cause the real significance level of the test to be far from the nominal value α.

Of course, when a one-sided test is implemented, the asymmetry of the distribution of D causes no problems. For instance, when the alternative H_1^+ is considered, the rule of rejecting H_0 whenever $D \geq d^+$ should applied (the critical value d^+ being such that, under H_0, $\Pr\{D \geq d^+\} \leq \alpha$). Note that the real significance level of the test can be far from the nominal value α as well. A similar reasoning applies when the alternative H_1^- is considered: in this case, H_0 will be rejected whenever $D \leq d^-$, where the critical value d^- is such that, under H_0, $\Pr\{D \leq d^-\} \leq \alpha$.

3.1 Performance of the test based on D

An important issue concerning the implementation of a test of independence based on D is the performance of the test, in comparison to other known nonparametric methods. At this aim, some results of a simulation study based on some specific families of bivariate distributions are here reported. As a natural point of start, the test based on the proposed test-statistic D was first compared to other classical nonparametric tests (Kendall's tau, Spearman's rho and Gini's cograduation index) when the samples are drawn from a bivariate standard nor-

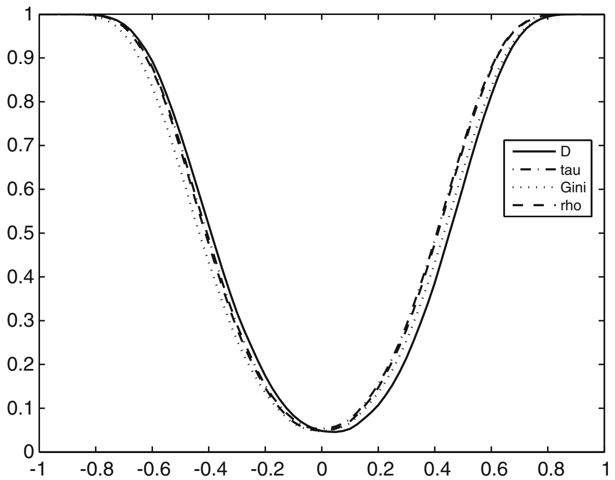

Fig. 2 Simulated power functions for the bivariate normal model ($n = 25$)

mal distribution with correlation coefficient r. For each fixed value of r, 50,000 bivariate samples were generated from the normal model and the power of the tests was estimated as the percentage of a correct rejection of the null hypothesis of independence. Figure 2 reports the estimated power functions (plotted against the value of r ranging from -1 to $+1$) for the four tests in their two-sided version, with a 5% significance level and a sample size $n = 25$.

From Fig. 2 it is clear that the power of the test based on D has an asymmetric behavior; indeed the proposed test is more powerful for discordance alternatives, while showing a poor performance when concordance is considered. Indeed the power function of D is the highest when $r < 0$, but it becomes the lowest for $r > 0$. To understand better this behavior, the distribution of the test-statistic D under the alternative hypothesis should be further studied; as a first hint, however, recall the discussion about the asymmetry of the null distribution of D and the role of the functional $\mathcal{D}(H)$ in the beginning of Sect. 2. To give some further comments on Fig. 2, consider that, as known, the power of τ and ρ result to be quite similar; D results to be slightly better than both τ and ρ for $r < 0$, but it becomes definitely worse when $r > 0$. Conversely, when G and D are compared, one notices that G performs quite worse than D when $r < 0$ and just slightly better when $r > 0$.

From the above-reported results, it is natural to conclude that D should be the best test-statistic when a one-sided test for independence against discordance is implemented; to appreciate the increase of power which can be gained, Table 5 reports some values of the estimated powers of a test for H_0 against H_1^- when the samples are drawn from a bivariate standard normal distribution and $n = 25$ (the significance level is again set to 5%). Of course similar reasonings lead to a bad performance of D when a one-sided test against concordance is considered; Table 6 reports, with the same settings, the estimated powers of a test for H_0 against H_1^+.

Table 5 Simulated powers for the one-sided test (bivariate normal, $n = 25$)

	D	tau	Gini	rho
$r = -1.00$	1.0000	1.0000	1.0000	1.0000
$r = -0.90$	1.0000	1.0000	1.0000	1.0000
$r = -0.80$	0.9992	0.9990	0.9981	0.9992
$r = -0.70$	0.9868	0.9846	0.9781	0.9859
$r = -0.60$	0.9292	0.9215	0.9015	0.9248
$r = -0.55$	0.8742	0.8622	0.8377	0.8671
$r = -0.50$	0.8019	0.7838	0.7582	0.7918
$r = -0.45$	0.7137	0.6894	0.6654	0.6998
$r = -0.40$	0.6165	0.5903	0.5683	0.6001
$r = -0.35$	0.5161	0.4884	0.4690	0.4986
$r = -0.30$	0.4179	0.3894	0.3776	0.3990
$r = -0.25$	0.3286	0.3028	0.2958	0.3118
$r = -0.20$	0.2470	0.2225	0.2235	0.2319
$r = -0.10$	0.1263	0.1098	0.1122	0.1139

Table 6 Simulated powers for the one-sided test (bivariate normal, $n = 25$)

	D	tau	Gini	rho
$r = 0.10$	0.0955	0.1128	0.1158	0.1169
$r = 0.20$	0.1938	0.2235	0.2209	0.2302
$r = 0.25$	0.2619	0.2985	0.2914	0.3062
$r = 0.30$	0.3482	0.3898	0.3767	0.3981
$r = 0.35$	0.4376	0.4835	0.4669	0.4943
$r = 0.40$	0.5403	0.5873	0.5647	0.5984
$r = 0.45$	0.6450	0.6880	0.6650	0.6973
$r = 0.50$	0.7436	0.7838	0.7572	0.7911
$r = 0.55$	0.8294	0.8615	0.8387	0.8681
$r = 0.60$	0.9007	0.9240	0.9047	0.9286
$r = 0.70$	0.9770	0.9847	0.9771	0.9853
$r = 0.80$	0.9980	0.9990	0.9979	0.9991
$r = 0.90$	1.0000	1.0000	1.0000	1.0000
$r = 1.00$	1.0000	1.0000	1.0000	1.0000

Clearly the differences among the powers of the considered tests tend to vanish when the sample size increases. Figure 3 shows the estimated power functions of the four tests for a bivariate standard normal distribution with $n = 50$. It can be easily noticed that, even if D keeps its first position for $r < 0$, it is now very slightly better than the other tests (except for G); the bad performance of D for $r > 0$, in addition, is not very evident, even if it is still present. When the sample size increase to $n = 200$ (see Fig. 4), the power function of D is almost coincident with the one of τ and ρ; a different situation, however, is observed with respect to G: even if not strongly, D performs still better than G when $r < 0$ and it even starts to do the same when $r > 0$ (notice that, to give more readability to the graph, the power functions are plotted just in their relevant part, i.e. when r ranges from -0.5 to $+0.5$).

The conclusion drawn for the bivariate normal model can be of course modified when a different structure of dependence is chosen for the two populations from which the samples are drawn. A commonly used model in simulations is

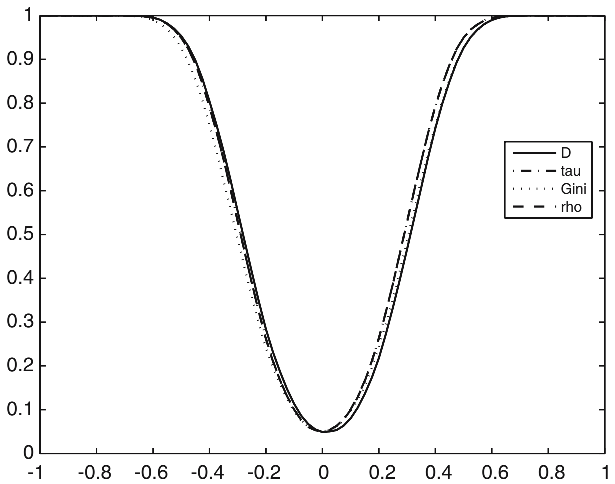

Fig. 3 Simulated power functions for the bivariate normal model ($n = 50$)

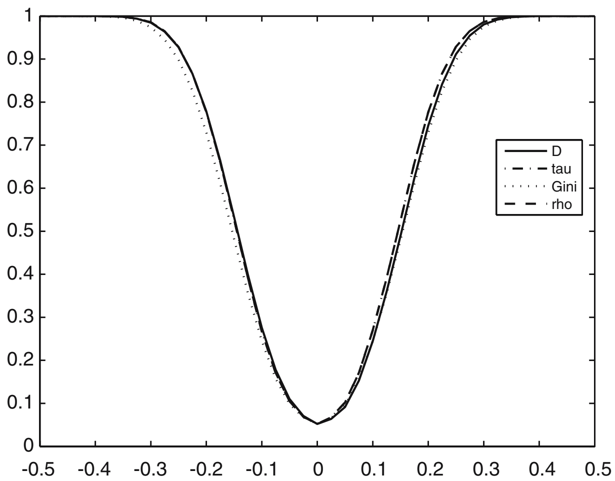

Fig. 4 Simulated power functions for the bivariate normal model ($n = 200$)

the Fairlie–Gumbel–Morgenstern distribution (with uniform marginals) with density:

$$f(x_1, x_2) = 1 + \alpha(2x_1 - 1)(2x_2 - 1); \qquad 0 \le x_1, x_2 \le 1; \quad -1 \le \alpha \le 1$$

(see Johnson 1987 for a quick reference and for details about the simulations from this model). It is fairly known that, as the parameter α ranges from -1 to $+1$, this distribution can model a limited level of dependence (when the marginal distributions are uniform, indeed, the correlation coefficient ranges from $-1/3$ to $1/3$); this fact is reflected in the results of our simulations: Figs. 5

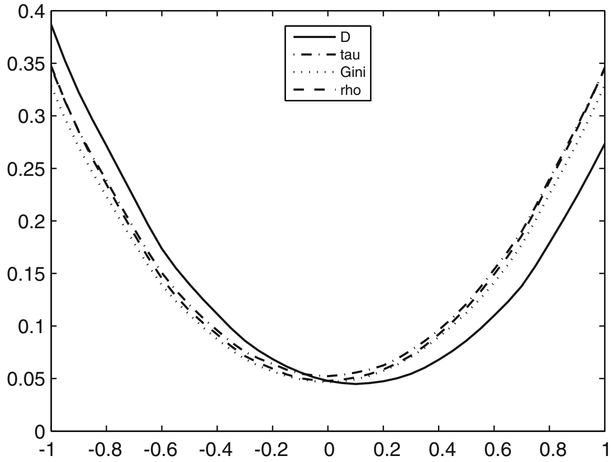

Fig. 5 Simulated power functions for the Farlie–Gumbel–Morgenstern model ($n = 25$)

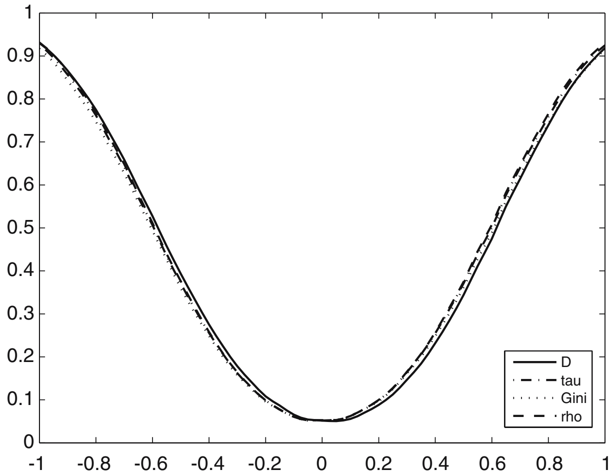

Fig. 6 Simulated power functions for the Farlie–Gumbel–Morgenstern model ($n = 100$)

and 6 report the estimated power functions of the four considered test, in their two-sided version, respectively, for $n = 25$ and $n = 100$ (the functions are plotted against the values of the parameters α; a 5% level of significance is again applied). When the sample size is small ($n = 25$), the situation observed in Fig. 2 is replied, with a stronger evidence due to the small level of correlation. When the sample sizes increases to $n = 100$, all the power functions tent to coincide, like in the normal case.

It is clear that the best performances of the test based on D can be observed for such families of distributions where the dependence structure is mainly discordant. To prove this conclusion, Figs. 7 ($n = 25$) and 8 ($n = 100$) report the estimated power functions for the Gumbel's bivariate exponential model, which has density:

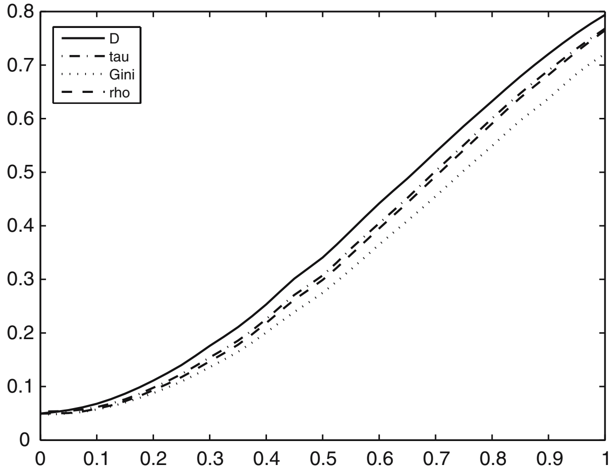

Fig. 7 Simulated power functions for the Gumbel's bivariate exponential model ($n = 25$)

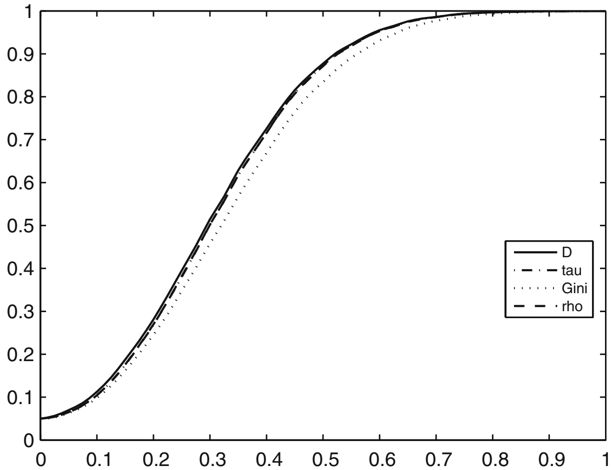

Fig. 8 Simulated power functions for the Gumbel's bivariate exponential model ($n = 100$)

$$f(x_1, x_2) = [(1 + \theta x_1)(1 + \theta x_2)] \exp\{-x_1 - x_2 - \theta x_1, x_2\}; \quad x_1, x_2 > 0$$

where $0 \le \theta \le 1$ (see Gumbel 1960 and again Johnson 1987). The parameter θ in the Gumbel's distribution regulates the dependence structure, giving independence for $\theta = 0$; when θ increases the components get more discordant, the correlation coefficient being -0.43 when $\theta = 1$.

Figure 7 [3] proves what previously conjectured: the test based on D performs definitely better than the other considered tests; notice, again, that the power

[3] The power functions are here plotted against the values of the parameter θ; a 5% level of significance is applied for all the tests, in their two-sided version.

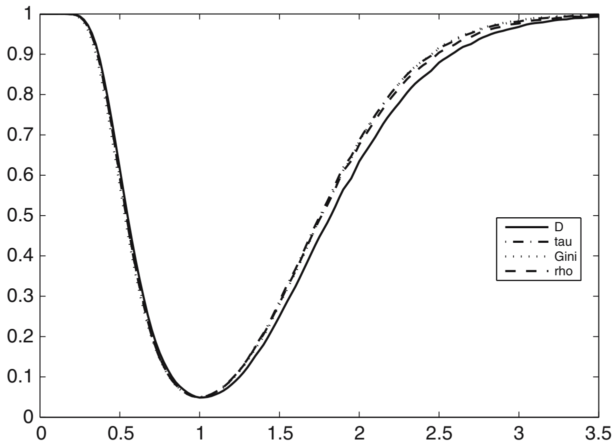

Fig. 9 Simulated power functions for the Plackett's bivariate uniform model ($n = 100$)

of τ and ρ is quite similar, while the power of G is very lower than the one of D. When the sample size increases (Fig. 8), the considered tests tend to perform similarly; however, G makes a remarkable exception to this general rule, as its power function remains definitely low.

As a final term of comparison Fig. 9 reports the results obtained with a flexible model, Plackett's bivariate uniform distribution (see Plackett 1965). The r.v. (X_1, X_2) follows a Plackett's distribution if it has density

$$f(x_1,x_2) = \frac{\psi[(\psi-1)(x_1+x_2-2x_1x_2)+1]}{\{[1+(x_1+x_2)(\psi-1)]^2 - 4\psi(\psi-1)x_1x_2\}^{3/2}}; \quad 0 < x_1, x_2 < 1$$

where $\psi > 0$. The parameter ψ can lead to different dependence situations as $X_1 = 1 - X_2$ if $\psi \to 0, X_1$ and X_2 are independent if $\psi = 1$ and $X_1 = X_2$ if $\psi \to \infty$. In Fig. 9 the estimated power functions for $n = 100$ (5% significance level; two-sided tests) are plotted against the value of the parameter ψ; as the values of ψ greater than 1 are the ones giving concordance, this part of the power function results to be amplified. Notice that, despite the large value of the sample size, the test based on D has here a remarkable worse performance for $\psi > 1$; differently to the other above-considered situations, in addition, the test based on G performs quite similarly to τ and ρ.

As a final remark concerning the performance of the test based on D, the reader may refer to Borroni and Cazzaro (2006), where the power function of the test is simulated under known "alterations" of the bivariate normal model and for small values of the sample size. Some of the therein reported simulations follow the approach of Vale and Maurelli (1983) (see also Kotz et al. 2000) where the marginal components of a bivariate normal distribution are subjected to nonlinear transformations with known coefficients; this procedure leads to a different dependence structure along with different kinds of marginal distributions with known levels of kurtosis and skewness. Even if the marginal distributions are not relevant in our context, as the considered tests do not depend on them, it has to be cited that some combinations of transformations give a higher power for the test based on D, even for concordance alternatives and for small sample sizes. Nevertheless, this fact is observed just occasionally and the conclusions drawn from the above-reported simulations are mainly confirmed, especially with respect to the relationship between D and G (see Borroni and Cazzaro 2006 for further details).

4 Concluding remarks and future research

In this paper a nonparametric procedure to test if two criteria leading to two sequences of ranks can be considered independent, concordant or discordant is introduced. Hence the proposed test D can be regarded as a competitor of other classical rank correlation methods, such as Spearman's rho and Kendall's tau. Some simulation studies show that the introduced procedure may perform better than its competitors, depending on the situations where it is applied; of course these comparisons can be extended and the situations where D has the best performance could be analytically characterized by studying the Pitman efficiency of the test. However, one immediate advantage of the presented approach is that the proposed test can be directly extended to deal with more than two criteria of ranking; we think that this extension can be considered as the most profitable direction of future research. Indeed, when $m > 2$ the index $D^{(m)}$ does not turn out to be a mean of known measures computed on the possible $\binom{m}{2}$ couples of rankings; hence it can be suspected that the information drawn by $D^{(m)}$ are at least different, and perhaps deeper, than the ones provided by other classical measures, such as Kendall's coefficient of concordance, which are essentially based on known pairwise comparisons of rankings. The link between $D^{(m)}$ and the concept of compensation, in addition, gives new hints to better understand the relationship between the independence of m rankings and its opposite extremes.

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