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Computing the moments of order statistics from nonidentical random variables

H. M. Barakat[1], **Y. H. Abdelkader**[2]

[1] Dept. of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt (e-mail: hbarakat2@hotmail.com)
[2] Dept. of Math., Faculty of Science, Alexandria University, Alexandria, Egypt (e-mail: yabdelkader@yahoo.com)

Abstract. In this paper, we establish new representations, identities and recurrence relations of order statistics (o.s.) arising from general independent nonidentically distributed random variables (r.v.'s). These recurrence relations will enable one to compute all moments of all o.s. in a simple manner. Applications for some known distributions are given.

Key words: Order statistics, moments, permanents, recurrence relations and identities.

1 Introduction

Let $X_1, X_2, \cdots, X_n$ be independent r.v.'s with distribution functions (d.f.'s) F_1, $F_2, \cdots, F_n$, respectively. Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the corresponding o.s. It is known that the distribution of the rth o.s. $X_{r:n} (1 \leq r \leq n)$ is conveniently expressed in terms of permanents (see, e.g., Bapat and Beg, 1989). That is

$$F_{r:n}(x) = \sum_{i=r}^{n} \frac{1}{i!(n-i)!} \operatorname{Per} \left[\underset{i}{\underline{F(x)}} \; \underset{n-i}{\underline{1-F(x)}}\right], \quad -\infty < x < \infty, \qquad (1.1)$$

where $\underline{F(x)}$ and $\underline{1-F(x)}$ denote the column vectors $(F_1(x)\ F_2(x)\ \cdots\ F_n(x))'$ and $(1-F_1(x)\ 1-F_2(x)\ \cdots\ 1-F_n(x))'$, respectively. Moreover, if $\underline{a}_1, \underline{a}_2, \cdots$ are column vectors, then

$$[\underset{i_1}{\underline{a}_1} \; \underset{i_2}{\underline{a}_2} \; \underset{\cdots}{\cdots}]$$

will denote the matrix obtained by taking i_1 copies of $\underline{a}_1$, i_2 copies of $\underline{a}_2$ and so on. Finally, in (1.1), Per$A$, where $A = (a_{ij})$ is $n \times n$ square matrix, denotes the permanent of A, i.e.,

$$\mathrm{Per}A = \sum_{\sigma \epsilon S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where S_n is the set of permutations of $1, 2, \cdots, n$. Thus the definition of the permanent is equivalent to the determinant except that all signs in the expansion are positive (see the survey papers Minc 1983, 1987).

The need for recurrence relations and identities is well established in the literature and for details we refer to the monograph by Arnold and Balakrishnan (1989). Recently, by using some known properties of permanents, general results on o.s. from nonidentically r.v.'s are coming out. For example, by using the property that the permanent is a multilinear function of the columns or the rows, Balakrishnan et al. (1992) and Cao and West (1997) have established several recurrence relations for the d.f.'s of o.s. of nonidentically r.v.'s. Bapat and Beg (1989), Beg (1991) and Balakrishnan (1992) used the same property of the permanents to obtain several recurrence relations and identities satisfied by the single and the product moments of o.s. Balakrishnan (1994) exploited the linear relation between the probability density function and the d.f. of the exponential r.v., which is equivalent to the constant hazard rate property. This lends itself to the use of integration by parts to get some elegant recurrence relations satisfied by the single and the product moments of o.s. Balakrishnan and Balasubramanian (1995) have applied the same procedure on o.s. from nonidentical power function r.v.'s to obtain some similar relations. However, most of these recurrence relations show that, it is enough to evaluate the kth moment of a single o.s. in a sample of size n, if these moments in samples of size less than n are already available. The kth moment of the remaining $n-1$ o.s. can then be determined by repeated use of these recurrence relations. For this purpose one could, for example, start with either $\mu_{1:n}^{(k)} = \mathrm{E}(X_{1:n}^k)$ or $\mu_{n:n}^{(k)} = \mathrm{E}(X_{n:n}^k)$. It is, therefore, useful to express the moment $\mu_{r:n}^{(k)} = \mathrm{E}(X_{r:n}^k)$ $(1 \le r \le n)$ purely in terms of the kth moment of the maximum o.s. or of the minimum o.s. from samples of size up to n. In the case of independent and identically distributed r.v.'s this problem is discussed in Arnold et al. (1992) (Theorems 5.3.2 and 5.3.3). Barakat and Abdelkader (2000) have considered this problem for o.s. arising from n independent nonidentically distributed Weibull r.v.'s. Namely, they computed recursively the kth moment of all o.s. from the kth moment of the maximum.

The main aim of this paper is to extend the result of Barakat and Abdelkader (2000) to any d.f. In the next section, an interesting result (Theorem 2.1) is given, which shows that the kth moment $(k = 1, 2, \cdots)$ of rth o.s. $(1 \le r \le n)$ of a sample of size n can be expressed purely in terms of the kth moment of the maximums and the minimums of all possible subsamples of the given sample. This result can then be utilized for the recursive computation of the single moments of o.s. arising from nonidentically distributed r.v.'s. Moreover, two new identities, which are satisfied by the single moments (Theorem 2.2), are derived. These identities are simple in nature and can be useful in checking the computations of these moments. Finally, in Sect. 3, some illustrative examples are given.

2 Main result

To lay the groundwork of this study, we begin with the following lemma.

Lemma 2.1. *Let X be an arbitrary r.v. with d.f. $F(x)$. Then, for any positive integer k,*

$$\int_0^{\infty} x^k d\,F(x) = k \int_0^{\infty} x^{k-1}(1-F(x))\,dx$$

and

$$\int_{-\infty}^{0} x^k d\,F(x) = -k \int_{-\infty}^{0} x^{k-1}F(x)\,dx.$$

Moreover, if the kth moment $\mu^{(k)}$ of X exists, then $\mu^{(k)} = \mu^{(k)+} - \mu^{(k)-}$, where $\mu^{(k)+} = k\int_0^{\infty} x^{k-1}(1-F(x))dx$, $\mu^{(k)-} = k\int_{-\infty}^{0} x^{k-1}F(x)dx$ and $\lim\limits_{x\to-\infty} x^kF(x) = \lim\limits_{x\to+\infty} x^k(1-F(x)) = 0$.

Proof. The proof of the first part can be found in Galambos (1987, p 375). The proof of the second part is similar to the first one.

Remark 2.1. It is convenient to call $\mu^{(k)+}$ and $\mu^{(k)-}$, for any r.v. X and positive integer k, the positive and the negative kth moment of X, respectively.

In the following, some conventions and notations will be used. For any $1 \leq i_1 < i_2 < \cdots < i_m \leq n$, let $Z_{i(m)} = \max(X_{i_1}, X_{i_2}, \cdots, X_{i_m})$ and $W_{i(m)} = \min(X_{i_1}, X_{i_2}, \cdots, X_{i_m})$. Then $H_{i(m)}(x) = \prod_{t=1}^{m} F_{i_t}(x)$ and $L_{i(m)}(x) = 1 - \prod_{t=1}^{m} G_{i_t}(x)$, where $G_i(x) = 1 - F_i(x)$, $\forall\, i = 1, 2, \cdots, n$, be the d.f.'s of $Z_{i(m)}$ and $W_{i(m)}$, respectively. Moreover, in view of Lemma 2.1,

$$\mu_{m:i(m)}^{(k)-} = -\int_{-\infty}^{0} x^k\, d\,H_{i(m)}(x) \;=\; k\int_{-\infty}^{0} x^{k-1}\prod_{t=1}^{m} F_{i_t}(x)\,dx$$

and

$$\mu_{1:i(m)}^{(k)+} = \int_0^{\infty} x^k\, d\,L_{i(m)}(x) \;=\; k\int_0^{\infty} x^{k-1}\prod_{t=1}^{m} G_{i_t}(x)\,dx$$

be the negative and the positive kth moment of the maximum and the minimum of $(X_{i_1}, X_{i_2}, \cdots, X_{i_m})$, respectively. Finally, let

$$\mathrm{I}_j^+ = \sum_{1\leq i_1<i_2<\cdots<i_j\leq n}\cdots\sum \mu_{1:i(j)}^{(k)+}, \quad j = 1, 2, \cdots, n, \tag{2.1}$$

and

$$\mathrm{I}_j^- = \sum_{1\leq i_1<i_2<\cdots<i_j\leq n}\cdots\sum \mu_{j:i(j)}^{(k)-}, \quad j = 1, 2, \cdots, n. \tag{2.2}$$

The two sequences $\mathbf{I}_j^+$ and $\mathbf{I}_j^-$ may be interpreted, respectively, as the sum of the positive kth moment of the largest and the negative kth moment of the smallest o.s. from all possible subsamples of size j of the given sample. The following theorem shows that $\mu_{r:n}^{(k)}, \forall\, 1 \leq r \leq n,\ k = 1, 2, \cdots$, can be expressed purely in terms of $\mathbf{I}_j^+$ and $\mathbf{I}_j^-$, for $j = 1, 2, ..., n$.

Theorem 2.1. *For $1 \leq r \leq n$ and $k = 1, 2, \cdots$, the positive and the negative kth moment of $X_{r:n}$ can be expressed as*

$$\mu_{r:n}^{(k)+} = \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r} \boldsymbol{I}_j^+ \tag{2.3}$$

and

$$\mu_{r:n}^{(k)-} = \sum_{j=r}^{n} (-1)^{j-r} \binom{j-1}{r-1} \boldsymbol{I}_j^- . \tag{2.4}$$

Moreover, $\mu_{r:n}^{(k)} = \mu_{r:n}^{(k)+} - \mu_{r:n}^{(k)-}$.

Remark 2.2. If $\check{x}_{i0} = \inf\{x : F_i(x) > 0\} \geq 0,\ i = 1, 2, \cdots, n$, then $\mu_{r:n}^{(k)} = \mu_{r:n}^{(k)+}$. Also, if $\hat{x}_{i0} = \sup\{x : F_i(x) < 1\} \leq 0,\ i = 1, 2, \cdots, n$, then $\mu_{r:n}^{(k)} = \mu_{r:n}^{(k)-}$.

Remark 2.3. If the d.f.'s $F_i(x), i = 1, 2, \cdots, n$, are symmetric, then $\mathbf{I}_j^+ = (-1)^{k-1}\, \mathbf{I}_j^-$.

Corollary 2.1. *Let $F_1 = F_2 = \cdots = F_n$. Then $\boldsymbol{I}_j^+ = \binom{n}{j} \mu_{1:j}^{(k)+}$ and $\boldsymbol{I}_j^- = \binom{n}{j} \mu_{j:j}^{(k)-}$, where $\mu_{1:j}^{(k)+} = k \int_0^\infty x^{k-1} G^j(x)\, dx$, $\mu_{j:j}^{(k)-} = k \int_{-\infty}^0 x^{k-1} F^j(x)\, dx$ and $G(x) = 1 - F(x)$. In this case (2.3) and (2.4), respectively, have the following forms*

$$\mu_{r:n}^{(k)+} = \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r} \binom{n}{j} \mu_{1:j}^{(k)+} \tag{2.5}$$

and

$$\mu_{r:n}^{(k)-} = \sum_{j=r}^{n} (-1)^{j-r} \binom{j-1}{r-1} \binom{n}{j} \mu_{j:j}^{(k)-} . \tag{2.6}$$

Moreover, (2.5), when $\check{x}_{i0} \geq 0$, and (2.6), when $\hat{x}_{i0} \leq 0$, reduce to (5.3.15), in Theorem 5.3.3 of Arnold et al. (1992), and (5.3.14), in Theorem 5.3.2 of Arnold et al. (1992), respectively.

Corollary 2.2. *If n is odd integer, then the kth moment of the median $\mu^{(k)}_{\frac{n+1}{2}:n}$ is given by*

$$\mu^{(k)}_{\frac{n+1}{2}:n} = \sum_{j=\frac{n+1}{2}}^{n} (-1)^{j-\frac{n+1}{2}} \binom{j-1}{\frac{n-1}{2}} \boldsymbol{I}_j,$$

where $\boldsymbol{I}_j = \boldsymbol{I}_j^{+} - \boldsymbol{I}_j^{-}$. Moreover, in view of Remark 2.3, if the d.f.'s $F_i(x),\ i = 1, 2, \cdots, n$ are symmetric and k is odd integer, then $\mu^{(k)}_{\frac{n+1}{2}:n} = 0$.

Proof of Theorem 2.1. From (1.1), it can be shown that, for any $1 < r \leq n$,

$$F_{r-1:n}(x) = F_{r:n}(x) + \frac{1}{(r-1)!(n-r+1)!} \operatorname{Per}\big[\underbrace{F(x)}_{r-1}\ \underbrace{1-F(x)}_{n-r+1}\big],$$

which is equivalent to

$$F_{r-1:n}(x) = F_{r:n}(x) + \sum_{\mathcal{P}} \prod_{j=1}^{r-1} F_{i_j}(x) \prod_{j=1}^{n-r+1} \Big(1 - F_{i_{n-j+1}}(x)\Big), \quad (2.7)$$

where the summation $\mathcal{P}$ extends over all permutations $(i_1, i_2, \cdots, i_n)$ of $(1, 2, \cdots, n)$ for which $1 \leq i_1 < i_2 < \cdots < i_{r-1} \leq n$, $1 \leq i_r < i_{r+1} < \cdots < i_n \leq n$.

An application of Lemma 2.1 thus yields $\mu^{(k)}_{r:n} = \mu^{(k)+}_{r:n} - \mu^{(k)-}_{r:n}$, where $\mu^{(k)+}_{r:n} = k \int_0^\infty x^{k-1} G_{r:n}(x)dx$, $\mu^{(k)-}_{r:n} = k \int_{-\infty}^0 x^{k-1} F_{r:n}(x)dx$ and $G_{r:n}(x) = 1 - F_{r:n}(x)$. To compute $\mu^{(k)}_{r:n}$, the Equation (2.7) can be used and this yields

$$\mu^{(k)+}_{r:n} = \mu^{(k)+}_{r-1:n} + \mathbf{J}^{(k)+}_{r:n} \quad (2.8)$$

and

$$\mu^{(k)-}_{r:n} = \mu^{(k)-}_{r-1:n} - \mathbf{J}^{(k)-}_{r:n},$$

where

$$\mathbf{J}^{(k)+}_{r:n} = k \int_0^\infty x^{k-1} \sum_{\mathcal{P}} \prod_{j=1}^{r-1} \left(1 - G_{i_j}(x)\right) \prod_{j=1}^{n-r+1} G_{i_{n-j+1}}(x)\, dx \quad (2.9)$$

and

$$\mathbf{J}^{(k)-}_{r:n} = k \int_{-\infty}^0 x^{k-1} \sum_{\mathcal{P}} \prod_{j=1}^{r-1} F_{i_j}(x) \prod_{j=1}^{n-r+1} \left(1 - F_{i_{n-j+1}}(x)\right) dx.$$

Now, (2.9) can be rewritten as

$$\begin{aligned}
&\mathbf{J}^{(k)+}_{r:n} \\
&= k\sum_{\mathcal{P}}\Bigg(\int_0^\infty x^{k-1}\prod_{j=r}^{n}G_{i_j}(x)\,dx-\sum_{j_1=1}^{r-1}\int_0^\infty x^{k-1}G_{i_{j_1}}(x)\prod_{j=r}^{n}G_{i_j}(x)\,dx \\
&\quad+\sum\sum_{1\le j_1<j_2\le r-1}\int_0^\infty x^{k-1}G_{i_{j_1}}(x)G_{i_{j_2}}(x)\prod_{j=r}^{n}G_{i_j}(x)\,dx \\
&\quad+\cdots+(-1)^{r-2} \\
&\quad\times\sum\cdots\sum_{1\le j_1<j_2<\cdots<j_{r-2}\le r-1}\int_0^\infty x^{k-1}\prod_{j=1}^{r-2}G_{i_j}(x)\prod_{j=r}^{n}G_{i_j}(x)dx \\
&\quad+(-1)^{r-1}\int_0^\infty x^{k-1}\prod_{j=1}^{n}G_{i_j}(x)\,dx\Bigg) \\
&= \sum_{j=1}^{r}(-1)^{j-1}A_j^+(r,n)\mathbf{I}^+_{n-r+j}, \qquad (2.10)
\end{aligned}$$

where $\mathbf{I}^+_j$ is defined by (2.1) and $A_j^+(r,n)$ is a suitable sequence of constants, which depends only on r and n. On account of (2.10) and by using the obvious relations $\sum_{\mathcal{P}}(1)=\binom{n}{r-1}$ and $\sum\cdots\sum_{1\le j_1<j_2<\cdots<j_m\le n}(1)=\binom{n}{m}$, for all $n\ge m$, an application of the multiplication principle of the combinatorial analysis on the left and the right hand sides of the jth term of (2.10), yields the following combinatorial identity

$$\binom{n}{r-1}\binom{r-1}{j-1}=A_j^+(r,n)\binom{n}{n-r+j}$$

(each of the left and the right hand sides of the above combinatorial identity represents the total number of integrations involving, respectively, in the left and the right hand sides of the jth term in (2.10)). Therefore, $A_j^+(r,n)=\binom{n-r+j}{j-1}$, from which, by using (2.8), we get

$$\mu^{(k)+}_{r:n}=\mu^{(k)+}_{r-1:n}+\sum_{j=1}^{r}(-1)^{j-1}\binom{n-r+j}{j-1}\mathbf{I}^+_{n-r+j},\quad 2\le r\le n. \qquad (2.11)$$

Note that $\mu^{(k)+}_{1:n}=\mathbf{I}^+_n$ and $\mu^{(k)+}_{2:n}=\mathbf{I}^+_{n-1}-(n-1)\mathbf{I}^+_n$. This show that (2.3) holds for $r=1,2$. Now, we prove (2.3) by induction over r. Let us assume that (2.3) has been proved for a fixed $r=\ell-1,\ 1\le\ell-1<n$. Then, by virtue of (2.8) and the

assumption of induction, we get

$$\begin{aligned}
\mu_{\ell:n}^{(k)+} &= \sum_{j=n-\ell+2}^{n} (-1)^{j-(n-\ell+2)} \binom{j-1}{n-\ell+1} \mathbf{I}_j^+ \\
&\quad + \sum_{j=1}^{\ell} (-1)^{j-1} \binom{n-\ell+j}{j-1} \mathbf{I}_{n-\ell+j}^+ \\
&= \mathbf{I}_{n-\ell+1}^+ + \sum_{j=n-\ell+2}^{n} (-1)^{j-(n-\ell+1)} \left[\binom{j}{n-\ell+1} - \binom{j-1}{n-\ell+1} \right] \mathbf{I}_j^+ \\
&= \sum_{j=n-\ell+1}^{n} (-1)^{j-(n-\ell+1)} \binom{j-1}{n-\ell} \mathbf{I}_j^+ ,
\end{aligned}$$

which completes the proof of (2.3). Similarly, it can be shown that

$$\mathbf{J}_{r:n}^{(k)-} = \sum_{j=r-1}^{n} (-1)^{j-r+1} B_j^-(r,n) \mathbf{I}_j^- ,$$

where $\mathbf{I}_j^-$ is defined by (2.2) and the sequence $\{B_j^+(r,n)\}$ satisfies the combinatorial relation

$$\binom{n}{r-1} \binom{n-r+1}{j-r+1} = B_j^-(r,n) \binom{n}{j} .$$

Thus,

$$\mu_{r-1:n}^{(k)-} = \mu_{r:n}^{(k)-} + \sum_{j=1}^{n-r+2} (-1)^{j-1} \binom{j+r-2}{j-1} \mathbf{I}_{j+r-2}^- , \quad 2 \le r \le n. \quad (2.12)$$

By induction over r, we get (2.4) and the proof is completed.

Remark 2.4. Note that the sequences $\left\{\mathbf{I}_j^+\right\}_{j=1}^{j=n}$ and $\left\{\mathbf{I}_j^-\right\}_{j=1}^{j=n}$ for many d.f.'s are simple to evaluate (see Sect. 3). Furthermore, by computing these sequences and recursively applying (2.11) and (2.12), the kth moment $(k = 1, 2, \cdots)$ of all o.s. can be evaluated with a simple algorithm. Table 1, which constitutes a lower triangle matrix, illustrates this algorithm for negative moments in (2.12).

Remark 2.5. In Table 1, if a_{rj}^- denotes the coefficient of $\mathbf{I}_j^-$ in the rth row, then

$$a_{rj}^- = \begin{cases} (-1)^{j-r}\binom{j-1}{r-1}, & 1 \le r \le j, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $a_{rj}^- = 0, \forall\, r > j, a_{rr}^- = 1,\ r = 1, 2, \cdots, n, \mid a_{rj}^- \mid + \mid a_{r+1j}^- \mid = \mid a_{r+1j+1}^- \mid, \forall\, r < j$, (since $\binom{j-1}{r-1} + \binom{j-1}{r} = \binom{j}{r}$) and $\sum_{r=1}^{j} a_{rj}^- = 0$ (since $\sum_{r=1}^{j} a_{rj}^- = \sum_{r=0}^{j-1} (-1)^{j-r-1}\binom{j-1}{r} = 0$).

Table 1. The negative moments $\mu_{r:n}^{(k)-}$, $r \leq n$, of o.s. arising from nonidentically r.v.'s

$\mu_{n:n}^{(k)-}$	=								$\mathbf{I}_n^-$
$\mu_{n-1:n}^{(k)-}$	=							$\mathbf{I}_{n-1}^-$	$-(n-1)\mathbf{I}_n^-$
$\cdots$		$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\mu_{5:n}^{(k)-}$	=					$\mathbf{I}_5^- -$	$\cdots$	$\frac{(-1)^{n-6}(n-2)\cdots(n-5)}{4!}\mathbf{I}_{n-1}^-$	$\frac{(-1)^{n-5}(n-1)\cdots(n-4)}{4!}\mathbf{I}_n^-$
$\mu_{4:n}^{(k)-}$	=				$\mathbf{I}_4^-$	$-4\mathbf{I}_5^- +$	$\cdots$	$\frac{(-1)^{n-5}(n-2)\cdots(n-4)}{3!}\mathbf{I}_{n-1}^-$	$\frac{(-1)^{n-4}(n-1)\cdots(n-3)}{3!}\mathbf{I}_n^-$
$\mu_{3:n}^{(k)-}$	=			$\mathbf{I}_3^-$	$-3\mathbf{I}_4^-$	$+6\mathbf{I}_5^- -$	$\cdots$	$\frac{(-1)^{n-4}(n-2)(n-3)}{2!}\mathbf{I}_{n-1}^-$	$\frac{(-1)^{n-3}(n-1)(n-2)}{2!}\mathbf{I}_n^-$
$\mu_{2:n}^{(k)-}$	=		$\mathbf{I}_2^-$	$-2\mathbf{I}_3^-$	$+3\mathbf{I}_4^-$	$-4\mathbf{I}_5^- +$	$\cdots$	$(-1)^{n-3}(n-2)\mathbf{I}_{n-1}^-$	$(-1)^{n-2}(n-1)\mathbf{I}_n^-$
$\mu_{1:n}^{(k)-}$	=	$\mathbf{I}_1^-$	$-\mathbf{I}_2^-$	$+\mathbf{I}_3^-$	$-\mathbf{I}_4^-$	$+\mathbf{I}_5^- -$	$\cdots$	$(-1)^{n-2}\mathbf{I}_{n-1}^-$	$(-1)^{n-1}\mathbf{I}_n^-$

Remark 2.6. A similar table, for positive moments in (2.11) is represented in Barakat and Abdelkader (2000), which constitutes an upper triangle matrix with vertex at $\mu_{1:n}^{(k)+}$ $(= \mathbf{I}_n^+)$ and base at $\mu_{n:n}^{(k)+}$ $(= \mathbf{I}_1^+ - \mathbf{I}_2^+ + \mathbf{I}_3^+ + \cdots + (-1)^{n+1}\mathbf{I}_n^+)$. Moreover, similar properties, as stated in Remark 2.5, are of course satisfied with obvious changes.

The following theorem presents two identities satisfied by the kth moment of o.s., which are interesting and simple in nature and, of course, can be useful in checking the computation of these moments.

Theorem 2.2. *For $n \geq 2$ and $k = 1, 2, \cdots$, we have*

$$\sum_{r=1}^{n} \frac{1}{r}\, \mu_{r:n}^{(k)-} = \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j} \boldsymbol{I}_j^- \tag{2.13}$$

and

$$\sum_{r=1}^{n} \frac{1}{n-r+1}\, \mu_{r:n}^{(k)+} = \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j} \boldsymbol{I}_j^+. \tag{2.14}$$

Proof. We prove (2.13), by induction over n (the coefficients in the table of negative moments). If $n = 2$, $\sum_{r=1}^{2} \frac{1}{r}\, \mu_{r:2}^{(k)-} = \mathbf{I}_1^- - \frac{1}{2}\mathbf{I}_2^- = \sum_{j=1}^{2} \frac{(-1)^{j-1}}{j}\, \mathbf{I}_j^-$. Assume (2.13) has been proved for a fixed n. Then, since (in view of (2.4)),

$$\begin{aligned}
\sum_{r=1}^{n+1} \frac{1}{r}\, \mu_{r:n+1}^{(k)-} &= \sum_{r=1}^{n+1} \frac{1}{r} \left(\sum_{j=r}^{n+1} (-1)^{j-r} \tbinom{j-1}{r-1} \mathbf{I}_j^- \right) \\
&= \sum_{r=1}^{n} \frac{1}{r}\, \mu_{r:n}^{(k)-} + \mathbf{I}_{n+1}^- \sum_{r=1}^{n} \frac{(-1)^{n+1-r}}{r} \tbinom{n}{r-1} + \frac{1}{n+1} \mathbf{I}_{n+1}^-,
\end{aligned}$$

an application of the assumption of induction yields

$$\begin{aligned}
&\sum_{r=1}^{n+1} \frac{1}{r}\, \mu_{r:n+1}^{(k)-} \\
&= \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j} \mathbf{I}_j^- + \mathbf{I}_{n+1}^- \sum_{r=1}^{n} \frac{(-1)^{n+1-r}}{n+1} \tbinom{n+1}{r} + \frac{1}{n+1} \mathbf{I}_{n+1}^- \\
&= \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j} \mathbf{I}_j^- + \mathbf{I}_{n+1}^- \sum_{r=0}^{n+1} \frac{(-1)^{n+1-r}}{n+1} \tbinom{n+1}{r} + \frac{(-1)^n}{n+1} \mathbf{I}_{n+1}^- \\
&= \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{j} \mathbf{I}_j^- .
\end{aligned}$$

Formula (2.13) is thus proved. The proof of (2.14) is similar to (2.13), for brevity we omit the proof. The theorem is thus established.

The identities (2.13) and (2.14) show that the knowledge of the sequences $\mathbf{I}_j$ and $\mathbf{I}_j^-$ are enough to find the kth moments of all o.s. For example, if $\mathbf{I}_j^-$ is known, then $\mu_{r:n}^{(k)-}$ is completely determined.

3 Applications

The moments of order statistics have assumed considerable interest in recent years and have been tabulated quite extensively for several distributions. There are many practical applications for the moments of order statistics arising from nonidentical d.f.'s. In stochastic activity networks, for example, these moments may be used to compute the network completion time when each activity represented by Weibull r.v. (see, Abdelkader, 2003c). Bendell et al. (1995) used the moments of o.s. from Erlang distribution in a narrow case. Namely, the first four central moments of $\max(X_1, X_2)$, where X_1 and X_2 are independent nonidentically r.v.'s, are derived and thus the network completion time is obtained. Abdelkader (2003a) generalized the result of Bendell et al. (1995) to n independent nonidentically Erlang variables. Furthermore, the Weibull and Erlang distributions may used in the reliability theory to find the expected life system of independent nonidentical parallel components. Childs and Balakrishnan (1998) studied the moments of o.s. arising from independent nonidentical Pareto r.v.'s and as a result of this study, they obtained some important results for the multiple-outlier model (with a slippage of p observations). Generally speaking, the moments of o.s. are quite extensively used in literature to address the problem of the efficient estimation of the parameters of the underlying distribution when the sample possibly contains one or more outliers (see, for example, Balakrishnan, 1994, and Childs and Balakrishnan, 1998). However, the computation of such moments, specially when arising from nonidentically distributed r.v.'s, is tremendous difficult. Clearly, Theorem 2.1 reduces the computation of these moments to compute merely the sequences $\mathbf{I}_j^+$ and $\mathbf{I}_j^-$, which are simple in nature. Moreover, Remarks 2.4, 2.5 and 2.6 enable one to compute these moments in a simple recursive manner. On the other hand, in most applications, such as the stochastic activity networks and the reliability theory, we deal only with the sequences of d.f.'s for which $\check{x}_{i0} = \inf\{x : F_i(x) > 0\} \geq 0,\ i = 1, 2, \cdots, n$. Therefore, in view of Remark 2.2, we have to compute $\mathbf{I}_j^+$ only. In this case, Theorem 2.1 reveals that the kth moment of the rth o.s. can be expressed as a linear combinations of the kth moment of the maximum o.s. for various subsets of $(X_1, X_2, ..., X_n)$. Therefore, we may argue that this will be hold for the d.f.'s itself. In this case, this simply means, in the terminology of the reliability theory, that the reliability of an r out- of n system can be expressed as a linear combination of the reliabilities of all possible series systems formed out of the n components. We now list explicit expressions for the sequences $\mathbf{I}_j^+$ and $\mathbf{I}_j^-$ of the above mentioned d.f.'s as well as the other important known d.f.'s.

(1) Nonidentically distributed Weibull variables:

$$F_i(x) = 1 - \exp(-\alpha_i\, x^{\beta}),\ x > 0; \alpha_i,\ \beta > 0,\ i = 1, 2, \cdots, n.$$

The sequence $\mathbf{I}_j^+$ is given by

$$\mathbf{I}_j^+ = \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} \cdots \sum \frac{k\Gamma(\frac{k}{\beta})}{\beta(\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_j})^{\frac{k}{\beta}}}$$

(see Barakat and Abdelkader, 2000).

(2) Nonidentically distributed Erlang variables:

$$F_i(x) = 1 - \sum_{s=0}^{m-1} \frac{(\lambda_i x)^s}{s!} \exp(-\lambda_i x),\ x > 0; \lambda_i > 0,\ m = 1, 2, \cdots .$$

The sequence $\mathbf{I}_j^+$ is given by

$$\mathbf{I}_j^+ = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \sum_{s_1=0}^{m-1} \cdots \sum_{s_j=0}^{m-1} \frac{k \lambda_{i_1}^{s_1} \lambda_{i_2}^{s_2} \cdots \lambda_{i_j}^{s_j} (s_1 + s_2 + \cdots + s_j + k - 1)!}{s_1! s_2! \cdots s_j! (\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_j})^{s_1 + s_2 + \cdots + s_j + k}}$$

(see Abdelkader, 2003b).

(3) Nonidentically distributed positive exponential variables:

$$F_i(x) = \exp(\alpha_i\, x),\ x < 0; \alpha_i > 0,\ i = 1, 2, \cdots, n.$$

The sequence $\mathbf{I}_j^-$ is given by

$$\mathbf{I}_j^- = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \frac{(-1)^{k-1} k!}{(\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_j})^k}.$$

(4) Nonidentically distributed Pareto variables:

$$F_i(x) = 1 - \frac{x_o^{\theta_i}}{x^{\theta_i}},\ \ x \ge x_o > 0;\ \theta_i > 0.$$

It is easy to show that $\mathbf{I}_j^+$ exists, for all $1 \le j \le n$, and k $(k = 1, 2, \cdots)$, only if $\sum_{i=1}^{n} \theta_i > k$. Moreover, $\mathbf{I}_j^+$ takes the following simple form

$$\mathbf{I}_j^+ = x_o^k \binom{n}{j} + k\, x_o^k \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \frac{1}{\sum_{t=1}^{j} \theta_{i_t} - k}.$$

(5) Nonidentically distributed Laplace variables:

$$F_i(x) = \begin{cases} \frac{1}{2} e^{\beta_i x}, & x \le 0, \\ 1 - \frac{1}{2} e^{-\beta_i x}, & x > 0. \end{cases}$$

The sequences $\mathbf{I}_j^+$ and $\mathbf{I}_j^-$ are given by

$$\mathbf{I}_j^+ = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \frac{k!}{2^j (\beta_{i_1} + \beta_{i_2} + \cdots + \beta_{i_j})^k}$$

and

$$\mathbf{I}_j^- = \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum \frac{(-1)^{k-1}k!}{2^j(\beta_{i_1}+\beta_{i_2}+\cdots+\beta_{i_j})^k}.$$

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