Acta Mathematicae Applicatae Sinica, English Series
Vol. 40, No. 3 (2024) 862–874
https://doi.org/10.1007/s10255-024-1127-2
http://www.ApplMath.com.cn & www.SpringerLink.com

Acta Mathematicae Applicatae Sinica, English Series

Strong Limit Theorems for Weighted Sums under the Sub-linear Expectations

Feng-xiang FENG[1,2], Ding-cheng WANG[2,†], Qun-ying WU[1], Hai-wu HUANG[3]

[1]School of Mathematics and Statistics, Guilin University of Technology, Guilin 541004, China

[2]School of Mathematical Science, University of Electronic Science and Technology of China, Chengdu 611731, China (†E-mail: wangdc@uestc.edu.cn)

[3]School of Science, Guilin University of Aerospace Technology, Guilin 541004, China

Abstract In this article, we study strong limit theorems for weighted sums of extended negatively dependent random variables under the sub-linear expectations. We establish general strong law and complete convergence theorems for weighted sums of extended negatively dependent random variables under the sub-linear expectations. Our results of strong limit theorems are more general than some related results previously obtained by Thrum (1987), Li et al. (1995) and Wu (2010) in classical probability space.

Keywords sub-linear expectation; complete convergence; complete moment convergence; the maximal weighted sums

2020 MR Subject Classification 60F15

1 Introduction

In the classical probability, the additivity of the probabilities and the expectations are assumed. But in practice, such additivity assumption is not feasible in many areas of applications because the uncertainty phenomena can not be modeled by using additive probabilities or additive expectations. Non-additive probabilities and non-additive expectations are useful tools for studying uncertainties in statistics, measures of risk, superhedging in finance and non-linear stochastic calculus, see Denis, Martini[5], Gilboa[8], Marinacci[13], Peng[15–17, 19]. Peng[17, 19, 20] introduced the general framework of the sub-linear expectation in a general function space by relaxing the linear property of the classical expectation to the sub-additivity and positive homogeneity (cf. Definition 2.1 below). Under Peng's sub-linear expectation framework, many limit theorems have been established recently, including the central limit theorem and weak law of large numbers[18, 20, 21], strong law of large numbers[2, 3, 9, 26], the law of the iterated algorithm[4, 26, 28], Donsker's invariance principle and Chung's law of the iterated logarithm[27], the moment inequalities for the maximum partial sums and the Kolomogov strong law of large numbers[29], and so on.

The limiting behavior of weighted sums is very important in many statistical problems such as least-squares estimators, nonparametric regression function estimators and jackknife estimators among others. Many limit properties of weighted sums have been obtained in classical probability space. In the following Theorem 1.1 and Theorem 1.2, let $\{X, X_n;\ n \geq 1\}$ be a sequence of i.i.d. random variables with expectation zero. Let $\{k_n, n \geq 1\}$ ($k_n \leq Mn$, where M is an integer not depending on n) be a sequence of positive integers and $\{a_{ni}, 1 \leq i \leq k_n\}$

Manuscript received October 29, 2018. Accepted on March 30, 2024.
This research is supported by the Natural Science Foundation of Guangxi (Grant No.2024GXNSFAA010476) and the National Natural Science Foundation of China (Grant No.12361031).
†Corresponding author.

be an array of real numbers. Define a weighted sum by $S_{k_n} = \sum_{i=1}^{k_n} a_{ni}X_i$. Thrum[24]obtained the following result:

Theorem 1.1. *If* $\sum_{i=1}^{n} a_{ni}^2 = 1$ *and* $E|X|^p < \infty$ $(p \geq 2)$, *then* $S_n/n^{1/p} \to 0$ *a.s.*

Li et al.[12] extended this result and obtained:

Theorem 1.2. *If* $\sup_{1\leq i\leq k_n} |a_{ni}| < \infty$, $\sum_{i=1}^{k_n} a_{ni}^2 = O(n^\delta)$ $(\delta < \min\{1, 2/p\})$ *and* $E|X|^p < \infty$ $(p \geq 1)$, *then* $S_{k_n}/n^{1/p} \to 0$ *a.s.*

Wu[25] established the complete convergence theorem for ND random variables:

Theorem 1.3. *Let* $\{X_n;\ n \geq 1\}$ *be a sequence of ND identically distributed random variables with*

$$E|X_1|^{2/\alpha} < \infty, \qquad \text{for some} \quad \alpha > 1.$$

Then $\sum_{i=1}^{n} X_i/n^\alpha \xrightarrow{c} 0$.

There are many other strong laws for weighted sums in classical probability space. We refer the reader to [10, 11], and so on. By Borel-Cantelli Lemma, complete convergence implies almost sure convergence. Complete convergence theorems for weighted sums are important limit theorems in probability theory. Many of related results have been obtained in the probability space. We refer the reader to Peligrad, Gut[14], Sung[23], Feng, Wang, Wu[6], Chen and Sung[1], and so on. Investigating the limit theorems in sub-linear expectation space is of great significance in the theory and application. Because sub-linear expectation and capacity are not additive, the study of the limit theorems under sub-linear expectations becomes much more complex and challenging. Although, Feng, Wang, Wu[7] and Zhong and Wu[31] establish the complete convergence theorems for weighted sums, there are few related strong limit theorems for weighted sums under sub-linear expectations. The main purpose of this article is to establish very extensive version strong laws and complete convergence theorem for weight sums of extended negatively dependent random variables under the sub-linear expectations.

Throughout this paper, C stands for a positive constant which may differ from one place to another. Let $a_n \ll b_n$ denote that there exists a constant $c > 0$ such that $a_n \leq cb_n$ for sufficiently large n, and $I(\cdot)$ denotes an indicator function.

2 Preliminaries

We use the framework and notations of Peng[20]. Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, \cdots, X_n \in \mathcal{H}$ then $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l.\mathrm{Lip}}(\mathbb{R}^n)$, where $C_{l.\mathrm{Lip}}(\mathbb{R}^n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})| \leq C(1 + |\boldsymbol{x}|^m + |\boldsymbol{y}|^m)|\boldsymbol{x} - \boldsymbol{y}|, \qquad \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n,$$

for some $C > 0, m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of "random variables". If X is an element of set $\mathcal{H}$, then we denote $X \in \mathcal{H}$.

Definition 2.1. *A sub-linear expectation* $\widehat{\mathbb{E}}$ *on* $\mathcal{H}$ *is a function* $\widehat{\mathbb{E}} : \mathcal{H} \to \bar{\mathbb{R}}$ *satisfying the following properties: for all* $X, Y \in \mathcal{H}$, *we have*
(a) Monotonicity: If $X \geq Y$ *then* $\widehat{\mathbb{E}}[X] \geq \widehat{\mathbb{E}}[Y]$;

(b) Constant preserving: $\widehat{\mathbb{E}}[c] = c$;
(c) Sub-additivity: $\widehat{\mathbb{E}}[X+Y] \le \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ *whenever* $\widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ *is not of the form* $+\infty - \infty$ *or* $-\infty + \infty$;
(d) Positive homogeneity: $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X], \lambda > 0$.
Here $\bar{\mathbb{R}} = [-\infty, +\infty]$. *The triple* $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ *is called a sub-linear expectation space. Given a sub-linear expectation* $\widehat{\mathbb{E}}$, *let us denote the conjugate expectation* $\widehat{\mathcal{E}}$ *of* $\widehat{\mathbb{E}}$ *by*

$$\widehat{\mathcal{E}}[X] := -\widehat{\mathbb{E}}[-X], \qquad \forall\, X \in \mathcal{H}.$$

From the definition, we can easily get that $\widehat{\mathcal{E}}[X] \le \widehat{\mathbb{E}}[X]$, $\widehat{\mathbb{E}}[X+c] = \widehat{\mathbb{E}}[X] + c$, $\widehat{\mathbb{E}}[X-Y] \ge \widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[Y]$ and $|\widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[Y]| \le \widehat{\mathbb{E}}[|X-Y|]$. Further, if $\widehat{\mathbb{E}}[|X|]$ is finite, then $\widehat{\mathcal{E}}[X]$ and $\widehat{\mathbb{E}}[X]$ are both finite.

Definition 2.2. *(i) (Identical distribution) Let* $\boldsymbol{X_1}$ *and* $\boldsymbol{X_2}$ *be two* n*-dimensional random vectors defined respectively in sub-linear expectation spaces* $(\Omega_1, \mathcal{H}_1, \widehat{\mathbb{E}}_1)$ *and* $(\Omega_2, \mathcal{H}_2, \widehat{\mathbb{E}}_2)$. *They are called identically distributed, denoted by* $\boldsymbol{X_1} \overset{d}{=} \boldsymbol{X_2}$, *if* $\widehat{\mathbb{E}}_1[\varphi(\boldsymbol{X_1})] = \widehat{\mathbb{E}}_2[\varphi(\boldsymbol{X_2})], \quad \forall \varphi \in C_{l.Lip}(\mathbb{R}^n)$, *whenever the sub-expectations are finite. A sequence* $\{X_n;\ n \ge 1\}$ *of random variables is said to be identically distributed if* $\boldsymbol{X_i} \overset{d}{=} \boldsymbol{X_1}$, *for each* $i \ge 1$.

(ii) (Extended negative dependence)[30] *A sequence of random variables* $\{X_n; n \ge 1\}$ *is said to be upper (resp. lower) extended negatively dependent if there is some dominating constant* $K \ge 1$ *such that*

$$\widehat{\mathbb{E}}\Big[\prod_{i=1}^{n} \varphi_i(X_i)\Big] \le K \prod_{i=1}^{n} \widehat{\mathbb{E}}\big[\varphi_i(X_i)\big], \qquad \forall\, n \ge 2,$$

whenever the non-negative functions $\varphi_i \in C_{b.\mathrm{Lip}}(\mathbb{R})$, $i = 1, 2, \cdots$, *are all non-decreasing (resp. all non-increasing). They are called extended negatively dependent if they are both upper extended negatively dependent and lower extended negatively dependent.*

It is obvious that, if $\{X_n; n \ge 1\}$ is a sequence of upper (resp. lower) extended negatively dependent random variables and $f_1(x), f_2(x), \cdots \in C_{l.\mathrm{Lip}}(\mathbb{R})$ are non-decreasing (resp. non-increasing) functions, then $\{f_n(X_n);\ n \ge 1\}$ is also a sequence of upper (resp. lower) extended negatively dependent random variables. If $\{X_n;\ n \ge 1\}$ is a sequence of upper (resp. lower) extended negatively dependent random variables, then $\{-X_n;\ n \ge 1\}$ is a sequence of lower (resp. upper) extended negatively dependent random variables. Hence, if $\{X_n;\ n \ge 1\}$ is a sequence of extended negatively dependent random variables, then $\{-X_n;\ n \ge 1\}$ is a sequence of extended negatively dependent random variables. It shall be noted that the extended negative dependence of $\{X_n;\ n \ge 1\}$ under $\widehat{\mathbb{E}}$ does not imply the extended negative dependence under $\widehat{\mathcal{E}}$.

Next, we introduce the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0, 1]$ is called a capacity if

$$V(\phi) = 0, \quad V(\Omega) = 1 \qquad \text{and} \qquad V(A) \le V(B), \qquad \forall\, A \subseteq B, \ \ A, B \in \mathcal{G}.$$

It is called to be sub-additive if $V(A \cup B) \le V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$.

Let $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ be a sub-linear expectation space, and $\widehat{\mathcal{E}}$ be the conjugate expectation of $\widehat{\mathbb{E}}$. We denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A) := \inf\{\widehat{\mathbb{E}}[\xi] : I(A) \le \xi, \xi \in \mathcal{H}\}, \qquad \mathcal{V}(A) := 1 - \mathbb{V}(A^c), \qquad \forall\, A \in \mathcal{F},$$

where A^c is the complement set of A. It is obvious that $\mathbb{V}$ is sub-additive and $\mathbb{V}(A) := \widehat{\mathbb{E}}[I(A)]$, $\mathcal{V}(A) := \widehat{\mathcal{E}}[I(A)]$, if $I(A) \in \mathcal{H}$,

$$\widehat{\mathbb{E}}[f] \le \mathbb{V}(A) \le \widehat{\mathbb{E}}[g], \quad \widehat{\mathcal{E}}[f] \le \mathcal{V}(A) \le \widehat{\mathcal{E}}[g], \qquad \text{if } f \le I(A) \le g, \ \ f, g \in \mathcal{H}. \tag{2.1}$$

This implies Markov inequality: $\forall X \in \mathcal{H}$,

$$\mathbb{V}(|X| \geq x) \leq \widehat{\mathbb{E}}[|X|^p]/x^p, \qquad \forall\, x > 0, \ \ p > 0$$

from $I(|X| \geq x) \leq |X|^p/x^p \in \mathcal{H}$. By Lemma 4.1 of Zhang[28], we have Hölder inequality: $\forall\, X, Y \in \mathcal{H},\ p, q > 1$, satisfying $p^{-1} + q^{-1} = 1$,

$$\widehat{\mathbb{E}}[|XY|] \leq (\widehat{\mathbb{E}}[|X|^p])^{\frac{1}{p}} (\widehat{\mathbb{E}}[|Y|^q])^{\frac{1}{q}},$$

particularly, Jensen inequality:

$$(\widehat{\mathbb{E}}[|X|^r])^{\frac{1}{r}} \leq (\widehat{\mathbb{E}}[|X|^s])^{\frac{1}{s}}, \qquad \text{for} \ \ 0 < r \leq s.$$

Definition 2.3 (see [28])**.** *A function $V : \mathcal{F} \to [0,1]$ is called to be countably sub-additive if*

$$V\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leq \sum_{n=1}^{\infty} V(A_n), \qquad \forall\, A_n \in \mathcal{F}.$$

We define the Choquet integrals/expecations $(C_{\mathbb{V}}, C_{\mathcal{V}})$ by

$$C_V(X) := \int_0^{\infty} V(X \geq x)dx + \int_{-\infty}^{0} (V(X \geq x) - 1)dx$$

with V being replaced by $\mathbb{V}$ and $\mathcal{V}$, respectively. If $\lim\limits_{c\to+\infty} \widehat{\mathbb{E}}[(|X| - c)^+] = 0$, then $\widehat{\mathbb{E}}[|X|] \leq C_{\mathbb{V}}(|X|)$ (see Lemma 4.5(iii) of [28]).

In order to prove our results, we need the following lemmas.

Lemma 2.4. *Let $\{X_n;\ n \geq 1\}$ be a sequence of upper extended negatively dependent random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, with $\widehat{\mathbb{E}}[X_n] \leq 0,\ n \geq 1$. Let $S_n = \sum\limits_{i=1}^{n} X_i,\ B_n = \sum\limits_{i=1}^{n} \widehat{\mathbb{E}}[X_i^2]$. Then for all $x > 0,\ a > 0$,*

$$\mathbb{V}(S_n > x) \leq \mathbb{V}\Big(\max_{1\leq k\leq n} X_k > a\Big) + K\exp\Big\{-\frac{x^2}{2(xa + B_n)}\Big\}. \tag{2.2}$$

By Theorem 3.1 of [30], we can have (2.2).

Lemma 2.5 (Borel-Cantelli Lemma[28])**.** *Let $\{A_n; n \geq 1\}$ be a sequence of events in $\mathcal{F}$. Suppose that V is a countably sub-additive capacity. If $\sum\limits_{n=1}^{\infty} V(A_n) < \infty$, then*

$$V(A_n,\ i.o.) = 0, \qquad \textit{where}\ (A_n, i.o.) = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$

Lemma 2.6. *Suppose $X \in \mathcal{H}$, Then for any $c > 0$,*

$$\text{(i)}\ C_{\mathbb{V}}(|X|^p) < \infty \Leftrightarrow \sum_{n=1}^{\infty} \mathbb{V}(|X| > cn^{1/p}) < \infty; \tag{2.3}$$

$$\text{(ii)}\ C_{\mathbb{V}}(|X|^p) < \infty \Leftrightarrow \sum_{n=1}^{\infty} n\mathbb{V}(|X| > cn^{2/p}) < \infty; \tag{2.4}$$

$$\text{(iii)}\ C_{\mathbb{V}}(|X|^p) < \infty \Rightarrow \lim_{x\to\infty} x^{4/3}\mathbb{V}(|X| > cx^{\frac{4}{3p}}) = 0. \tag{2.5}$$

Proof. (i) Obviously, $C_{\mathbb{V}}(|X|^p) < \infty$ is equivalent to $C_{\mathbb{V}}(|X|^p/c^p) < \infty$ for any $c > 0$. Note that

$$C_{\mathbb{V}}(|X|^p/c^p) < \infty \Leftrightarrow \int_0^\infty \mathbb{V}(|X| > cx^{1/p})dx < \infty \Leftrightarrow \sum_{n=1}^\infty \mathbb{V}(|X| > cn^{1/p}) < \infty.$$

(ii)

$$C_{\mathbb{V}}(|X|^p/c^p) = \int_0^\infty \mathbb{V}(|X| > cx^{1/p})dx = \int_0^\infty 2x\mathbb{V}(|X| > cx^{2/p})dx,$$

$$C_{\mathbb{V}}(|X|^p/c^p) < \infty \Leftrightarrow \sum_{n=1}^\infty n\mathbb{V}(|X| > cn^{2/p}) < \infty.$$

Hence, (2.4) holds.

(iii) By $\int_0^\infty \mathbb{V}(|X| > cx^{1/p})dx < \infty$ and $\mathbb{V}(|X| > cx^{1/p}) \downarrow$, we have

$$\lim_{x\to\infty} x\mathbb{V}(|X| > cx^{\frac{1}{p}}) = 0,$$

which is equivalent to

$$\lim_{x\to\infty} x^{4/3}\mathbb{V}(|X| > cx^{\frac{4}{3p}}) = 0.$$

□

3 Strong Limit Theorems

In this section we show the main result–strong limit theorems. Firstly, we define $X_n \to X$ a.s. $\mathbb{V}(\mathcal{V})$ and $X_n \to X$ complete converge as follows:

Definition 3.1. *A sequence of random variables $\{X_n; n \geq 1\}$ is said to converge to X almost surely V (a.s. V), denoted by $X_n \to X$ a.s. V as $n \to \infty$, if $V(X_n \nrightarrow X) = 0$.*

V can be replaced by $\mathbb{V}$ and $\mathcal{V}$ respectively. By $\mathcal{V} \leq \mathbb{V}$ and $\mathcal{V}(A) + \mathbb{V}(A^c) = 1$ for any $A \in \mathcal{F}$, it is obvious that $X_n \to X$ a.s. $\mathbb{V}$ implies $X_n \to X$ a.s. $\mathcal{V}$, but $X_n \to X$ a.s. $\mathcal{V}$ does not imply $X_n \to X$ a.s. $\mathbb{V}$. Further

$$X_n \to X, \quad \text{a.s. } \mathbb{V} \Leftrightarrow \mathcal{V}(X_n \to X) = 1 \Leftrightarrow \mathbb{V}(|X_n - X| > \varepsilon, \text{ i.o.}) = 0, \qquad \text{for } \forall\varepsilon > 0,$$

and

$$X_n \to X, \quad \text{a.s. } \mathcal{V} \Leftrightarrow \mathcal{V}(X_n \nrightarrow X) = 0 \Leftrightarrow \mathbb{V}(X_n \to X) = 1.$$

Definition 3.2. *A sequence of random variables $\{X_n; n \geq 1\}$ is said to converge completely to X, denoted by $X_n \xrightarrow{c} X$ as $n \to \infty$, if $\sum_{n=1}^\infty \mathbb{V}(|X_n - X| > \varepsilon) < \infty$, for any $\varepsilon > 0$.*

Our results are as follows.

Theorem 3.3. *Suppose that $\mathbb{V}$ is countably sub-additive. Let $\{X_n; n \geq 1\}$ be a sequence of extended negatively dependent and identically distributed random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ satisfying*

$$\widehat{\mathbb{E}}[|X_1|^p] \leq C_{\mathbb{V}}(|X_1|^p) < \infty, \qquad \text{for some } \ p > 0. \tag{3.1}$$

If $p > 1$, suppose that

$$\widehat{\mathbb{E}}[X_1] = \widehat{\mathcal{E}}[X_1] = 0. \tag{3.2}$$

Let $\{k_n, n \geq 1\}$ ($k_n \leq Mn$, where M is an integer not depending on n) be a sequence of positive integers and $\{b_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ is an array of real positive numbers satisfying

$$\max_{1\leq i\leq k_n} b_{ni} = O(n^{-1/p}). \tag{3.3}$$

(1) If $p > 2$ and

$$\sum_{i=1}^{k_n} b_{ni}^2 = o((\log n)^{-1}), \tag{3.4}$$

then

$$\lim_{n\to\infty} T_{k_n} = \lim_{n\to\infty} \sum_{i=1}^{k_n} b_{ni}X_i = 0, \quad a.s. \ \ \mathbb{V}. \tag{3.5}$$

(2) If $0 < p \leq 2$ and

$$\sum_{i=1}^{k_n} b_{ni}^p = O(n^{-\delta}), \qquad \text{for some} \ \ \delta > 0, \tag{3.6}$$

then (3.5) holds.

Remark 3.4. Theorem 3.3 is a very general result. If we take $b_{ni} = a_{ni}/n^{1/p}$, where $\{a_{ni}\}$ satisfying $\max\limits_{1\leq i\leq n} |a_{ni}| = O(1)$, then we can obtain the results of Theorem 1.1 and Theorem 1.2 for array of real positive numbers $\{a_{ni}\}$ in sub-linear expectations space from our Theorem 3.3.

Theorem 3.5. *Suppose that $0 < p \leq 2$. Let $\{X_n; n \geq 1\}$ be a sequence of extended negatively dependent and identically distributed random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. Suppose (3.1) for $0 < p \leq 2$ and (3.2) holds. Assume that $\{b_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ is an array of real positive numbers satisfying*

$$\max_{1\leq i\leq k_n} b_{ni} = O(n^{-2/p}), \tag{3.7}$$

and

$$\sum_{i=1}^{k_n} b_{ni}^p = o((\log n)^{-1}). \tag{3.8}$$

Then

$$T_{k_n} = \sum_{i=1}^{k_n} b_{ni}X_i \xrightarrow{c} 0. \tag{3.9}$$

Remark 3.6. Theorem 3.5 is a very general result. If we take $p = 2/\alpha,\ \alpha > 1,\ b_{ni} = n^{-\alpha} = n^{-2/p},\ \sum\limits_{i=1}^{n} b_{ni}^p = n^{-1} = o((\log n)^{-1})$ in Theorem 3.5, then we have $\sum\limits_{i=1}^{n} b_{ni}X_i = \sum\limits_{i=1}^{n} X_i/n^{\alpha} \xrightarrow{c} 0$. Hence, we can obtain the result of Theorem 1.3.

Proof of Theorem 3.3. Without loss of generality, assume that $k_n = n$ for every $n \geq 1$. In order to prove (3.5), we need to prove:

$$\limsup_{n\to\infty} T_n = \limsup_{n\to\infty} \sum_{i=1}^{n} b_{ni}X_i \leq 0, \quad \text{a.s.} \ \ \mathbb{V}, \tag{3.10}$$

and

$$\liminf_{n\to\infty} T_n = \liminf_{n\to\infty} \sum_{i=1}^{n} b_{ni}X_i \geq 0, \quad \text{a.s. } \mathbb{V}. \tag{3.11}$$

We only need to prove (3.10). Because of using $\{-X_n; n \geq 1\}$ instead of $\{X_n; n \geq 1\}$ in (3.10), we can obtain (3.11).

For any $1 \leq i \leq n,\ n \geq 1$ and any $\varepsilon > 0$, we choose some small $\eta > 0$ and large N. Let

$$\begin{aligned}
X_{ni}^{(1)} &= -b_{ni}^{-1}n^{-\eta}I(b_{ni}X_i < -n^{-\eta}) + X_iI(|b_{ni}X_i| \leq n^{-\eta}) + b_{ni}^{-1}n^{-\eta}I(b_{ni}X_i > n^{-\eta}),\\
X_{ni}^{(2)} &= (X_i + b_{ni}^{-1}n^{-\eta})I(b_{ni}X_i \leq -\varepsilon/N) + (X_i - b_{ni}^{-1}n^{-\eta})I(b_{ni}X_i \geq \varepsilon/N),\\
X_{ni}^{(3)} &= (X_i - b_{ni}^{-1}n^{-\eta})I(n^{-\eta} < b_{ni}X_i < \varepsilon/N),\\
X_{ni}^{(4)} &= (X_i + b_{ni}^{-1}n^{-\eta})I(-\varepsilon/N < b_{ni}X_i < -n^{-\eta}),\\
T_n^{(l)} &= \sum_{i=1}^{n} b_{ni}X_{ni}^{(l)}, \qquad l = 1, 2, 3, 4.
\end{aligned}$$

Then $T_n = \sum_{i=1}^{n} b_{ni}X_i = T_n^{(1)} + T_n^{(2)} + T_n^{(3)} + T_n^{(4)}$. Obviously, $\{X_{ni}^{(1)},\ 1 \leq i \leq n,\ n \geq 1\}$ is a sequence of upper extended negatively dependent random variables.

Proof of part (1). In order to prove (3.10), it suffices to verify that

$$\limsup_{n\to\infty} T_n^{(1)} \leq 0, \quad \text{a.s. } \mathbb{V} \qquad \text{and} \qquad T_n^{(l)} = o(1), \quad \text{a.s. } \mathbb{V}, \qquad \text{for } \ l = 2, 3, 4.$$

We first to show $\limsup_{n\to\infty} T_n^{(1)} \leq 0$ a.s. $\mathbb{V}$. We will show $\sum_{i=1}^{n} \widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}] \to 0$ and $\limsup_{n\to\infty}(T_n^{(1)} - \sum_{i=1}^{n} \widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]) \leq 0$ a.s. $\mathbb{V}$.

In the probability space, there is an equality $EI(|X| \leq a) = P(|X| \leq a)$. However, in the sub-linear expectation space $\widehat{\mathbb{E}}$ is defined through continuous functions in $C_{l.\mathrm{Lip}}(\mathbb{R})$ and the indicator function $I(|x| \leq a)$ is not continuous. Therefore, the expression $\widehat{\mathbb{E}}[I(|X| \leq a)]$ does not exist. This needs to modify the indicator function by functions in $C_{l.\mathrm{Lip}}(\mathbb{R})$. To this end, we define the function $g(x) \in C_{l.\mathrm{Lip}}(\mathbb{R})$ as follows. For $0 < \mu < 1$, let $g(x) \in C_{l.\mathrm{Lip}}(\mathbb{R}),\ 0 \leq g(x) \leq 1$ for all x, $g(x) = 1$ if $|x| \leq \mu$, $\ g(x) = 0$ if $|x| > 1$, $g(x)$ is non-decreasing if $x < 0$ and $g(x)$ is non-increasing if $x > 0$. Then

$$I(|x| \leq \mu) \leq g(x) \leq I(|x| \leq 1),\ I(|x| > 1) \leq 1 - g(x) \leq I(|x| > \mu). \tag{3.12}$$

Note that

$$\begin{aligned}
|b_{ni}X_i - b_{ni}X_{ni}^{(1)}| &= |(b_{ni}X_i + n^{-\eta})I(b_{ni}X_i < -n^{-\eta}) + (b_{ni}X_i - n^{-\eta})I(b_{ni}X_i > n^{-\eta})\\
&\ll |b_{ni}X_i|I(|b_{ni}X_i| > n^{-\eta}).
\end{aligned}$$

Hence by the fact $\widehat{\mathbb{E}}[X_i] = 0$, (3.12), (3.1), (3.3) and (3.4), we have

$$\begin{aligned}
\Big|\sum_{i=1}^{n} \widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]\Big| &\leq \sum_{i=1}^{n} |\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]|\\
&= \sum_{i=1}^{n} |\widehat{\mathbb{E}}[b_{ni}X_i] - \widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]| \leq \sum_{i=1}^{n} \widehat{\mathbb{E}}[|b_{ni}X_i - b_{ni}X_{ni}^{(1)}|]
\end{aligned}$$

$$\ll \sum_{i=1}^{n} \widehat{\mathbb{E}}[|b_{ni}X_i|\big(1-g(n^{\eta}b_{ni}X_i)\big)] \ll \sum_{i=1}^{n} \widehat{\mathbb{E}}[|b_{ni}X_i|^p n^{\eta(p-1)}]$$

$$\ll n^{\eta(p-1)}(\max_{1\le i\le n}|b_{ni}|)^{p-2}\sum_{i=1}^{n} b_{ni}^2 \ll n^{\eta(p-1)-1+2/p}(\log n)^{-1} \to 0, \tag{3.13}$$

if η is chosen small enough such that $\eta(p-1) < 1-2/p$. Thus, $\sum_{i=1}^{n}\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}] \to 0$.

We will prove

$$\sum_{n=1}^{\infty}\mathbb{V}\Big(\sum_{i=1}^{n}(b_{ni}X_{ni}^{(1)} - \widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]) > \varepsilon\Big) < \infty, \qquad \forall\, \varepsilon > 0. \tag{3.14}$$

We now apply Lemma 2.4. Take $x=\varepsilon,\ \ a=3n^{-\eta}$. Note that $\max_{1\le i\le n}|b_{ni}X_{ni}^{(1)} - \widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]| \le 2n^{-\eta}$, $B_n = \sum_{i=1}^{n}\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)} - \widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]]^2 = o((\log n)^{-1})$ from (3.4). Then by Lemma 2.4, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\mathbb{V}\Big(\sum_{i=1}^{n}(b_{ni}X_{ni}^{(1)} - \widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]) > \varepsilon\Big)\\
\le&\sum_{n=1}^{\infty} C\exp\Big\{-\frac{\varepsilon^2}{2(3\varepsilon n^{-\eta}+o((\log n)^{-1}))}\Big\}\\
\le& C\sum_{n=1}^{\infty}\exp\{-2\log n\} < \infty.
\end{aligned} \tag{3.15}$$

By Borel-Cantelli Lemma, we have $\mathbb{V}\big(\sum_{i=1}^{n}(b_{ni}X_{ni}^{(1)} - \widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]) > \varepsilon,\ \text{i.o.}\big) = 0$. Therefore, we have $\limsup_{n\to\infty} T_n^{(1)} \le 0$ a.s. $\mathbb{V}$.

Next, we look at $T_n^{(2)}$. Note that $C_{\mathbb{V}}(|X_i|^p) < \infty$ is equivalent to $\sum_{n=1}^{\infty}\mathbb{V}(|X_i| \ge cn^{1/p}) < \infty$ for any $c>0$. By Borel-Cantelli Lemma, we have $\mathbb{V}(|X_i| \ge cn^{1/p},\ \text{i.o.}) = 0$. Hence, by the definition of $X_{ni}^{(2)}$, we can get $\sum_{i=1}^{n}|X_{ni}^{(2)}|$ is bounded a.s. It follows that

$$|T_n^{(2)}| \le \max_{1\le i\le n}|b_{ni}|\sum_{i=1}^{n}|X_{ni}^{(2)}| = O(n^{-1/p})\sum_{i=1}^{n}|X_{ni}^{(2)}| \to 0, \qquad \text{a.s. } \mathbb{V}. \tag{3.16}$$

We should note that the identical distribution is defined under $\widehat{\mathbb{E}}$, not under $\mathbb{V}$ (see Definition 2.2). The identical distribution of X_i implies $\widehat{\mathbb{E}}[f(X_i)] = \widehat{\mathbb{E}}[f(X_1)]$ for $f(.) \in C_{l.Lip}(\mathbb{R})$, but does not imply $\mathbb{V}(f(X_i)\in A) = \mathbb{V}(f(X_1)\in A)$. Therefore, in the calculation of $\mathbb{V}(f(X_i)\in A)$, we need to convert $\mathbb{V}$ to $\widehat{\mathbb{E}}$. As to $T_n^{(3)}$, by the definition of $X_{ni}^{(3)}$, the definition of extended negative dependence, the fact $g(x)$ is non-increasing if $x>0$, (3.12), (3.1), Markov inequality, (3.3) and (3.4), we have

$$\begin{aligned}
\mathbb{V}(|T_n^{(3)}| > \varepsilon) &\le \mathbb{V}(\text{there exist at least } N \text{ indices } i \text{ such that } b_{ni}X_i > n^{-\eta})\\
&\le \sum_{1\le i_1<i_2<\cdots<i_N\le n}\mathbb{V}\big(b_{ni_1}X_{i_1} > n^{-\eta}, \cdots, b_{ni_N}X_{i_N} > n^{-\eta}\big)
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{1\le i_1\le n}\sum_{1\le i_2\le n}\cdots\sum_{1\le i_N\le n}\widehat{\mathbb{E}}\Big[\prod_{j=1}^{N}(1-g(b_{ni_j}n^{\eta}X_{i_j}))\Big]\\
&\le C\sum_{1\le i_1\le n}\sum_{1\le i_2\le n}\cdots\sum_{1\le i_N\le n}\prod_{j=1}^{N}\widehat{\mathbb{E}}[1-g(b_{ni_j}n^{\eta}X_{i_j})]\\
&= C\Big(\sum_{i=1}^{n}\widehat{\mathbb{E}}[1-g(b_{ni}n^{\eta}X_i)]\Big)^N \le C\Big(\sum_{i=1}^{n}\mathbb{V}(b_{ni}X_i>\mu n^{-\eta})\Big)^N\\
&\le C\Big(\sum_{i=1}^{n}n^{p\eta}b_{ni}^{p}\widehat{\mathbb{E}}[|X_1|^p]\Big)^N \le C\Big(n^{p\eta}\max_{1\le i\le n}|b_{ni}|^{p-2}\sum_{i=1}^{n}b_{ni}^2\Big)^N\\
&\ll n^{-(1-2/p-p\eta)N}(\log n)^{-N}. \qquad (3.17)
\end{aligned}
$$

Choosing some small η and large $N>1$ such that $(1-2/p-p\eta)N\ge 1$, we get $\sum_{n=1}^{\infty}\mathbb{V}(|T_n^{(3)}|>\varepsilon)<\infty$, which implies $T_n^{(3)}\to 0$, a.s. $\mathbb{V}$.

The proof of $T_n^{(4)}\to 0$, a.s. $\mathbb{V}$ is similar to that of $T_n^{(3)}$, and hence we omit it.

Proof of part (2). We first prove that $\sum_{i=1}^{n}\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]\to 0$.

If $1<p\le 2$, By (3.13) and (3.6), we have

$$
\begin{aligned}
|\sum_{i=1}^{n}\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]| &\le \sum_{i=1}^{n}|\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]| \ll \sum_{i=1}^{n}\widehat{\mathbb{E}}[|b_{ni}X_i|^p n^{\eta(p-1)}]\\
&\le n^{\eta(p-1)-\delta}\to 0
\end{aligned}
$$

if η is chosen small enough such that $\eta(p-1)<\delta$.

If $0<p\le 1$, by (3.12), Markov inequality and (3.1), we have

$$
\begin{aligned}
\Big|\sum_{i=1}^{n}\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]\Big| &\le \sum_{i=1}^{n}|\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]|\\
&\le n^{-\eta}\sum_{i=1}^{n}\mathbb{V}(|b_{ni}X_i|>n^{-\eta})+\sum_{i=1}^{n}\widehat{\mathbb{E}}[|b_{ni}X_i|g(\mu n^{\eta}b_{ni}X_i)]\\
&\ll n^{-(1-p)\eta}\sum_{i=1}^{n}\widehat{\mathbb{E}}[|b_{ni}X_i|^p]+n^{-(1-p)\eta}\sum_{i=1}^{n}\widehat{\mathbb{E}}[|b_{ni}X_i|^p]\\
&\ll n^{-(1-p)\eta}\sum_{i=1}^{n}b_{ni}^p \ll n^{-(1-p)\eta-\delta}\to 0, \qquad \text{for any } \eta>0. \qquad (3.18)
\end{aligned}
$$

Next, we show that $\limsup_{n\to\infty}(T_n^{(1)}-\sum_{i=1}^{n}\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}])\le 0$ a.s. $\mathbb{V}$.

By (3.12), Markov inequality and (3.1), we get

$$
\begin{aligned}
B_n &= \sum_{i=1}^{n}\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}-\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]]^2\\
&\le 2\sum_{i=1}^{n}\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]^2\\
&\ll \sum_{i=1}^{n}\widehat{\mathbb{E}}[(b_{ni}X_i)^2 g(\mu n^{\eta}b_{ni}X_i)]+n^{-2\eta}\sum_{i=1}^{n}\mathbb{V}(|b_{ni}X_i|>n^{-\eta})
\end{aligned}
$$

$$\ll n^{-(2-p)\eta}\sum_{i=1}^{n}\widehat{\mathbb{E}}[|b_{ni}X_i|^p]+n^{-(2-p)\eta}\sum_{i=1}^{n}\widehat{\mathbb{E}}[|b_{ni}X_i|^p]$$
$$\ll n^{-(2-p)\eta-\delta}=o((\log n)^{-1}). \tag{3.19}$$

Therefore (3.15) remains true. Hence, $\limsup_{n\to\infty} T_n^{(1)}\le 0$ a.s. $\mathbb{V}$.

The proof of $T_n^{(2)}=o(1)$ a.s. $\mathbb{V}$ is similar to that of (3.16). Similar to the proof of (3.17), we have

$$\mathbb{V}(|T_n^{(3)}|>\varepsilon)\le C\Big(n^{p\eta}\sum_{i=1}^{n}{b_{ni}}^p\Big)^N\le Cn^{-(\delta-p\eta)N}.$$

Choosing some small η and large $N>1$ such that $(\delta-p\eta)N\ge 1$, we get $\sum_{n=1}^{\infty}\mathbb{V}(|T_n^{(3)}|>\varepsilon)<\infty$, which implies $T_n^{(3)}\to 0$, a.s. $\mathbb{V}$.

The proof of $T_n^{(4)}\to 0$, a.s. $\mathbb{V}$ is similar to that of $T_n^{(3)}$, and hence we omit it. We complete the proof of Theorem 3.3. □

Proof of Theorem 3.5. Without loss of generality, assume that $k_n=n$ for every $n\ge 1$. In order to prove (3.9), for any $\varepsilon>0$, we need to prove:

$$\sum_{n=1}^{\infty}\mathbb{V}(T_n>\varepsilon)=\sum_{n=1}^{\infty}\mathbb{V}\Big(\sum_{i=1}^{n}b_{ni}X_i>\varepsilon\Big)<\infty \tag{3.20}$$

and

$$\sum_{n=1}^{\infty}\mathbb{V}(-T_n>\varepsilon)=\sum_{n=1}^{\infty}\mathbb{V}\Big(\sum_{i=1}^{n}-b_{ni}X_i>\varepsilon\Big)<\infty. \tag{3.21}$$

We only need to prove (3.20). Because of using $\{-X_n;n\ge 1\}$ instead of $\{X_n;n\ge 1\}$ in (3.20), we can obtain (3.21).

For any $1\le i\le n$, $n\ge 1$ and any ε, we choose $N\ge 4$. Let

$$\begin{aligned}
X_{ni}^{(1)}&=-b_{ni}^{-1}n^{-\frac{2}{3p}}I(b_{ni}X_i<-n^{-\frac{2}{3p}})+X_iI(|b_{ni}X_i|\le n^{-\frac{2}{3p}})+b_{ni}^{-1}n^{-\frac{2}{3p}}I(b_{ni}X_i>n^{-\frac{2}{3p}}),\\
X_{ni}^{(2)}&=(X_i+b_{ni}^{-1}n^{-\frac{2}{3p}})I(b_{ni}X_i\le-\varepsilon/N)+(X_i-b_{ni}^{-1}n^{-\frac{2}{3p}})I(b_{ni}X_i\ge\varepsilon/N),\\
X_{ni}^{(3)}&=(X_i-b_{ni}^{-1}n^{-\frac{2}{3p}})I(n^{-\frac{2}{3p}}<b_{ni}X_i<\varepsilon/N),\\
X_{ni}^{(4)}&=(X_i+b_{ni}^{-1}n^{-\frac{2}{3p}})I(-\varepsilon/N<b_{ni}X_i<-n^{-\frac{2}{3p}}),\\
T_n^{(l)}&=\sum_{i=1}^{n}b_{ni}X_{ni}^{(l)},\qquad l=1,2,3,4.
\end{aligned} \tag{3.22}$$

Then $T_n=\sum_{i=1}^{n}b_{ni}X_i=T_n^{(1)}+T_n^{(2)}+T_n^{(3)}+T_n^{(4)}$. Obviously, $\{X_{ni}^{(1)},\ 1\le i\le n,\ n\ge 1\}$ is a sequence of upper extended negatively dependent random variables.

In order to prove (3.20), it suffices to verify that

$$\sum_{i=1}^{n}\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]\to 0, \tag{3.23}$$

$$\sum_{n=1}^{\infty}\mathbb{V}\Big(T_n^{(1)}-\sum_{i=1}^{n}\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]>\varepsilon\Big)<\infty, \tag{3.24}$$

$$\sum_{n=1}^{\infty}\mathbb{V}\big(|T_n^{(l)}|>\varepsilon\big)<\infty,\qquad l=2,3,4. \tag{3.25}$$

We first show $\sum_{i=1}^{n}\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]\to 0$. If $0<p\le 1$, similar to the proof of (3.18), we have

$$\begin{aligned}\Big|\sum_{i=1}^{n}\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]\Big| &\le \sum_{i=1}^{n}|\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]|\\ &\ll n^{-(1-p)\frac{2}{3p}}\sum_{i=1}^{n}\widehat{\mathbb{E}}[|b_{ni}X_i|^p]\ll n^{-(1-p)\frac{2}{3p}}o((\log n)^{-1})\to 0.\end{aligned} \tag{3.26}$$

Note that for $1<p\le 2$

$$\begin{aligned}|X_i-X_{ni}^{(1)}| &= |(X_i+b_{ni}^{-1}n^{-\frac{2}{3p}})I(b_{ni}X_i<-n^{-\frac{2}{3p}})+(X_i-b_{ni}^{-1}n^{-\frac{2}{3p}})I(b_{ni}X_i>n^{-\frac{2}{3p}})\\ &\ll |X_i|I(|X_i|>Cn^{\frac{4}{3p}})\ll |X_i|^p/n^{\frac{4(p-1)}{3p}}.\end{aligned}$$

Hence, for $1<p\le 2$, by the fact $\widehat{\mathbb{E}}[X_i]=0$ and (3.12), we have

$$\begin{aligned}\Big|\sum_{i=1}^{n}\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]\Big| &\le \sum_{i=1}^{n}|\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]|\\ &= \sum_{i=1}^{n}|\widehat{\mathbb{E}}[b_{ni}X_i]-\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]|\le \sum_{i=1}^{n}b_{ni}\widehat{\mathbb{E}}[|X_i-X_{ni}^{(1)}|]\\ &\ll \sum_{i=1}^{n}b_{ni}\widehat{\mathbb{E}}[|X_1|^p]/n^{\frac{4(p-1)}{3p}}\\ &\ll n^{1-2/p}/n^{\frac{4(p-1)}{3p}}=n^{-1/3-\frac{2}{3p}}\to 0.\end{aligned} \tag{3.27}$$

Next, we show that $\sum_{n=1}^{\infty}\mathbb{V}\big(T_n^{(1)}-\sum_{i=1}^{n}\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]>\varepsilon\big)<\infty$. Similar to (3.19), we have

$$\begin{aligned}B_n &= \sum_{i=1}^{n}\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}-\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]]^2\\ &\ll n^{-(2-p)\frac{2}{3p}}\sum_{i=1}^{n}\widehat{\mathbb{E}}[|b_{ni}X_i|^p]\\ &\ll n^{-(2-p)\frac{2}{3p}}o((\log n)^{-1})=o((\log n)^{-1}).\end{aligned} \tag{3.28}$$

Similar to the proof of (3.15), we have $\sum_{n=1}^{\infty}\mathbb{V}\big(T_n^{(1)}-\sum_{i=1}^{n}\widehat{\mathbb{E}}[b_{ni}X_{ni}^{(1)}]>\varepsilon\big)<\infty$.

As to $T_n^{(2)}$, by the definition of $X_{ni}^{(2)}$, (3.12) and (2.4), we get

$$\begin{aligned}\sum_{n=1}^{\infty}\mathbb{V}\big(|T_n^{(2)}|>\varepsilon\big) &\le \sum_{n=1}^{\infty}\sum_{i=1}^{n}\mathbb{V}\big(|X_i|>\varepsilon/(Nb_{ni})\big)\\ &\le \sum_{n=1}^{\infty}\sum_{i=1}^{n}\widehat{\mathbb{E}}\big(1-g(Nb_{ni}X_i/\varepsilon)\big)\\ &\le \sum_{n=1}^{\infty}\sum_{i=1}^{n}\widehat{\mathbb{E}}\big(1-g(Nb_{ni}X_1/\varepsilon)\big)\end{aligned}$$

$$\leq \sum_{n=1}^{\infty} n\mathbb{V}\big(|X_1| > \mu\varepsilon C n^{2/p}/N\big) < \infty. \tag{3.29}$$

As to $T_n^{(3)}$, by the definition of $X_{ni}^{(3)}$, the definition of extended negative dependence, (3.7), (3.12) and (2.5), we have

$$\begin{aligned}
\mathbb{V}(|T_n^{(3)}| > \varepsilon) &\leq \mathbb{V}(\text{there exist at least } N \text{ indices } i \text{ such that } b_{ni}X_i > n^{-\frac{2}{3p}}) \\
&\leq \sum_{1\leq i_1<i_2<\cdots<i_N\leq n} \mathbb{V}\big(b_{ni_1}X_{i_1} > n^{-\frac{2}{3p}}, \cdots, b_{ni_N}X_{i_N} > n^{-\frac{2}{3p}}\big) \\
&\leq \sum_{1\leq i_1<i_2<\cdots<i_N\leq n} \mathbb{V}\big(X_{i_1} > Cn^{\frac{4}{3p}}, \cdots, X_{i_N} > Cn^{\frac{4}{3p}}\big) \\
&\leq C \sum_{1\leq i_1<i_2<\cdots<i_N\leq n} \widehat{\mathbb{E}}\Big(\prod_{j=1}^{N}(1-g(Cn^{-\frac{4}{3p}}X_{i_j}))\Big) \\
&\leq C \sum_{1\leq i_1<i_2<\cdots<i_N\leq n} \prod_{j=1}^{N} \widehat{\mathbb{E}}(1-g(Cn^{-\frac{4}{3p}}X_{i_j})) \\
&= CC_n^N\big[\widehat{\mathbb{E}}(1-g(Cn^{-\frac{4}{3p}}X_1))\big]^N \leq CC_n^N\big[\mathbb{V}(|X_1| > Cn^{\frac{4}{3p}})\big]^N \\
&\leq Cn^N\big[\mathbb{V}(|X_1| > Cn^{\frac{4}{3p}})\big]^N \leq Cn^{-N/3}\big[n^{4/3}\mathbb{V}(|X_1| > Cn^{\frac{4}{3p}})\big]^N \\
&\leq Cn^{-N/3}.
\end{aligned} \tag{3.30}$$

Hence, choosing large $N \geq 4$, we get $\sum_{n=1}^{\infty} \mathbb{V}(|T_n^{(3)}| > \varepsilon) < \infty$.

The proof of $\sum_{n=1}^{\infty} \mathbb{V}(|T_n^{(4)}| > \varepsilon) < \infty$ is similar to that of $T_n^{(3)}$, and hence we omit it. We complete the proof of Theorem 3.5. □

Acknowledgments. The authors would like to thank the editors and the reviewers for their careful reading our paper and constructive comments, which have led to significant improvements of this paper.

Conflict of Interest

The authors declare no conflict of interest.

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