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A Global Optimality Principle for Fully Coupled Mean-field Control Systems

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Abstract This paper concerns a global optimality principle for fully coupled mean-field control systems. Both the first-order and the second-order variational equations are fully coupled mean-field linear FBSDEs. A new *linear relation* is introduced, with which we successfully decouple the fully coupled first-order variational equations. We give a new second-order expansion of Y^ε that can work well in mean-field framework. Based on this result, the stochastic maximum principle is proved. The comparison with the stochastic maximum principle for controlled mean-field stochastic differential equations is supplied.

Keywords optimal control; global maximum principle; fully coupled general mean-field FBSDE; adjoint equation; recursive utility
2020 MR Subject Classification 93E20; 60H10; 35K15

1 Introduction

The purpose of this paper is to investigate a global stochastic maximum principle (SMP) for optimal problem governed by the following fully coupled mean-field control system

$$\begin{cases} dX^v(t) = b(t, \Pi^v(t), \mathbb{P}_{\Lambda^v(t)}, v(t))dt + \sigma(t, \Pi^v(t), \mathbb{P}_{\Lambda^v(t)}, v(t))dW(t), \ t \in [0, T], \\ dY^v(t) = -f(t, \Pi^v(t), \mathbb{P}_{\Lambda^v(t)}, v(t))dt + Z^v(t)dW(t), \ t \in [0, T], \\ X^v(0) = x_0, \ Y^v(T) = \Phi(X^v(T), \mathbb{P}_{X^v(T)}), \end{cases} \tag{1.1}$$

where $\Pi^v(t) = (X^v(t), Y^v(t), Z^v(t))$, $\Lambda^v(t) = (X^v(t), Y^v(t))$; W is a standard d-dimensional Brownian motion; $\mathbb{P}_\xi = \mathbb{P} \circ \xi^{-1}$ is the law of random variable $\xi \in L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$; v is a control process taking values in a set $U \subset \mathbb{R}^l$, not necessarily convex; the coefficients (b, σ, f) : $[0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathcal{P}_2(\mathbb{R}^{n+m}) \times U \to \mathbb{R}$, $\Phi : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}^m$. The accurate assumptions on b, σ, f, Φ are given in Section 3. The cost functional is defined by $J(v(\cdot)) = Y^v(0)$, where $(X^v(\cdot), Y(\cdot), Z(\cdot))$ is the unique solution of the above equation.

Define admissible control set

$$\mathcal{U}_{ad} = \Big\{ v(\cdot) \big| v(\cdot) \text{ is an } \mathcal{F}_t\text{-adapted process with value in } U \text{ such that } \sup_{0\le t\le T} \mathbb{E}|u(t)|^8 < +\infty \Big\}. \tag{1.2}$$

Our control problem can be described as:

Problem (MFFC). Find an admissible control $u^*(\cdot)$ such that

$$J(u^*(\cdot)) = \min_{v\in\mathcal{U}_{ad}} J(v(\cdot)),$$

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subject to (1.1). u^* is called optimal control and $(X^*(\cdot), Y^*(\cdot), Z^*(\cdot))$, the solution of (1.1) with $u^*(\cdot)$, is the optimal trajectory.

The motivation of our work comes from two aspects. i) Recently, the rapid development of the theory of fully coupled general mean-field forward-backward stochastic differential equations (FBSDEs) has made many scholars pay attention to the investigation in related fields, see Chassagneux, Crisan, Delarue[9], Li[20], Pham, Wei[28, 29], Shi, Wen, Xiong[30]. ii) Following Peng's open problem being solved completely by Hu[16], Hu, Ji, Xue[17], it becomes possible to investigate the necessary condition of optimality of system (1.1).

As everyone knows, a powerful tool to study optimal control problems is stochastic maximum principle (SMP). We refer to Kushner [18], Bismut[3], Bensoussan[2] for an early investigation on this topic; refer to Peng[25] for the case where the diffusion coefficients of SDEs depend on control and the control domain is unnecessarily convex. In 1997, El Karoui, Peng, Quenez[11] proposed the notion of more general recursive utilities via the solutions of BSDEs. For those recursive stochastic optimal control problems, a lot of works have been published in the last few decades, such as, Peng[26] obtained a local SMP when the control space is convex. The control problem for nonconvex case is proposed by Peng[27] as an open problem. By regarding $Z(\cdot)$ as a control process and the terminal condition $Y(T) = \Phi(X(T))$ as a constraint, Yong[34] obtained an optimality variational principle by means of Ekeland variation. With similar argument, Wu[33] considered a stochastic recursive optimal control problem. Note that the SMPs obtained in the last two works above contain unknown parameters. In fact, Peng's open problem has not been solved completely by Hu[16] until 2017. Hu, Ji, Xue[17] generalized Hu's work from the decoupled control system to the fully coupled control system. It should be pointed out that in[16, 17] an important observation is the following equality

$$Y^{1,\varepsilon}(t) = p(t)X^{1,\varepsilon}(t), \qquad t \in [0, T], \tag{1.3}$$

where $(X^{1,\varepsilon}(\cdot), Y^{1,\varepsilon}(\cdot))$ is the solution of the first-order variational equation, which is a fully coupled linear FBSDE; $p(\cdot)$ is the solution of the first-order adjoint equation.

As for the optimal problems for mean-field systems, this direction has also drawn great attention, for example, when the control domain is convex, Andersson, Djehiche[1] proved a maximum principle for SDE of mean-field type. In the same action space, Li[19] obtained the SMP in the mean-field controls. If the control domain is unnecessarily convex, we refer to Buckdahn, Djehiche, Li[5] for a general SMP for mean-field SDEs in expectation form, and Buckdahn, Li, Ma[6] for mean-field SDEs in law form, and Hao, Meng[15] for general mean-field forward-backward stochastic systems. The SMP of mean-field type for other various problems were investigated in Du, Huang, Qin[10], Shen, Meng, Shi[31], Guo, Xiong[12] and so on.

There is only a few literature on the SMP of mean-field FBSDEs. Min, Peng, Qin[23] studied fully coupled mean-field FBSDEs and related SMP with convex control domain. Li and Liu[21] considered an optimal control problem for fully coupled mean-field FBSDE in the case where the diffusion coefficient depends on control and the control domain is not assumed to be convex. Hafayed, Tabet, Boukaf[13] proved a SMP for mean-field FBSDE with jump. Wang, Xiao, Xing[32] investigated an optimal control problem for mean-field FBSDE with noisy observation. In all of the above works, the coefficients of the forward-backward systems depend on the expectation of the solution, but not the law of the solution. To our knowledge, up to now, there is no works published on the SMP for fully coupled general mean-field FBSDEs in the existing literature.

Since we need to deal with the fully coupled forward-backward mean-field control system (1.1), there are some potential obstacles met in our analysis. Let us explain it in detail.

First, in[6], the first-order adjoint equation is a mean-field BSDE, which can be obtained by Fubini Theorem. We argue that for the solution of their first-order adjoint equation, we only have

$$\mathbb{E}[Y^{1,\varepsilon}(t)] = \mathbb{E}[p(t)X^{1,\varepsilon}(t)], \qquad t \in [0,T], \tag{1.4}$$

but not the relation (1.3). However, (1.4) is not enough for some estimations in our case, see Remark 4.4. Inspired by the work of Hu, Ji, Xue[17], we propose to split the single adjoint equation into two decoupled equations (see (3.1)) and establish the following *linear relation*:

$$Y^{1,\varepsilon}(t) = p_0(t)X^{1,\varepsilon}(t) + \hat{\mathbb{E}}[\hat{\mathring{p}}_1(t)\hat{X}^{1,\varepsilon}(t)], \qquad t \in [0,T]. \tag{1.5}$$

where $(p_0(\cdot), \hat{\mathring{p}}_1(\cdot))$ is the solution of (3.1). (1.5) plays an very important role in our calculation. Clearly, (1.5) is slightly "stronger" than (1.4) and it is in fact the counterpart of (1.3) in mean-field case. Besides, according to Fubini Theorem, for the process $p(\cdot)$ (the solution of first-order adjoint equation (3.11)[6]) and the pair $(p_0(\cdot), \hat{\mathring{p}}_1(\cdot))$, we have $p(t) = p_0(t) + \hat{\mathbb{E}}[\hat{\mathring{p}}_1(t)],\ t \in [0,T]$ (see (6.4)).

Second, due to the mean-field feature of our system, the second-order expansion of Y^ε given by Hu, Ji, Xue (see Lemma 3.17[17]) does not work in our case. By adopting two new and split adjoint equations, we make the second-order expansion of Y^ε:

$$\begin{aligned} Y^\varepsilon(t) &= p_0(t)(X^{1,\varepsilon}(t) + X^{2,\varepsilon}(t)) + \hat{\mathbb{E}}[\hat{\mathring{p}}_1(t)(\hat{X}^{1,\varepsilon}(t) + \hat{X}^{2,\varepsilon}(t))] + \frac{1}{2}P_0(t)(X^{1,\varepsilon}(t))^2 \\ &\quad + \frac{1}{2}\hat{\mathbb{E}}[\hat{\mathring{P}}_1(t)(\hat{X}^{1,\varepsilon}(t))^2] + \mathcal{M}(t), \end{aligned}$$

where $(p_0(\cdot), \hat{\mathring{p}}_1(\cdot))$ and $(P_0(\cdot), \hat{\mathring{P}}_1(\cdot))$ are the solutions of the first- and second-order adjoint systems, respectively; $\mathcal{M}(\cdot)$ is the solution of some auxiliary mean-field BSDE.

Third, the fact that our control system is a fully coupled mean-field FBSDE leads to the auxiliary BSDE (4.22) appearing in the expansion of Y^ε, which is different to the case of mean-field free[17]. It is difficult to get its precise solution of (4.22). Hence, we use the comparison principle of mean-field SDEs to prove our SMP.

Our paper contributes to the literature in at least three points. To begin with, we propose a method of splitting adjoint equations, and, thereby, establish the *linear relation* between of $X^{1,\varepsilon}$ and $Y^{1,\varepsilon}$. What's more, we show the second-order expansion of Y^ε in mean-field framework with the help of two new adjoint systems. Last but not least, the SMP for optimal control problems governed by fully coupled general mean-field FBSDEs is proved.

This paper is arranged as follows. The preliminaries and Lions' derivative are recalled in Section 2. Section 3 is devoted to the introduction of two new and split adjoint equations and the main result–SMP. In section 4 we list the first- and second-order variational equations as well as show the proof of Theorem 3.4. In section 5 we consider the square integrable case. The relation between Buckdahn et al.'s SMP and our SMP is stated in Section 6. An auxiliary result is given in the last section for closing our paper.

2 Preliminaries

2.1 Notations

Let $\mathbb{R}^n$, $\mathbb{R}^{n\times d}$ denote the n-dimensional real Euclidean space and the space of $n \times d$ real matrices, respectively, on which the scalar product $\langle \cdot, \cdot \rangle$ and the norm $|\cdot|$ are defined as usual, i.e, for $a = (a_i),\ b = (b_i) \in \mathbb{R}^n,\ \langle a, b\rangle = \sum\limits_{i=1}^n a_i b_i,\ \ ||a|| = \sqrt{\sum\limits_{i=1}^n (a_i)^2}$; for $A = (a_{ij}),\ B = (b_{ij}) \in \mathbb{R}^{n\times d}$, $\langle A, B\rangle = \mathrm{tr}\{AB^\intercal\}, ||A|| = \sqrt{\mathrm{tr}\{AA^\intercal\}}$, where $\intercal$ denotes the transpose of matrices or vectors.

Next let us introduce some usual spaces. For $\alpha \geq 1$,

- $L^\alpha(\mathcal{F};\mathbb{R}^n) = \{\mathcal{F}\text{-measurable } \mathbb{R}^n\text{-value random variables } \xi \text{ with } ||\xi||^\alpha_{L^\alpha} = E|\xi|^\alpha < +\infty\}$,
- $\mathcal{S}^\alpha_{\mathbb{F}}(0,T;\mathbb{R}^n) = \Big\{\mathcal{F}_t\text{-adapted } \alpha\text{-th integrable processes } \varphi(\cdot) \text{ over } [0,T] \text{ with } \mathbb{E}\Big[\sup_{0\leq t\leq T}|\varphi(t)|^\alpha\Big] < +\infty\Big\}$,
- $\mathcal{H}^{\alpha,\beta}_{\mathbb{F}}(0,T;\mathbb{R}^n) = \Big\{\mathcal{F}_t\text{-adapted stochastic processes } \varphi(\cdot) \text{ over } [0,T] \text{ with } \Big\{\mathbb{E}\Big[\Big(\int_0^T|\varphi(t)|^\alpha dt\Big)^{\frac{\beta}{\alpha}}\Big]\Big\}^{\frac{1}{\beta}} < +\infty\Big\}$.

Throughout the paper by δ_x we denote the Dirac measure at x; $\rho:(0,+\infty)\to(0,+\infty)$ denotes a function with $\rho(\varepsilon)\to 0$ as $\varepsilon\to 0$; L is a positive constant, which maybe change from line to line; for $p\geq 2$, we define

$$
\begin{aligned}
&\Lambda^p := \Big\{(\varphi,\psi)\Big|\mathbb{E}\Big[\sup_{0\leq t\leq T}|\varphi(t)|^p + \Big(\int_0^T|\psi(t)|^2dt\Big)^{\frac{p}{2}}\Big] < +\infty\Big\},\\
&\Gamma^p := \Big\{(\phi,\varphi,\psi)\Big|\mathbb{E}\Big[\sup_{0\leq t\leq T}(|\phi(t)|^p + |\varphi(t)|^p) + \Big(\int_0^T|\psi(t)|^2dt\Big)^{\frac{p}{2}}\Big] < +\infty\Big\}.
\end{aligned}
\tag{2.1}
$$

2.2 L^p Estimation for Decoupled Mean-field FBSDEs

Suppose the mappings

$$
\begin{aligned}
&b:\Omega\times[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathcal{P}_2(\mathbb{R}^{n+m+m\times d})\to\mathbb{R}^n,\\
&\sigma:\Omega\times[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathcal{P}_2(\mathbb{R}^{n+m+m\times d})\to\mathbb{R}^{n\times d},\\
&f:\Omega\times[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathcal{P}_2(\mathbb{R}^{n+m+m\times d})\to\mathbb{R}^m,\\
&\Phi:\Omega\times\mathbb{R}^n\times\mathcal{P}_2(\mathbb{R}^n)\to\mathbb{R}^m
\end{aligned}
\tag{2.2}
$$

satisfies

Assumption 2.1. i) For given adapted process $(y(\cdot),z(\cdot))$ and $p\geq 2$,

$$
\begin{aligned}
&\mathbb{E}\Big\{|\Phi(0,\delta_0)|^p + \Big(\int_0^T|b(t,0,y(t),z(t),\mathbb{P}_{(0,y(t),z(t))})| + f(t,0,0,0,\delta_{\mathbf{0}})|dt\Big)^p\\
&+\Big(\int_0^T|\sigma(t,0,y(t),z(t),\mathbb{P}_{(0,y(t),z(t))})|^2dt\Big)^{\frac{p}{2}}\Big\} < +\infty,
\end{aligned}
$$

where $\mathbf{0}=(0,0,0)$.
ii) For $x,\bar{x}\in\mathbb{R}^n, y,\bar{y}\in\mathbb{R}^m, z,\bar{z}\in\mathbb{R}^{m\times d}$, $t\in[0,T]$, $\xi,\bar{\xi}\in L^2(\mathcal{F};\mathbb{R}^n),\eta,\bar{\eta}\in L^2(\mathcal{F};\mathbb{R}^m)$, $\zeta,\bar{\zeta}\in L^2(\mathcal{F};\mathbb{R}^{m\times d})$ and $h=b,\sigma$, there exists a constant $C_1>0$ such that $\mathbb{P}$-a.s.,

$$
\begin{aligned}
&|h(t,x,y,z,\mathbb{P}_{(\xi,\eta,\zeta)}) - h(t,\bar{x},y,z,\mathbb{P}_{(\bar{\xi},\eta,\zeta)})| \leq C_1(|x-\bar{x}| + ||\xi-\bar{\xi}||_{L^2}),\\
&|f(t,x,y,z,\mathbb{P}_{(\xi,\eta,\zeta)}) - f(t,\bar{x},\bar{y},\bar{z},\mathbb{P}_{(\bar{\xi},\bar{\eta},\bar{\zeta})})|\\
\leq& C_1(|x-\bar{x}| + |y-\bar{y}| + |z-\bar{z}| + ||\xi-\bar{\xi}||_{L^2} + ||\eta-\bar{\eta}||_{L^2} + ||\zeta-\bar{\zeta}||_{L^2}),\\
&|\Phi(x,\mathbb{P}_\xi) - \Phi(\bar{x},\mathbb{P}_{\bar{\xi}})| \leq C_1(|x-\bar{x}| + ||\xi-\bar{\xi}||_{L^2}).
\end{aligned}
$$

Lemma 2.1. *Let Assumption 2.1 be in force, for $p\geq 2$ and for any given a pari of adapted* *process* $(y(\cdot), z(\cdot))$*, the following decoupled mean-field BSDE:*

$$\begin{cases} dX(t) = b(t, X(t), y(t), z(t), \mathbb{P}_{(X(t),y(t),z(t))})dt \\ \qquad\qquad + \sigma(t, X(t), y(t), z(t), \mathbb{P}_{(X(t),y(t),z(t))})dW(t), \\ dY(t) = -f(t, X(t), Y(t), Z(t), \mathbb{P}_{(X(t),Y(t),Z(t))})dt + Z(t)dW(t), \qquad t \in [0, T], \\ X(0) = x_0, \ Y(T) = \Phi(X(T), \mathbb{P}_{X(T)}) \end{cases} \tag{2.3}$$

exists a unique adapted solution

$$(X(\cdot), Y(\cdot), Z(\cdot)) \in \mathcal{S}^p_{\mathbb{F}}(0, T; \mathbb{R}^n) \times \mathcal{S}^p_{\mathbb{F}}(0, T; \mathbb{R}^m) \times \mathcal{H}^{2,p}_{\mathbb{F}}(0, T; \mathbb{R}^{m\times d}),$$

and, moreover, there exists a constant $K_p > 0$ *depending only on* p, T, C_1 *such that*

$$\begin{aligned} &\mathbb{E}\Big[\sup_{0\le t\le T}(|X(t)|^p + |Y(t)|^p) + \Big(\int_0^T |Z(t)|^2 dt\Big)^{\frac{p}{2}}\Big] \\ \le &K_p\mathbb{E}\Big\{\Big[\int_0^T |b(t, 0, y(t), z(t), \mathbb{P}_{(0,y(t),z(t))})| + |f(t, 0, 0, 0, \delta_{\mathbf{0}})|dt\Big]^p \\ &+ \Big[\int_0^T \sigma(t, 0, y(t), z(t), \mathbb{P}_{(0,y(t),z(t))})|^2 dt\Big]^{\frac{p}{2}} + |\Phi(0, \delta_0)|^p + |x_0|^p\Big\}, \end{aligned} \tag{2.4}$$

where $\mathbf{0} = (0, 0, 0)$.

Proof. Define, for $(t, x, \xi) \in [0, T] \times \mathbb{R}^n \times L^2(\mathcal{F}; \mathbb{R}^n)$,

$$\begin{aligned} \bar{b}(t, x, [\mathbb{P} \circ (y(\cdot), z(\cdot))^{-1}]_\xi) :&= b(t, x, y(t), z(t), \mathbb{P}_{(\xi,y(t),z(t))}), \\ \bar{\sigma}(t, x, [\mathbb{P} \circ (y(\cdot), z(\cdot))^{-1}]_\xi) :&= \sigma(t, x, y(t), z(t), \mathbb{P}_{(\xi,y(t),z(t))}), \end{aligned}$$

where $[\mathbb{P} \circ (y(\cdot), z(\cdot))^{-1}]$ denotes the law induced by the pair $(y(\cdot), z(\cdot))$. From Assumption 2.1 we know, for $h = \bar{b}, \bar{\sigma}$, and $(t, x) \in [0, T] \times \mathbb{R}^n$,

$$|h(t, x, [\mathbb{P} \circ (y(\cdot), z(\cdot))^{-1}]_\xi) - h(t, x, [\mathbb{P} \circ (y(\cdot), z(\cdot))^{-1}]_{\bar{\xi}})| \le C_1\big(|x - \bar{x}| + ||\xi - \bar{\xi}||_{L^2}\big),$$

and

$$\mathbb{E}\Big[\Big(\int_0^T |\bar{b}(t, 0, \delta_0^{(y(\cdot),z(\cdot))})|dt\Big)^p + \Big(\int_0^T |\bar{\sigma}(t, 0, \delta_0^{(y(\cdot),z(\cdot))})|^2 dt\Big)^{\frac{p}{2}}\Big] < +\infty,$$

where $\delta^{(y(\cdot),z(\cdot))}_\cdot$ denotes the Dirac measure corresponding to the induced measure $[\mathbb{P}\circ(y(\cdot), z(\cdot))^{-1}]$. From Burkholder-Davis-Gundy inequality and Gronwall lemma, we know that, for $p \ge 2$, the e-quation (2.3) possesses a unique solution $X \in \mathcal{S}^p_{\mathbb{F}}(0, T; \mathbb{R}^n)$ and, moreover, there exists a $K_p > 0$ depending only on p, T, C_1 such that

$$\begin{aligned} &\mathbb{E}\Big[\sup_{0\le t\le T}|X(t)|^p\Big] \\ \le &K_p\mathbb{E}\Big\{\Big(\int_0^T |\bar{b}(t, 0, \delta_0^{(y(\cdot),z(\cdot))})|dt\Big)^p + \Big(\int_0^T |\bar{\sigma}(t, 0, \delta_0^{(y(\cdot),z(\cdot))})|^2 dt\Big)^{\frac{p}{2}} + |x_0|^p\Big\}, \end{aligned}$$

i.e.,

$$\mathbb{E}\Big[\sup_{0\le t\le T}|X(t)|^p\Big] \le K_p\mathbb{E}\Big\{\Big(\int_0^T |b(t, 0, y(t), z(t), \mathbb{P}_{(0,y(t),z(t))})|dt\Big)^p$$

$$+\Big(\int_0^T|\sigma(t,0,y(t),z(t),\mathbb{P}_{(0,y(t),z(t))})|^2dt\Big)^{\frac{p}{2}}+|x_0|^p\Big\}. \tag{2.5}$$

Once knowing $X(\cdot)$, the second equation in (2.3) becomes a mean-field BSDE. By Corollary 5.3[24] (setting $dN(t)=(X(t)+||X(t)||_{L^2}+|f(t,0,0,0,\delta_{\mathbf{0}})|)dt, dV(t)=dt, dR(t)=0, dD(t)=0$) and (2.5), we have (2.4). □

2.3 L^p Estimation for Coupled Mean-field FBSDEs

In this subsection we prove L^p estimation for fully coupled mean-field FBSDEs on a short time interval via Lemma 2.1.

Let the mappings given in (2.2) satisfy the following assumptions:

Assumption 2.2. i) There exist three constants $L_i,\ i=1,2,3$ such that, for $x,\bar{x}\in\mathbb{R}^n, y,\bar{y}\in\mathbb{R}^m, z,\bar{z}\in\mathbb{R}^{m\times d}$, $\xi,\bar{\xi}\in L^2(\mathcal{F};\mathbb{R}^n)$, $\eta,\bar{\eta}\in L^2(\mathcal{F};\mathbb{R}^m)$, $\zeta,\bar{\zeta}\in L^2(\mathcal{F};\mathbb{R}^{m\times d})$, $t\in[0,T]$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&|b(t,x,y,z,\mathbb{P}_{(\xi,\eta,\zeta)})-b(t,\bar{x},\bar{y},\bar{z},\mathbb{P}_{(\bar{\xi},\bar{\eta},\bar{\zeta})})|\\
\le& C_1(|x-\bar{x}|+||\xi-\bar{\xi}||_{L^2})+C_2(|y-\bar{y}|+|z-\bar{z}|+||\eta-\bar{\eta}||_{L^2}+||\zeta-\bar{\zeta}||_{L^2}),\\
&|\sigma(t,x,y,z,\mathbb{P}_{(\xi,\eta,\zeta)})-\sigma(t,\bar{x},\bar{y},\bar{z},\mathbb{P}_{(\bar{\xi},\bar{\eta},\bar{\zeta})})|\\
\le& C_1(|x-\bar{x}|+||\xi-\bar{\xi}||_{L^2})+C_2(|y-\bar{y}|+||\eta-\bar{\eta}||_{L^2})+C_3(|z-\bar{z}|+||\zeta-\bar{\zeta}||_{L^2}),\\
&|f(t,x,y,z,\mathbb{P}_{(\xi,\eta,\zeta)})-f(t,\bar{x},\bar{y},\bar{z},\mathbb{P}_{(\bar{\xi},\bar{\eta},\bar{\zeta})})|\\
\le& C_1(|x-\bar{x}|+||\xi-\bar{\xi}||_{L^2}+|y-\bar{y}|+||\eta-\bar{\eta}||_{L^2}+|z-\bar{z}|+||\zeta-\bar{\zeta}||_{L^2}).
\end{aligned}$$

ii) For some given real constant $\beta>1$, $\Phi(0,\delta_0)\in L^\beta(\mathcal{F};\mathbb{R}^m)$, $b(t,0,0,0,\delta_{\mathbf{0}})\in\mathcal{H}^{1,\beta}_{\mathbb{F}}(0,T;\mathbb{R}^n)$, $f(t,0,0,0,\delta_{\mathbf{0}})\in\mathcal{H}^{1,\beta}_{\mathbb{F}}(0,T;\mathbb{R}^m)$, $\sigma(t,0,0,0,\delta_{\mathbf{0}})\in\mathcal{H}^{2,\beta}_{\mathbb{F}}(0,T;\mathbb{R}^{n\times d})$, where $\mathbf{0}=(0,0,0)$.

For $p\ge 2$, define

$$\Theta_p:=K_p4^p(1+T)^p(\max\{C_2,C_3\})^p,$$

where K_p is given in (2.4).

Theorem 2.2. *Under Assumption 2.2, for $p\ge2$, if $\Theta_p<1$, the following fully coupled mean-field FBSDE:*

$$\begin{cases}
dX(t)=b(t,\Pi(t),\mathbb{P}_{\Pi(t)})dt+\sigma(t,\Pi(t),\mathbb{P}_{\Pi(t)})dW(t), & t\in[0,T],\\
dY(t)=-f(t,\Pi(t),\mathbb{P}_{\Pi(t)})dt+Z(t)dW(t), & t\in[0,T],\\
X_0=x_0, & Y(T)=\Phi(X(T),\mathbb{P}_{X(T)})
\end{cases} \tag{2.6}$$

admits a unique solution $(X(\cdot),Y(\cdot),Z(\cdot))\in\mathcal{S}^p_{\mathbb{F}}(0,T;\mathbb{R}^n)\times\mathcal{S}^p_{\mathbb{F}}(0,T;\mathbb{R}^m)\times\mathcal{H}^{2,p}_{\mathbb{F}}(0,T;\mathbb{R}^{m\times d})$, and there exists a positive constant $K_p>0$ depending on p,T,L_1,L_2,L_3 such that

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T}(|X(t)|^p+|Y(t)|^p)+\Big(\int_0^T|Z(t)|^2dt\Big)^{\frac{p}{2}}\Big]\\
\le& K_p\mathbb{E}\Big\{\Big(\int_0^T|b(t,\mathbf{0},\delta_{\mathbf{0}})|dt\Big)^p+\Big(\int_0^T|f(t,\mathbf{0},\delta_{\mathbf{0}})|dt\Big)^p\\
&+\Big(\int_0^T|\sigma(t,\mathbf{0},\delta_{\mathbf{0}})|^2dt\Big)^{\frac{p}{2}}+|\Phi(0,\delta_0)|^p+|x_0|^p\Big\},
\end{aligned} \tag{2.7}$$

where $\Pi(t)=(X(t),Y(t),Z(t))$, $\mathbf{0}=(0,0,0)$.

Proof. Given a pair of adapted process $(y(\cdot), z(\cdot))$, consider

$$\begin{cases} dX(t) = b(t, X(t), y(t), z(t), \mathbb{P}_{(X(t),y(t),z(t))})dt \\ \qquad\qquad + \sigma(t, X(t), y(t), z(t), \mathbb{P}_{(X(t),y(t),z(t))})dW(t), \qquad t \in [0,T], \\ dY(t) = -f(t, X(t), Y(t), Z(t), \mathbb{P}_{(X(t),Y(t),Z(t))})dt + Z(t)dW(t), \qquad t \in [0,T], \\ X(0) = x_0, \qquad Y(T) = \Phi(X(T), \mathbb{P}_{X(T)}). \end{cases} \tag{2.8}$$

Suppose $(y(\cdot), z(\cdot)) \in \Lambda^p$ (see (2.1)). Thanks to Assumption 2.2 and Lemma 2.1, we have $X(\cdot) \in \mathcal{S}^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$ and $(Y(\cdot), Z(\cdot)) \in \Lambda^p$, which allows to define a mapping $\Upsilon : \Lambda^p \to \Lambda^p$ by $\Upsilon(y(\cdot), z(\cdot)) = (Y(\cdot), Z(\cdot))$.

Next let us show that Υ is contractive. In fact, let $(y^i(\cdot), z^i(\cdot)) \in \Lambda^p$, $i = 1, 2$ and by $(X^i(\cdot), Y^i(\cdot), Z^i(\cdot)), i = 1, 2$ we denote the solution of (2.8) with $(y^i(\cdot), z^i(\cdot))$, $i = 1, 2$.

Set $\Delta X = X_1 - X_2$, $\Delta Y = Y_1 - Y_2$, $\Delta Z = Z_1 - Z_2$, $\Delta y = y_1 - y_2, \Delta z = z_1 - z_2$. Then

$$\begin{cases} d\Delta X(t) = \big\{\alpha_1(t)\Delta X(t) + \beta_1(t)\Delta y(t) + \gamma_1(t)\Delta z(t) \\ \qquad\qquad + \bar{\alpha}_1(t)||\Delta X(t)||_{L^2} + \bar{\beta}_1(t)||\Delta Y(t)||_{L^2} + \bar{\gamma}_1(t)||\Delta z(t)||_{L^2}\big\}dt \\ \qquad\qquad + \big\{\alpha_2(t)\Delta X(t) + \beta_2(t)\Delta y(t) + \gamma_2(t)\Delta z(t) \\ \qquad\qquad\quad + \bar{\alpha}_2(t)||\Delta X(t)||_{L^2} + \bar{\beta}_2(t)||\Delta Y(t)||_{L^2} + \bar{\gamma}_2(t)||\Delta z(t)||_{L^2}\big\}dW(t), \\ d\Delta Y(t) = -\big\{\alpha_3(t)\Delta X(t) + \beta_3(t)\Delta y(t) + \gamma_3(t)\Delta z(t) \\ \qquad\qquad + \bar{\alpha}_3(t)||\Delta X(t)||_{L^2} + \bar{\beta}_3(t)||\Delta Y(t)||_{L^2} + \bar{\gamma}_3(t)||\Delta z(t)||_{L^2}\big\} + \Delta Z(t)dW(t), \\ \Delta X(0) = 0, \qquad \Delta Y(t) = \alpha_4(T)\Delta X(T) + \bar{\alpha}_4(T)||\Delta X(T)||_{L^2}, \end{cases}$$

where

$$\alpha_1(t) = \begin{cases} \dfrac{b(t,\pi_1(t),\mathbb{P}_{\pi_1(t)}) - b(t,\pi_2(t),\mathbb{P}_{\pi_1(t)})}{X_1(t) - X_2(t)}, & \text{if } X_1(t) \neq X_2(t), \\ 0, & \text{if } X_1(t) = X_2(t), \end{cases}$$

$$\bar{\alpha}_1(t) = \begin{cases} \dfrac{b(t,\pi_2(t),\mathbb{P}_{\pi_1(t)}) - b(t,\pi_2(t),\mathbb{P}_{\pi_2(t)})}{||X_1(t) - X_2(t)||_{L^2}}, & \text{if } ||X_1(t) - X_2(t)||_{L^2} \neq 0, \\ 0, & \text{if } ||X_1(t) - X_2(t)||_{L^2} = 0, \end{cases}$$

and $\pi_1(t) = (X_1(t), y_1(t), z_1(t)), \pi_2(t) = (X_2(t), y_1(t), z_1(t))$. $\beta_1, \gamma_1, \cdots \bar{\alpha}_4$ can be understood in the same manner.

From Assumption 2.2, we know that $\alpha_i, \beta_i, \gamma_i, \bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i, i = 1, 2, 3, 4$ are bounded. Since for any square integrable variable ξ, $g(\mathbb{P}_\xi) := \mathbb{E}|\xi|^2 = \int_{\mathbb{R}^n} x^2 \mathbb{P}_\xi(dx)$, thanks to Lemma 2.1, it follows, for $p \geq 2$,

$$\begin{aligned} &\mathbb{E}\Big[\sup_{0\le t\le T}\big(|\Delta X(t)|^p + |\Delta Y(t)|^p\big) + \Big(\int_0^T |\Delta Z(t)|^2 dt\Big)^{\frac{p}{2}}\Big] \\ \le& \mathbb{E}\Big\{\Big(\int_0^T (|\beta_1(t)||\Delta y(t)| + |\gamma_1(t)||\Delta z(t)| + |\bar{\beta}_1(t)|||\Delta y||_{L^2} + |\bar{\gamma}_1(t)|||\Delta z||_{L^2})dt\Big)^p \\ &+ \Big(\int_0^T (|\beta_2(t)||\Delta y(t)| + |\gamma_2(t)||\Delta z(t)| + |\bar{\beta}_2(t)|||\Delta y||_{L^2} + |\bar{\gamma}_2(t)|||\Delta z||_{L^2})^2 dt\Big)^{\frac{p}{2}}\Big\} \\ \le& K_p 4^p (1+T)^p (\max\{C_2, C_3\})^p \mathbb{E}\Big\{\sup_{0\le t\le T}|\Delta y(t)|^p + \Big(\int_0^T |\Delta z(t)|^2 dt\Big)^{\frac{p}{2}}\Big\}. \end{aligned}$$

Due to $\Theta_p = K_p 4^p (1+T)^p (\max\{C_2, C_3\})^p < 1$, the contractive mapping theorem allows to show that the mapping Υ exists a unique fixed point $(Y(\cdot), Z(\cdot)) \in \Lambda^p$. Then according to the existence and uniqueness theorem of mean-field SDEs (see, for example, Hao and Li[14] for jump case), the forward equation in (2.8) possesses a unique solution $X(\cdot)$ for this fixed point $(Y(\cdot), Z(\cdot))$. From this, one can see that $(X(\cdot), Y(\cdot), Z(\cdot))$ is the unique solution of (2.6).

(2.7) can be obtained following the argument of the proof of Theorem 2.2[17]. Hence, we omit it. □

2.4 Lions' Derivative

Let $\mathcal{P}_2(\mathbb{R}^d)$ be the space of all square integrable probability measures over $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, which is endowed with 2-Wasserstein metric: for $\nu_1, \nu_2 \in \mathbb{R}^d$,

$$W_2(\nu_1,\nu_2) = \inf\Big\{\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} |y_1-y_2|^2\rho(dy_1,dy_2)\Big)^{\frac{1}{2}},\ \rho\in\mathcal{P}_2(\mathbb{R}^{2d})\ \text{ satisfying}$$
$$\rho(A\times\mathbb{R}^d)=\nu_1(A),\ \rho(\mathbb{R}^d\times B)=\nu_2(B),\ A,B\in\mathcal{B}(\mathbb{R}^d)\Big\}.$$

Now we introduce the differentiability of a function with respect to a measure following the idea of Lions. Suppose the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is "rich enough", i.e., for each $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, there exists a random variable $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ ($L^2(\mathcal{F}; \mathbb{R}^d)$ for short) such that $\mathbb{P}_\xi = \mu$. Let $f : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and define the "lift" function $\bar f$ by $\bar f(\xi) := f(\mathbb{P}_\xi),\ \xi \in L^2(\mathcal{F}; \mathbb{R}^d)$. We call that f is differentiable in $\mu_0 = \mathbb{P}_{\xi_0}$, if the "lift" function $\bar f$ is differentiable at ξ_0 in Fréchet sense. That means there exists a linear continuous mapping $D\bar f(\xi_0) : L^2(\mathcal{F}; \mathbb{R}^d) \to \mathbb{R}$ such that for $\zeta \in L^2(\mathcal{F}; \mathbb{R}^d)$,

$$\bar f(\xi_0+\zeta) - \bar f(\xi_0) = D\bar f(\xi_0)(\zeta) + o(||\zeta||_{L^2}),$$

with $||\zeta||_{L^2} \to 0$. According to Riesz's Representation Theorem, there exists an $\eta \in L^2(\mathcal{F}; \mathbb{R}^d)$ such that $D\bar f(\xi_0)(\zeta) = \mathbb{E}[\eta \cdot \zeta]$. The random variable η is in fact of the form $h(\xi_0)$, where $h(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is Borel function depending on the law of ξ_0, but not the random variable ξ_0 itself. Hence, we have, for $\zeta \in L^2(\mathcal{F}; \mathbb{R}^d)$,

$$f(\mathbb{P}_{\xi_0+\zeta}) - f(\mathbb{P}_{\xi_0}) = \mathbb{E}[h(\xi_0)\cdot\zeta] + o(||\zeta||_{L^2}).$$

The function $\partial_\mu f(\mathbb{P}_{\xi_0}; a) := h(a), a \in \mathbb{R}^d$ is called the derivative of $f : \mathcal{P}_2(\mathbb{R}^2) \to \mathbb{R}$ at $\mathbb{P}_{\xi_0}$. Note that $\partial_\mu f(\mathbb{P}_{\xi_0}; a)$ is only $\mathbb{P}_{\xi_0}(da)$-a.e. uniquely determined (see[8] for more detail).

Now we explain the Lions' derivative by an example.

Example 1. Assume $\varphi : \mathbb{R} \to \mathbb{R},\ \psi : \mathbb{R} \to \mathbb{R},\ \phi : \mathbb{R}^2 \to \mathbb{R}$ are three continuously differentiable functions with bounded derivatives. Define for $\xi, \eta \in L^2(\mathcal{F}; \mathbb{R})$,

$$h(\mathbb{P}_\xi) := \varphi(\mathbb{E}[\psi(\xi)]), \qquad g(\mathbb{P}_{(\xi,\eta)}) := \varphi(\mathbb{E}[\phi(\xi,\eta)]).$$

Then

$$\begin{aligned}
&\partial_\nu h(\mathbb{P}_\xi) = \varphi'(\mathbb{E}[\psi(\xi)])\psi'(a),\ \partial_{\nu a} h(\mathbb{P}_\xi) = \varphi'(\mathbb{E}[\psi(\xi)])\psi''(a),\ a\in\mathbb{R},\\
&\partial_{\mu_1} g(\mathbb{P}_{(\xi,\eta)}; a_1,a_2) = (\partial_\mu g)_1(\mathbb{P}_{(\xi,\eta)}; a_1,a_2) = \varphi'(\mathbb{E}[\psi(\xi,\eta)])\phi_{a_1}(a_1,a_2),\\
&\partial_{\mu_1 a_1} g(\mathbb{P}_{(\xi,\eta)}; a_1,a_2) = \varphi'(\mathbb{E}[\psi(\xi,\eta)])\phi_{a_1a_1}(a_1,a_2),\ a_1,a_2\in\mathbb{R}.
\end{aligned}$$

$\partial_{\mu_2} g(\mathbb{P}_{(\xi,\eta)}; a_1, a_2)$ and $\partial_{\mu_2 a_2} g(\mathbb{P}_{(\xi,\eta)}; a_1, a_2)$ can be understood similarly.

In particular, if $\varphi(x) = x, \psi(x) = x,\ \phi(x_1, x_2) = x_1 + x_2,\ x, x_1, x_2 \in \mathbb{R}$, i.e., $h(\mathbb{P}_\xi) := \mathbb{E}[\xi]$, $g(\mathbb{P}_{(\xi,\eta)}) := \mathbb{E}[\xi] + \mathbb{E}[\eta]$, we have

$$\partial_\nu h(\mathbb{P}_\xi) = 1, \quad \partial_{\nu a} h(\mathbb{P}_\xi) = 0, \quad \partial_{\mu_1} g(\mathbb{P}_{(\xi,\eta)}; a_1,a_2) = 1, \quad \partial_{\mu_1 a_1} g(\mathbb{P}_{(\xi,\eta)}; a_1,a_2) = 0.$$

3 SMP for Bounded Cases

In this section we show the main result–SMP. For simplicity of editing, let us restrict $m = n = d = 1$. However, our results also hold true for multidimensional case. Recall that U is a subset of $\mathbb{R}$, unnecessarily convex.

Suppose the mappings

$$\begin{aligned}&(b, \sigma, f) : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}^2) \times U \to \mathbb{R},\\ &\Phi : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}\end{aligned}$$

satisfy

Assumption 3.1. For $h = b, \sigma, f$,

i) h, h_x, h_y, h_z, h_μ and Φ, Φ_ν are continuous with respect to (x, y, z, μ, u) and (x, ν), separately; $h_x, h_y, h_z, h_\mu, \Phi_\nu$ are bounded; h and Φ are linear growth with respect to their respective variable, i.e., there exists a constant $C_0 > 0$ such that

$$\begin{aligned}&|h(t, x, y, z, \mu, v)| \leq C_0\Big(1 + |x| + |y| + |z| + \Big(\int_{\mathbb{R}^2} a^2 \mu(da)\Big)^{\frac{1}{2}} + |v|\Big),\\ &|\Phi(x, \nu)| \leq C_0\Big(1 + |x| + \Big(\int_{\mathbb{R}} a^2 \nu(da)\Big)^{\frac{1}{2}}\Big),\end{aligned}$$

and for $z \in \mathbb{R}$, $v, \bar{v} \in U$, $t \in [0, T]$,

$$|h(t, 0, 0, z, \delta_0, v) - h(t, 0, 0, z, \delta_0, \bar{v})| \leq C_0(1 + |v| + |\bar{v}|).$$

ii) For arbitrary $2 \leq \beta \leq 8$, $\Theta_\beta = K_\beta 4^\beta (1 + T^\beta)(\max\{C_2, C_3\})^\beta$, where K_β is given in (2.4) with $C_1 = \max\{||b_x||_\infty, ||b_{\mu_1}||_\infty, ||\sigma_x||_\infty, ||\sigma_{\mu_1}||_\infty, ||f_x||_\infty, ||f_y||_\infty, ||f_z||_\infty, ||f_{\mu_1}||_\infty, ||f_{\mu_2}||_\infty, ||\Phi_x||_\infty, ||\Phi_\nu||_\infty\}$, $C_2 = \max\{||b_y||_\infty, ||b_z||_\infty, ||b_{\mu_2}||_\infty, ||\sigma_y||_\infty, ||\sigma_{\mu_2}||_\infty\}$, $C_3 = ||\sigma_z||_\infty$.

iii) All the second-order derivatives of h and Φ with respect to (x, y, z, μ) are bounded and continuous in (x, y, z, μ, v), and (x, ν), respectively.

Hamiltonian Functions: For $x, y, z \in \mathbb{R}, \mu \in \mathcal{P}(\mathbb{R}^2), p_0, q_0, p_1, q_{12} \in \mathbb{R}$, we define

$$\begin{aligned}&H^0(t, x, y, z, \mu, u, p_0, q_0) = b(t, x, y, z, \mu, u)p_0 + \sigma(t, x, y, z, \mu, u)q_0 + f(t, x, y, z, \mu, u),\\ &H^1(t, x, y, z, \mu, u, p_1, q_{12}) = b(t, x, y, z, \mu, u)p_1 + \sigma(t, x, y, z, \mu, u)q_{12}.\end{aligned}$$

Next let us introduce some notations used in our setting. Let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ be an intermediate probability space and independent of $(\Omega, \mathcal{F}, \mathbb{P})$, $\bar{B}$ a 1-dimensional Brownian motion over this space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, $\bar{\mathbb{E}}$ the expectation under probability $\bar{\mathbb{P}}$. Let $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{B}, \hat{\mathbb{E}})$ be the independent copy of $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{B}, \bar{\mathbb{E}})$, which means that

i) $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ is independent of $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$;

ii) $\hat{\mathbb{P}}_{\hat{\xi}} = \bar{\mathbb{P}}_{\bar{\xi}}$, $\bar{\xi} \in L^1(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, $\hat{\xi} \in L^1(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$.

$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{B}, \tilde{\mathbb{E}})$ can be understood similarly.

By $\bar{\varphi}(\cdot)$ we denote the stochastic process defined on space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, i.e., $\bar{\varphi}(t) = \varphi(t, \bar{\omega}),\ t \in [0, T],\ \bar{\omega} \in \bar{\Omega}$; $\hat{\bar{\varphi}}(\cdot)$ the stochastic process over product space $(\hat{\Omega} \times \bar{\Omega}, \hat{\mathcal{F}} \otimes \bar{\mathcal{F}}, \hat{\mathbb{P}} \otimes \bar{\mathbb{P}})$, i.e., $\hat{\bar{\varphi}}(t) = \varphi(t, \hat{\omega}, \bar{\omega}), t \in [0, T], (\hat{\omega}, \overline{\omega}) \in \hat{\Omega} \times \bar{\Omega}$; $\bar{\mathring{\varphi}}(\cdot)$ the stochastic process over product space $(\Omega \times \bar{\Omega}, \mathcal{F} \otimes \bar{\mathcal{F}}, \mathbb{P} \otimes \bar{\mathbb{P}})$, i.e., $\bar{\mathring{\varphi}}(t) = \varphi(t, \omega, \bar{\omega}),\ t \in [0, T], (\omega, \overline{\omega}) \in \Omega \times \bar{\Omega}$. Similarly, $\mathring{\bar{\varphi}}(\cdot)$ denotes the stochastic process over product space $(\bar{\Omega} \times \Omega, \bar{\mathcal{F}} \otimes \mathcal{F}, \bar{\mathbb{P}} \otimes \mathbb{P})$, i.e., $\mathring{\bar{\varphi}}(t) = \varphi(t, \bar{\omega}, \omega),\ t \in [0, T], (\overline{\omega}, \omega) \in \bar{\Omega} \times \Omega$. Moreover, due to the independence of $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ and $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$, the expectation of any random variable defined on product space $(\hat{\Omega} \times \bar{\Omega}, \hat{\mathcal{F}} \otimes \bar{\mathcal{F}}, \hat{\mathbb{P}} \otimes \bar{\mathbb{P}})$ can be calculated as follows:

$$\bar{\mathbb{E}}\hat{\mathbb{E}}[\hat{\bar{\xi}}] = \bar{\mathbb{E}}\Big[\int_{\hat{\Omega}} \xi(\bar{\omega}, \hat{\omega}) d\hat{\mathbb{P}}\Big] = \int_{\bar{\Omega}} \Big[\int_{\hat{\Omega}} \xi(\bar{\omega}, \hat{\omega}) d\hat{\mathbb{P}}\Big] d\bar{\mathbb{P}}.$$

Suppose $v(\cdot)$ is an admissible control. For $\phi = b, \sigma, f, \Phi,\ \ell = x, y, z$, define

$$
\begin{aligned}
&\phi(t) = \phi(t, X^*(t), Y^*(t), Z^*(t), \mathbb{P}_{(X^*(t),Y^*(t))}, u^*(t)),\\
&\phi_\ell(t) = \phi_\ell(t, X^*(t), Y^*(t), Z^*(t), \mathbb{P}_{(X^*(t),Y^*(t))}, u^*(t)),\\
&\delta\phi(t) = \phi(t, X^*(t), Y^*(t), Z^*(t), \mathbb{P}_{(X^*(t),Y^*(t))}, v(t)) - \phi(t),\\
&\delta\phi_\ell(t) = \phi_\ell(t, X^*(t), Y^*(t), Z^*(t), \mathbb{P}_{(X^*(t),Y^*(t))}, v(t)) - \phi_\ell(t),\\
&\delta\phi(t,\Xi) = \phi(t, X^*(t), Y^*(t), Z^*(t) + \Xi(t), \mathbb{P}_{(X^*(t),Y^*(t))}, v(t)) - \phi(t),\\
&\delta\phi_\ell(t,\Xi) = \phi_\ell(t, X^*(t), Y^*(t), Z^*(t) + \Xi(t), \mathbb{P}_{(X^*(t),Y^*(t))}, v(t)) - \phi_\ell(t),
\end{aligned}
$$

where $\Xi(\cdot)$ is an $\mathcal{F}_t$-adapted process, and for $\theta = \mu_1, \mu_2$,

$$
\begin{aligned}
&\hat{\mathring{\phi}}_\theta(t) := \phi_\theta(t, X^*(t), Y^*(t), Z^*(t), \mathbb{P}_{(X^*(t),Y^*(t))}, u^*(t); \hat{X}^*(t), \hat{Y}^*(t)),\\
&\delta\hat{\mathring{\phi}}_\theta(t) := \phi_\theta(t, X^*(t), Y^*(t), Z^*(t), \mathbb{P}_{(X^*(t),Y^*(t))}, v(t); \hat{X}^*(t), \hat{Y}^*(t)) - \hat{\mathring{\phi}}_\theta(t).
\end{aligned}
$$

Our first-order adjoint system consists of the following two BSDEs

$$
\begin{cases}
dp_0(t) = -\big\{H_x^0(t) + p_0(t)H_y^0(t) + k_0(t)H_z^0(t)\big\}dt + q_0(t)dW(t), \qquad t\in[0,T],\\
d\hat{\mathring{p}}_1(t) = -\hat{\mathring{F}}_1(t)dt + \hat{\mathring{q}}_{11}(t)dW(t) + \hat{\mathring{q}}_{12}(t)d\hat{W}(t), \qquad t\in[0,T],\\
p_0(T) = \Phi_x(T), \qquad \hat{\mathring{p}}_1(T) = \hat{\mathring{\Phi}}_\nu(T),
\end{cases}
\tag{3.1}
$$

where

$$
\begin{aligned}
\hat{\mathring{F}}_1(t) &= H_y^0(t)\hat{\mathring{p}}_1(t) + H_z^0(t)\hat{\mathring{k}}_1(t) + \hat{\mathring{H}}^0_{\mu_1}(t) + \hat{\mathring{H}}^0_{\mu_2}(t)\hat{p}_0(t) + \bar{\mathbb{E}}[\bar{\mathring{H}}^0_{\mu_2}(t)\hat{\bar{p}}_1(t)]\\
&\quad + \hat{\mathring{H}}^1_x(t) + \hat{\mathring{H}}^1_y(t)\hat{p}_0(t) + \hat{\mathring{H}}^1_z(t)\hat{k}_0(t) + \bar{\mathbb{E}}[\bar{\mathring{H}}^1_y(t)\hat{\bar{p}}_1(t)] + \bar{\mathbb{E}}[\bar{\mathring{H}}^1_z(t)\hat{\bar{k}}_1(t)]\\
&\quad + \bar{\mathbb{E}}[\hat{\bar{\mathring{H}}}^1_{\mu_1}(t)] + \bar{\mathbb{E}}[\hat{\bar{\mathring{H}}}^1_{\mu_2}(t)\bar{p}_0(t)] + \bar{\mathbb{E}}\tilde{\mathbb{E}}[\tilde{\bar{\mathring{H}}}^1_{\mu_2}(t)\hat{\tilde{p}}_1(t)],\\
k_0(t) &= (1-\sigma_z(t)p_0(t))^{-1}(p_0(t)\sigma_x(t) + \sigma_y(t)(p_0(t))^2 + q_0(t)),\\
\hat{\mathring{k}}_1(t) &= (1-\sigma_z(t)p_0(t))^{-1}\big\{\sigma_y(t)p_0(t)\hat{\mathring{p}}_1(t) + \hat{\mathring{\sigma}}_{\mu_1}(t)p_0(t) + \hat{\mathring{\sigma}}_{\mu_2}(t)p_0(t)\hat{p}_0(t)\\
&\qquad\qquad + \hat{\mathring{q}}_{11}(t) + p_0(t)\bar{\mathbb{E}}[\bar{\mathring{\sigma}}_{\mu_2}(t)\hat{\bar{p}}_1(t)]\big\},\\
H_x^0(t) &= b_x(t)p_0(t) + \sigma_x(t)q_0(t) + f_x(t),\ \ \hat{\mathring{H}}^0_{\mu_1}(t) = \hat{\mathring{b}}_{\mu_1}(t)p_0(t) + \hat{\mathring{\sigma}}_{\mu_1}(t)q_0(t) + \hat{\mathring{f}}_{\mu_1}(t),\\
\hat{\mathring{H}}^1_x(t) &= \hat{b}_x(t)\hat{\mathring{p}}_1(t) + \hat{\sigma}_x(t)\hat{\mathring{q}}_{12}(t), \qquad \hat{\bar{\mathring{H}}}^1_{\mu_1}(t) = \hat{\bar{b}}_{\mu_1}(t)\bar{\mathring{p}}_1(t) + \hat{\bar{\sigma}}_{\mu_1}(t)\bar{\mathring{q}}_{12}(t),\\
\Pi^*(t,\Xi\mathbf{I}_{E_\varepsilon}) &= (X^*(t), Y^*(t), Z^*(t) + \Xi(t)\mathbf{I}_{E_\varepsilon}(t)).
\end{aligned}
\tag{3.2}
$$

$H_y^0(t), H_z^0(t), \hat{\mathring{H}}^0_{\mu_2}(t), \hat{\mathring{H}}^1_y(t), \hat{\mathring{H}}^1_z(t), \hat{\bar{\mathring{H}}}^1_{\mu_2}(t)$ can be understood similarly.

Remark 3.1. Buckdahn, Li and Ma[6] investigated the following optimal control problem (without recursive utility) governed by a general mean-field control system:

Problem (BLM). Minimize $J(v(\cdot)) = \mathbb{E}[\int_0^T f(t, X^v(t), \mathbb{P}_{X^v(t)}, v(t))dt + \Phi(X^v(T), \mathbb{P}_{X^v(T)})]$, subject to

$$
\begin{cases}
dX^v(t) = b(t, X^v(t), \mathbb{P}_{X^v(t)}, v(t))dt + \sigma(t, X^v(t), \mathbb{P}_{X^v(t)}, v(t))dW(t),\ \ t\in[0,T],\\
X^v(0) = x_0.
\end{cases}
\tag{3.3}
$$

By Fubini Theorem, a single adjoint equation is built to deal with the first-order variation of X^ε, which is described as follows:

$$\begin{cases} dp(t) = -\big\{b_x(t)p(t) + \hat{\mathbb{E}}[\hat{\mathring{b}}_\nu(t)\cdot\hat{p}(t)] + \sigma_x(t)q(t) + \hat{\mathbb{E}}[\hat{\mathring{\sigma}}_\nu(t)\cdot\hat{q}(t)] \\ \qquad\qquad + f_x(t) + \hat{\mathbb{E}}[\hat{\mathring{f}}_\nu(t)]\big\}dt + q(t)dW(t), t\in[0,T], \\ p(T) = \Phi_x(T) + \hat{\mathbb{E}}[\hat{\mathring{\Phi}}_\nu(T)]. \end{cases} \tag{3.4}$$

If we define

$$Y^v(t) = \mathbb{E}^{\mathcal{F}_t}\Big[\Phi(X^v(T), \mathbb{P}_{X^v(T)}) + \int_t^T f(t, X^v(t), \mathbb{P}_{X^v(t)}, v(t))dt\Big],$$

following the scheme of El Karoui, Peng and Quenez[11] there exists an adapted process $Z^v(\cdot)$ such that

$$\begin{aligned} Y^v(t) =& \Phi(X^v(T), \mathbb{P}_{X^v(T)}) + \int_t^T f(t, X^v(t), \mathbb{P}_{X^v(t)}, v(t))dt \\ &- \int_t^T Z^v(s)dW(s),\ t\in[0,T]. \end{aligned} \tag{3.5}$$

By $Y^{1,\varepsilon}$ we denote the first-order variation of Y^ε, where $(Y^\varepsilon, Z^\varepsilon)$ is the solution of (3.5) with $v^\varepsilon(\cdot) := u^*(\cdot)\mathbf{I}_{(E_\varepsilon)^c} + v(\cdot)\mathbf{I}_{E_\varepsilon}$ instead of $v(\cdot)$. For the solution of the above adjoint equation (3.6) one can check

$$\mathbb{E}[Y^{1,\varepsilon}(t)] = \mathbb{E}[p(t)X^{1,\varepsilon}(t)], \qquad t\in[0,T]. \tag{3.6}$$

It should be pointed out that because we have to deal with the fully coupled mean-field control system, the equality (3.6) is not sufficient for our case (see Remark 4.4). In fact, we need a slightly "strong" result

$$Y^{1,\varepsilon}(t) = p_0(t)X^{1,\varepsilon}(t) + \hat{\mathbb{E}}[\hat{\mathring{p}}_1(t)\hat{X}^{1,\varepsilon}(t)], \qquad t\in[0,T],\quad \mathbb{P}\text{-a.s.}$$

This is why we introduce the split first-order adjoint equation (3.1).

The first equation in (3.1) is a classical BSDE, whose coefficient does not satisfy Lipschitz condition. However, from Lemma 3.3, Assumption 3.4, Remark 3.5[17], we know that if C_2 and C_3 are small enough, the first equation in (3.1) possesses a unique solution (p_0, q_0) with

$$|p_0(t)| \le L_0, \qquad t\in[0,T],\ \mathbb{P}\text{-a.s.},\quad q_0\in\mathcal{H}^{2,\beta}_{\mathbb{F}}(0,T), \tag{3.7}$$

where L_0 is a positive constant depending on C_1 and C_2.

The second equation in (3.1) is mean-field BSDE with non-Lipschitz coefficient, we make the following assumption:

Assumption 3.2. Suppose the second equation in (3.1) exists a unique solution $(\hat{\mathring{p}}_1, \hat{\mathring{q}}_{11}, \hat{\mathring{q}}_{12})$ with $|\hat{\mathring{p}}_1(t)| \le L_0, t\in[0,T]$, $\mathbb{P}\otimes\hat{\mathbb{P}}$-a.s., and $\hat{\mathring{q}}_{11}, \hat{\mathring{q}}_{12} \in \mathcal{H}^{2,\beta}_{\mathbb{F}\otimes\hat{\mathbb{F}}}(0,T)$, where L_0 is positive constant depending on C_1 and C_2.

Assumption 3.3. Suppose $q_0(\cdot)$ and $\hat{\mathring{q}}_{11}(\cdot), \hat{\mathring{q}}_{12}(\cdot)$ are bounded.

Next let us introduce the split second-order adjoint equation

$$\begin{cases} dP_0(t) = -\big\{P_0(t)[(D\sigma(t)^\intercal[1, p_0(t), k_0(t)]^\intercal)^2 + 2Db(t)^\intercal[1, p_0(t), k_0(t)]^\intercal + H^0_y] \\ \qquad\qquad + 2Q_0(t)D\sigma(t)^T[1, p_0(t), k_0(t)]^\intercal + H^0_z(t)K_0(t) \\ \qquad\qquad + [1, p_0(t), k_0(t)]D^2H^0(t)[1, p_0(t), k_0(t)]^\intercal\big\}dt + Q_0(t)dW(t), \\ P_0(T) = \Phi_{xx}(T), \end{cases} \tag{3.8}$$

$$
\begin{cases}
d\hat{\mathring{P}}_1(t) = -\big\{\hat{\mathring{P}}_1(t)[(D\hat{\sigma}(t)^\intercal[1,\hat{p}_0(t),\hat{k}_0(t)]^\intercal)^2 + 2D\hat{b}(t)^\intercal[1,\hat{p}_0(t),\hat{k}_0(t)]^\intercal + \hat{H}^0_y(t)] \\
\qquad + \bar{\mathbb{E}}[\bar{\hat{\mathring{P}}}_1(t)(\bar{\mathring{H}}^0_{\mu_2}(t) + \bar{\mathring{H}}^1_y(t) + \tilde{\mathbb{E}}[\tilde{\bar{\mathring{H}}}^1_{\mu_2}(t)])] + 2\hat{\mathring{Q}}_{12}(t)D\hat{\sigma}(t)^\intercal[1,\hat{p}_0(t),\hat{k}_0(t)]^\intercal \\
\qquad + [1,\hat{p}_0(t),\hat{k}_0(t)]D^2\hat{\mathring{H}}^1(t)[1,\hat{p}_0(t),\hat{k}_0(t)]^\intercal + \hat{\mathring{H}}^1_z(t)\hat{K}_0(t) + \hat{H}^0_z(t)\hat{\mathring{K}}_1(t) \\
\qquad + \bar{E}[\bar{\mathring{H}}^1_z(t)\bar{\hat{\mathring{K}}}_1(t)] + (\hat{\mathring{H}}^0_{\mu_2}(t) + \hat{\mathring{H}}^1_y(t) + \bar{\mathbb{E}}[\bar{\hat{\mathring{H}}}^1_{\mu_2}(t)])\hat{P}_0(t) \\
\qquad + \hat{\mathring{H}}^0_{\mu_1 a_1}(t) + \hat{\mathring{H}}^0_{\mu_2 a_2}(t)(\hat{p}_0(t))^2 + \bar{\mathbb{E}}[\bar{\hat{\mathring{H}}}^1_{\mu_1 a_1}(t)] + \bar{\mathbb{E}}[\bar{\hat{\mathring{H}}}^1_{\mu_2 a_2}(t)(\bar{p}_0(t))^2]\big\}dt \\
\quad + \hat{\mathring{Q}}_{11}(t)dW(t) + \hat{\mathring{Q}}_{12}(t)d\hat{W}(t),\ t \in [0,T], \\
\hat{\mathring{P}}_1(T) = \hat{\mathring{\Phi}}_{\nu a}(T),
\end{cases}
$$

where

$$
\begin{aligned}
K_0(t) &= (1 - p_0(t)\sigma_z(t))^{-1}\big\{p_0(t)\sigma_y(t) + 2[\sigma_x(t) + \sigma_y(t)p_0(t) + \sigma_z(t)k_0(t)]\big\}P_0(t) \\
&\quad + (1 - p_0(t)\sigma_z(t))^{-1}\big\{Q_0(t) + p_0(t)[1,p_0(t),k_0(t)]D^2\sigma[1,p_0(t),k_0(t)]^\intercal\big\}, \\
\hat{\mathring{K}}_1(t) &= (1 - p_0(t)\sigma_z(t))^{-1}\big\{p_0(t)\hat{\mathring{\sigma}}_{\mu_2}(t)\hat{P}_0(t) + p_0(t)\sigma_y(t)\hat{\mathring{P}}_1(t) + p_0(t)\bar{\mathbb{E}}[\bar{\mathring{\sigma}}_{\mu_2}(t)\bar{\hat{P}}_1(t)] \\
&\qquad + \hat{\mathring{Q}}_{11}(t) + p_0(t)\hat{\mathring{\sigma}}_{\mu_1 a_1}(t) + p_0(t)\hat{\mathring{\sigma}}_{\mu_2 a_2}(t)(\hat{p}_0(t))^2\big\}, \\
\bar{\mathring{H}}^0_{\mu_2}(t) &= \bar{\mathring{b}}_{\mu_2}(t)p_0(t) + \bar{\mathring{\sigma}}_{\mu_2}(t)q_0(t) + \bar{\mathring{f}}_{\mu_2}(t), \quad \bar{\mathring{H}}^1_y(t) = b_y(t)\bar{\mathring{p}}_1(t) + \sigma_y(t)\bar{\mathring{q}}_{12}(t), \\
\bar{\hat{\mathring{H}}}^1_{\mu_2}(t) &= \bar{\hat{b}}_{\mu_2}(t)\bar{\mathring{p}}_1(t) + \bar{\hat{\sigma}}_{\mu_2}(t)\bar{\mathring{q}}_{12}(t), \\
\hat{\mathring{H}}^0_{\mu_2 a_2}(t) &= \hat{\mathring{b}}_{\mu_2 a_2}(t)p_0(t) + \hat{\mathring{\sigma}}_{\mu_2 a_2}(t)q_0(t) + \hat{\mathring{f}}_{\mu_2 a_2}(t), \\
\bar{\hat{\mathring{H}}}^1_{\mu_2 a_2}(t) &= \bar{\hat{b}}_{\mu_2 a_2}(t)\hat{\mathring{p}}_1(t) + \bar{\hat{\sigma}}_{\mu_2 a_2}(t)\hat{\mathring{q}}_{12}(t).
\end{aligned}
\tag{3.9}
$$

Under Assumption (A3.1), Assumption (A3.2) and Assumption (A3.3), the first equation in (3.8) is a BSDE with Lipschitz coefficient. Hence, it possesses a unique solution $(P_0, Q_0) \in \mathcal{S}^4_{\mathbb{F}}(0,T) \times \mathcal{H}^{2,2}_{\mathbb{F}}(0,T)$. Once knowing (P_0, Q_0), the second equation in (3.8) is a mean-field BSDE over product space $(\Omega \times \hat{\Omega}, \mathbb{F} \otimes \hat{\mathbb{F}}, \mathbb{P} \otimes \hat{\mathbb{P}})$. From Theorm 7.1 (see Appendix), the second equation in (3.8) exists a unique solution $(\hat{\mathring{P}}_1, \hat{\mathring{Q}}_{11}, \hat{\mathring{Q}}_{12}) \in \mathcal{S}^4_{\mathbb{F}\otimes\hat{\mathbb{F}}}(0,T) \times \mathcal{H}^{2,2}_{\mathbb{F}\otimes\hat{\mathbb{F}}}(0,T) \times \mathcal{H}^{2,2}_{\mathbb{F}\otimes\hat{\mathbb{F}}}(0,T)$.

Let us consider an algebra equation

$$
\begin{aligned}
\Xi(t) = &p_0(t)\big(\sigma(t, X^*(t), Y^*(t), Z^*(t) + \Xi(t), \mathbb{P}_{(X^*(t),Y^*(t))}, v(t)) \\
&- \sigma(t, X^*(t), Y^*(t), Z^*(t), \mathbb{P}_{(X^*(t),Y^*(t))}, u^*(t))\big).
\end{aligned}
\tag{3.10}
$$

Clearly, $\Xi(t)$ depends on $p_0(t), v(t)$ and $u^*(t)$.

Lemma 3.2. *Let Assumption (A3.1) and Assumption (A3.2) holds true, the algebra equation (3.10) exists a unique solution $\Xi(\cdot)$ and*

$$
\begin{aligned}
&|\Xi(t)| \le L_0(1 + |X^*(t)| + |Y^*(t)| + ||X^*(t)||_{L^2} + ||Y^*(t)||_{L^2} + |v(t)| + |u^*(t)|), \\
&\sup_{0\le t\le T} \mathbb{E}[|\Xi(t)|^8] < +\infty.
\end{aligned}
\tag{3.11}
$$

The proof is similar to that of Lemma 3.9[17]. Hence, we omit it.

Define

$$
\mathcal{H}(t, x, y, z, \mu, v, p_0(t), \hat{\mathbb{E}}[\mathring{p}_1(t)], q_0(t), \hat{\mathbb{E}}[\mathring{q}_{12}(t)], P_0(t), \hat{\mathbb{E}}[\mathring{P}_1(t)])
$$

$$
\begin{aligned}
=&(p_0(t)+\hat{\mathbb{E}}[\mathring{\hat{p}}_1(t)])b(t,x,y,z+\Xi(t),\mu,v)+f(t,x,y,z+\Xi(t),\mu,v)\\
&+(q_0(t)+\hat{\mathbb{E}}[\mathring{\hat{q}}_{12}(t)])\sigma(t,x,y,z+\Xi(t),\mu,v)+\frac{1}{2}(P_0(t)+\hat{\mathbb{E}}[\mathring{\hat{P}}_1(t)])\\
&\cdot(\sigma(t,x,y,z+\Xi(t),\mu,v)-\sigma(t,X^*(t),Y^*(t),Z^*(t),\mathbb{P}_{(X^*(t),Y^*(t))},u^*(t)))^2,
\end{aligned}
\tag{3.12}
$$

where $\Xi(t)$ is introduced in (3.10), but with v instead of $v(t)$.

Theorem 3.3. *Let Assumption (A3.1), Assumption (A3.2) and Assumption (A3.3) be in force, and let u^* be the optimal control. By (X^*,Y^*,Z^*) we denote the optimal trajectory. Let $((p_0(\cdot),q_0(\cdot)),(\mathring{\hat{p}}_1(\cdot),\mathring{\hat{q}}_{11}(\cdot),\mathring{\hat{q}}_{12}(\cdot)))$ and $((P_0(\cdot),Q_0(\cdot)),(\mathring{\hat{P}}_1(\cdot),\mathring{\hat{Q}}_{11}(\cdot),\mathring{\hat{Q}}_{12}(\cdot)))$ be the solutions of the first- and second-order adjoint equations, respectively. Moreover, we assume $\mathbb{P}\otimes\hat{\mathbb{P}}$-a.s.,*

$$
\begin{cases}
\mathring{\hat{H}}^0_{\mu_2}(t)+\mathring{\hat{H}}^1_y(t)+\bar{\mathbb{E}}[\bar{\mathring{\hat{H}}}^1_{\mu_2}(t)]+p_0(t)f_z(t)\mathring{\hat{\sigma}}_{\mu_2}(t)\sigma_y(t)(1-p_0(t)\sigma_z(t))^{-1}\geq 0,\\
\mathring{\hat{H}}^1_z(t)=\hat{b}_z(t)\mathring{\hat{p}}_1(t)+\hat{\sigma}_z(t)\mathring{\hat{q}}_{12}(t)=0,\ \ t\in[0,T],
\end{cases}
\tag{3.13}
$$

where $\mathring{\hat{H}}^0_{\mu_2}(t),\mathring{\hat{H}}^1_y(t),\bar{\mathring{\hat{H}}}^1_{\mu_2}(t)$ is introduced in (3.2). Then

$$
\begin{aligned}
&\mathcal{H}(t,X^*(t),Y^*(t),Z^*(t),\mathbb{P}_{(X^*(t),Y^*(t))},v,p_0(t),\hat{\mathbb{E}}[\mathring{\hat{p}}_1(t)],q_0(t),\hat{\mathbb{E}}[\mathring{\hat{q}}_{12}(t)],P_0(t),\hat{\mathbb{E}}[\mathring{\hat{P}}_1(t)])\\
\geq&\mathcal{H}(t,X^*(t),Y^*(t),Z^*(t),\mathbb{P}_{(X^*(t),Y^*(t))},u^*(t),p_0(t),\hat{\mathbb{E}}[\mathring{\hat{p}}_1(t)],q_0(t),\hat{\mathbb{E}}[\mathring{\hat{q}}_{12}(t)],P_0(t),\hat{\mathbb{E}}[\mathring{\hat{P}}_1(t)]),\\
&v\in U,\ a.e.,\ a.s,
\end{aligned}
$$

Remark 3.4. i) If the coefficients b,σ,f,Φ are mean-field free, one can check $\mathring{\hat{p}}_1(\cdot)=\mathring{\hat{q}}_{11}(\cdot)=\mathring{\hat{q}}_{12}(\cdot)=0$, which means $\mathbb{P}\times\hat{\mathbb{P}}$-a.s.,

$$
\begin{cases}
\mathring{\hat{H}}^0_{\mu_2}(t)+\mathring{\hat{H}}^1_y(t)+\bar{\mathbb{E}}[\bar{\mathring{\hat{H}}}^1_{\mu_2}(t)]+p_0(t)f_z(t)\mathring{\hat{\sigma}}_{\mu_2}(t)\sigma_y(t)(1-p_0(t)\sigma_z(t))^{-1}=0,\\
\mathring{\hat{H}}^1_z(t)=0,\ \ t\in[0,T].
\end{cases}
\tag{3.14}
$$

Hence, from this point of view, the SMP obtained by Hu, Ji and Xue[17] is a special case of our SMP.

ii) Obviously, if b,σ are independent of z, the assumption $\mathring{\hat{H}}^1_z(t)=0,\ t\in[0,T],\ \mathbb{P}\otimes\hat{\mathbb{P}}$-a.s. holds true.

4 Variational Equations

Two variational equations are studied in this section, which are the building materials of our SMP. In view of the fact that the control domain is not necessarily convex in our case, the method of "spike variation" is borrowed to investigate our optimal problem. Let E_ε be a subset of $[0,T]$ with Lebesgue measure $|E_\varepsilon|=\varepsilon$. For any $v(\cdot)\in\mathcal{U}_{ad}$, define

$$
v^\varepsilon(\cdot):=u^*(\cdot)\mathbf{I}_{(E_\varepsilon)^c}+v(\cdot)\mathbf{I}_{E_\varepsilon},
$$

where $u^*(\cdot)$ is the optimal control. By $(X^\varepsilon,Y^\varepsilon,Z^\varepsilon)$ we denote the solution of (1.1) with v^ε, i.e., $(X^\varepsilon,Y^\varepsilon,Z^\varepsilon):=(X^{v^\varepsilon}(\cdot)Y^{v^\varepsilon}(\cdot),Z^{v^\varepsilon}(\cdot))$.

4.1 First-order Variational Equation

The first order variational equations can be written as

$$
dX^{1,\varepsilon}(t)=\big\{b_x(t)X^{1,\varepsilon}(t)+b_y(t)Y^{1,\varepsilon}(t)+b_z(t)(Z^{1,\varepsilon}(t)-\Xi(t)\mathbf{I}_{E_\varepsilon}(t))
$$

$$
\begin{aligned}
&\quad + \hat{\mathbb{E}}[\hat{\mathring{b}}_{\mu_1}(t)\hat{X}^{1,\varepsilon}(t)] + \hat{\mathbb{E}}[\hat{\mathring{b}}_{\mu_2}(t)\hat{Y}^{1,\varepsilon}(t)]\big\}dt \\
&\quad + \big\{\sigma_x(t)X^{1,\varepsilon}(t) + \sigma_y(t)Y^{1,\varepsilon}(t) + \sigma_z(t)(Z^{1,\varepsilon}(t) - \Xi(t)\mathbf{I}_{E_\varepsilon}(t)) \\
&\quad + \hat{\mathbb{E}}[\hat{\mathring{\sigma}}_{\mu_1}(t)\hat{X}^{1,\varepsilon}(t)] + \hat{\mathbb{E}}[\hat{\mathring{\sigma}}_{\mu_2}(t)\hat{Y}^{1,\varepsilon}(t)] + \delta\sigma(t,\Xi)\mathbf{I}_{E_\varepsilon}(t)\big\}dW(t), \qquad (4.1)\\
dY^{1,\varepsilon}(t) &= -\big\{f_x(t)X^{1,\varepsilon}(t) + f_y(t)Y^{1,\varepsilon}(t) + f_z(t)(Z^{1,\varepsilon}(t) - \Xi(t)\mathbf{I}_{E_\varepsilon}(t)) \\
&\quad + \hat{\mathbb{E}}[\hat{\mathring{f}}_{\mu_1}(t)\hat{X}^{1,\varepsilon}(t)] + \hat{\mathbb{E}}[\hat{\mathring{f}}_{\mu_2}(t)\hat{Y}^{1,\varepsilon}(t)] - q_0(t)\delta\sigma(t,\Xi)\mathbf{I}_{E_\varepsilon}(t) \\
&\quad - \hat{\mathbb{E}}[\hat{\mathring{q}}_{12}(t)\delta\hat{\sigma}(t,\Xi)\mathbf{I}_{E_\varepsilon}(t)]\big\}dt + Z^{1,\varepsilon}(t)dW(t),\ t\in[0,T], \\
X^{1,\varepsilon}(0) &= 0, \quad Y^{1,\varepsilon}(T) = \Phi_x(T)X^{1,\varepsilon}(T) + \hat{\mathbb{E}}[\hat{\mathring{\Phi}}_\nu(T)\hat{X}^{1,\varepsilon}(T)].
\end{aligned}
$$

(4.1) is a fully coupled linear mean-field FBSDE. According to Theorem 6[23], (4.1) exists a unique solution.

Proposition 4.1. *Under Assumptions (A3.1)-(A3.3) and suppose (3.7) hold true, then for $t\in[0,T]$, $\mathbb{P}$-a.s.,*

$$
\begin{aligned}
Y^{1,\varepsilon}(t) &= p_0(t)X^{1,\varepsilon}(t) + \hat{\mathbb{E}}[\hat{\mathring{p}}_1(t)\hat{X}^{1,\varepsilon}(t)], \\
Z^{1,\varepsilon}(t) &= k_0(t)X^{1,\varepsilon}(t) + \hat{\mathbb{E}}[\hat{\mathring{k}}_1(t)\hat{X}^{1,\varepsilon}(t)] + \Xi(t)\mathbf{I}_{E_\varepsilon}(t),
\end{aligned}
\tag{4.2}
$$

where k_0 and $\hat{\mathring{k}}_1$ are introduced in (3.2); $\Xi(\cdot)$ is the solution of (3.10).

Proof. Consider the following linear mean-field SDE:

$$
\begin{cases}
dx(t) = \big\{x(t)(b_x(t) + b_y(t)p_0(t) + b_z(t)k_0(t)) \\
\qquad + \hat{\mathbb{E}}[\hat{x}(t)(b_y(t)\hat{\mathring{p}}_1(t) + b_z(t)\hat{\mathring{k}}_1(t) + \hat{\mathring{b}}_{\mu_1}(t) + \hat{\mathring{b}}_{\mu_2}(t)\hat{p}_0(t) + \bar{\mathbb{E}}[\bar{\hat{\mathring{b}}}_{\mu_2}(t)\hat{\bar{p}}_1(t)])]\big\}dt \\
\qquad + \big\{x(t)(\sigma_x(t) + \sigma_y(t)p_0(t) + \sigma_z(t)k_0(t)) + \delta\sigma(t,\Xi)\mathbf{I}_{E_\varepsilon}(t) \\
\qquad + \hat{\mathbb{E}}[\hat{x}(t)(\sigma_y(t)\hat{\mathring{p}}_1(t) + \sigma_z(t)\hat{\mathring{k}}_1(t) + \hat{\mathring{\sigma}}_{\mu_1}(t) + \hat{\mathring{\sigma}}_{\mu_2}(t)\hat{p}_0(t) \\
\qquad + \bar{\mathbb{E}}[\bar{\hat{\sigma}}_{\mu_2}(t)\hat{\bar{p}}_1(t)])]\big\}dW(t), \\
x(0) = 0.
\end{cases}
\tag{4.3}
$$

From Assumption 3.1, Assumption 3.2 and (3.7), we know that (4.3) exists a unique solution, refer to Theorem 6[23].

Define $y(t) = p_0(t)x(t) + \hat{\mathbb{E}}[\hat{\mathring{p}}_1(t)\hat{x}(t)]$, $z(t) = k_0(t)x(t) + \hat{\mathbb{E}}[\hat{\mathring{k}}_1(t)\hat{x}(t)] + \Xi(t)\mathbf{I}_{E_\varepsilon}(t)$. Applying Itô's formula to $\hat{\mathbb{E}}[\hat{\mathring{p}}_1(t)\hat{x}(t)]$ we have

$$
\begin{aligned}
&d\hat{\mathbb{E}}[\hat{\mathring{p}}_1(t)\hat{x}(t)] \\
=&\hat{\mathbb{E}}[\hat{x}(t)\hat{\mathring{p}}_1(t)(\hat{b}_x(t) + \hat{b}_y(t)\hat{p}_0(t) + \hat{b}_z(t)\hat{k}_0(t)) \\
&\quad + \hat{x}(t)\hat{\mathring{q}}_{12}(t)(\hat{\sigma}_x(t) + \hat{\sigma}_y(t)\hat{p}_0(t) + \hat{\sigma}_z(t)\hat{k}_0(t)) - \hat{\mathring{F}}_1(t)] \\
&+ \bar{\mathbb{E}}\hat{\mathbb{E}}[\bar{x}(t)\hat{\mathring{p}}_1(t)(\hat{b}_y(t)\bar{\hat{p}}_1(t) + \hat{b}_z(t)\bar{\hat{k}}_1(t) + \bar{\hat{b}}_{\mu_1}(t) + \bar{\hat{b}}_{\mu_2}(t)\bar{p}_0(t) + \tilde{\mathbb{E}}[\tilde{\hat{b}}_{\mu_2}(t)\bar{\tilde{p}}_1(t)])] \\
&+ \bar{\mathbb{E}}\hat{\mathbb{E}}[\bar{x}(t)\hat{\mathring{q}}_{12}(t)(\hat{\sigma}_y(t)\bar{\hat{p}}_1(t) + \hat{\sigma}_z(t)\bar{\hat{k}}_1(t) + \bar{\hat{\sigma}}_{\mu_1}(t) + \bar{\hat{\sigma}}_{\mu_2}(t)\bar{p}_0(t) + \tilde{\mathbb{E}}[\tilde{\hat{\sigma}}_{\mu_2}(t)\bar{\tilde{p}}_1(t)])] \\
&+ \hat{\mathbb{E}}[\hat{\mathring{q}}_{12}(t)\delta\hat{\sigma}(t,\Xi)\mathbf{I}_{E_\varepsilon}(t)] + \hat{\mathbb{E}}[\hat{x}(t)\hat{\mathring{q}}_{11}(t)]dW(t),
\end{aligned}
\tag{4.4}
$$

where $\hat{\mathring{F}}_1(t)$ is given in (3.2). Notice

$$
\bar{\mathbb{E}}\hat{\mathbb{E}}[\bar{x}(t)\hat{\mathring{p}}_1(t)\hat{b}_y(t)\bar{\hat{p}}_1(t)] = \hat{\mathbb{E}}\bar{\mathbb{E}}[\hat{x}(t)\hat{\bar{p}}_1(t)\bar{b}_y(t)\bar{\hat{p}}_1(t)] = \hat{\mathbb{E}}[\bar{\mathbb{E}}[\hat{\bar{p}}_1(t)\bar{b}_y(t)\bar{\hat{p}}_1(t)]\hat{x}(t)],
$$

$$\bar{\mathbb{E}}\hat{\mathbb{E}}[\bar{x}(t)\hat{\mathring{p}}_1(t)\bar{\bar{b}}_{\mu_1}(t)] = \hat{\mathbb{E}}\bar{\mathbb{E}}[\hat{x}(t)\bar{\mathring{p}}_1(t)\hat{\bar{b}}_{\mu_1}(t)] = \hat{\mathbb{E}}[\bar{\mathbb{E}}[\bar{\mathring{p}}_1(t)\hat{\bar{b}}_{\mu_1}(t)]\hat{x}(t)],$$
$$\bar{\mathbb{E}}\hat{\mathbb{E}}[\bar{x}(t)\hat{\mathring{p}}_1(t)\tilde{\mathbb{E}}[\tilde{\bar{b}}_{\mu_2}(t)\tilde{\bar{p}}_1(t)]] = \bar{\mathbb{E}}\hat{\mathbb{E}}\tilde{\mathbb{E}}[\bar{x}(t)\hat{\mathring{p}}_1(t)\tilde{\bar{b}}_{\mu_2}(t)\tilde{\bar{p}}_1(t)]$$
$$=\hat{\mathbb{E}}\bar{\mathbb{E}}\tilde{\mathbb{E}}[\hat{x}(t)\bar{\mathring{p}}_1(t)\tilde{\bar{b}}_{\mu_2}(t)\tilde{\hat{p}}_1(t)] = \hat{\mathbb{E}}\big[\bar{\mathbb{E}}\tilde{\mathbb{E}}[\bar{\mathring{p}}_1(t)\tilde{\bar{b}}_{\mu_2}(t)\tilde{\hat{p}}_1(t)]\hat{x}(t)\big],$$

(4.4) can be written as

$$\begin{aligned}
&d\hat{\mathbb{E}}[\hat{\mathring{p}}_1(t)\hat{x}(t)]\\
=&\hat{\mathbb{E}}\big[\hat{x}(t)\big\{\hat{\mathring{p}}_1(t)(\hat{b}_x(t)+\hat{b}_y(t)\hat{p}_0(t)+\hat{b}_z(t)\hat{k}_0(t))+\hat{\mathring{q}}_{12}(t)(\hat{\sigma}_x(t)+\hat{\sigma}_y(t)\hat{p}_0(t)+\hat{\sigma}_z(t)\hat{k}_0(t))\\
&\qquad+\bar{\mathbb{E}}[\bar{\mathring{p}}_1(t)(\bar{b}_y(t)\hat{\bar{p}}_1(t)+\bar{b}_z(t)\hat{\bar{k}}_1(t)+\hat{\bar{b}}_{\mu_1}(t)+\hat{\bar{b}}_{\mu_2}(t)\bar{p}_0(t)+\tilde{\mathbb{E}}[\tilde{\bar{b}}_{\mu_2}(t)\tilde{\hat{p}}_1(t)]\big)\\
&\qquad+\bar{\mathring{q}}_{12}(t)(\bar{\sigma}_y(t)\hat{\bar{p}}_1(t)+\bar{\sigma}_z(t)\hat{\bar{k}}_1(t)+\hat{\bar{\sigma}}_{\mu_1}(t)+\hat{\bar{\sigma}}_{\mu_2}(t)\bar{p}_0(t)+\tilde{\mathbb{E}}[\tilde{\bar{\sigma}}_{\mu_2}(t)\tilde{\hat{p}}_1(t)])]\big\}\\
&-\hat{\mathring{F}}_1(t)]dt+\hat{\mathbb{E}}[\hat{\mathring{q}}_{12}(t)\delta\hat{\sigma}(t,\Xi)\mathbf{I}_{E_\varepsilon}(t)]dt+\hat{\mathbb{E}}[\hat{x}(t)\hat{\mathring{q}}_{11}(t)]dW(t).
\end{aligned}\tag{4.5}$$

For convenience, we denote

$$F_0(t) = H_x^0(t) + p_0(t)H_y^0(t) + k_0(t)H_z^0(t).$$

The Itô's formula to $p_0(t)x(t)$ allows to show

$$\begin{aligned}
&dp_0(t)x(t)\\
=&x(t)\big\{p_0(t)(b_x(t)+b_y(t)p_0(t)+b_z(t)k_0(t))+q_0(t)(\sigma_x(t)+\sigma_y(t)p_0(t)\\
&\quad+\sigma_z(t)k_0(t))-F_0(t)\big\}dt+\hat{\mathbb{E}}\big[\hat{x}(t)\big\{p_0(t)(b_y(t)\hat{\mathring{p}}_1(t)+b_z(t)\hat{\mathring{k}}_1(t)\\
&+\hat{\mathring{b}}_{\mu_1}(t)+\hat{\mathring{b}}_{\mu_2}(t)\hat{p}_0(t)+\bar{\mathbb{E}}[\bar{\mathring{b}}_{\mu_2}(t)\bar{\hat{p}}_1(t)])+q_0(t)(\sigma_y(t)\hat{\mathring{p}}_1(t)+\sigma_z(t)\hat{\mathring{k}}_1(t)\\
&+\hat{\mathring{\sigma}}_{\mu_1}(t)+\hat{\mathring{\sigma}}_{\mu_2}(t)\hat{p}_0(t)+\bar{\mathbb{E}}[\bar{\mathring{\sigma}}_{\mu_2}(t)\bar{\hat{p}}_1(t)])\big\}\big]dt\\
&+\big\{x(t)p_0(t)(\sigma_x(t)+\sigma_y(t)p_0(t)+\sigma_z(t)k_0(t))+p_0(t)\delta\sigma(t,\Xi)\mathbf{I}_{E_\varepsilon}(t)\\
&\quad+p_0(t)\hat{\mathbb{E}}[\hat{x}(t)(\sigma_y(t)\hat{\mathring{p}}_1(t)+\sigma_z(t)\hat{\mathring{k}}_1(t)+\hat{\mathring{\sigma}}_{\mu_1}(t)\\
&\quad+\hat{\mathring{\sigma}}_{\mu_2}(t)\hat{p}_0(t)+\bar{\mathbb{E}}[\bar{\mathring{\sigma}}_{\mu_2}(t)\bar{\hat{p}}_1(t)])]\big\}dW(t).
\end{aligned}\tag{4.6}$$

Combining (4.5)–(4.6) and the definition of $\hat{\mathring{F}}_1$ (see (3.2)), we arrive at

$$\begin{cases}
dy(t) = -\big\{f_x(t)x(t)+f_y(t)y(t)+f_z(t)z(t)+\hat{\mathbb{E}}[\hat{\mathring{f}}_{\mu_1}(t)\hat{x}(t)]+\hat{\mathbb{E}}[\hat{\mathring{f}}_{\mu_2}(t)\hat{y}(t)]\\
\qquad\qquad -q_0(t)\delta\sigma(t,\Xi)\mathbf{I}_{E_\varepsilon}(t)-\hat{\mathbb{E}}[\hat{\mathring{q}}_{12}(t)\delta\hat{\sigma}(t,\Xi)\mathbf{I}_{E_\varepsilon}(t)]\big\}dt\\
\qquad\quad +z(t)dW(t),\ t\in[0,T],\\
y(T) = \Phi_x(T)x(T)+\hat{\mathbb{E}}[\hat{\mathring{\Phi}}_\mu(T)\hat{x}(T)],
\end{cases}$$

which means that (x, y, z) solves (4.1). Then Theorem 2.2 allows to show $(x, y, z) = (X^{1,\varepsilon}, Y^{1,\varepsilon}, Z^{1,\varepsilon})$. □

Remark 4.2. It should be pointed out that (4.2) plays an important role in our analysis. As mentioned in Remark 3.1, the relation (3.6) established by Buckdahn, Li and Ma[6] is not enough to handle the fully coupled mean-field control systems. We need a bit "strong" relation (4.2).

Proposition 4.3. *Let Assumption (A3.1) i)–ii), Assumption (A3.2) and Assumption (A3.3) be in force, then for arbitrary $2 \le \beta < 8$, there exists a constant $L > 0$ depending on $C_0, C_1, C_2, C_3, L_0, \beta, T$ such that*

$$\begin{aligned}
&\text{i) } \mathbb{E}\Big[\sup_{t\in[0,T]}\big(|X^{1,\varepsilon}(t)|^\beta + |Y^{1,\varepsilon}(t)|^\beta\big)\Big] + \mathbb{E}\Big[\Big(\int_0^T |Z^{1,\varepsilon}(t)|^2 dt\Big)^{\frac{\beta}{2}}\Big] \le L\varepsilon^{\frac{\beta}{2}};\\
&\text{ii) } \mathbb{E}\Big[\sup_{t\in[0,T]}\big(|X^{\varepsilon}(t)-X^*(t)|^\beta + |Y^{\varepsilon}(t)-Y^*(t)|^\beta\big)\Big]\\
&\quad + \mathbb{E}\Big[\Big(\int_0^T |Z^{\varepsilon}(t)-Z^*(t)|^2 dt\Big)^{\frac{\beta}{2}}\Big] \le L\varepsilon^{\frac{\beta}{2}};\\
&\text{iii) } \mathbb{E}\Big[\sup_{t\in[0,T]}\big(|X^{\varepsilon}(t)-X^*(t)-X^{1,\varepsilon}(t)|^4 + |Y^{\varepsilon}(t)-Y^*(t)-Y^{1,\varepsilon}(t)|^4\big)\Big]\\
&\quad + \mathbb{E}\Big[\Big(\int_0^T |Z^{\varepsilon}(t)-Z^*(t)-Z^{1,\varepsilon}(t)|^2 dt\Big)^{2}\Big] \le L\varepsilon^{4}.
\end{aligned} \tag{4.7}$$

Proof. i) From Theorem 2.2, one has

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\big(|X^{1,\varepsilon}(t)|^\beta + |Y^{1,\varepsilon}(t)|^\beta\big)\Big] + \mathbb{E}\Big[\Big(\int_0^T |Z^{1,\varepsilon}(t)|^2 dt\Big)^{\frac{\beta}{2}}\Big]\\
\le& L\mathbb{E}\Big[\Big(\int_0^T (|b_z(t)| + |f_z(t)|)|\Xi(t)|\mathbf{I}_{E_\varepsilon}(t) + |\hat{\mathbb{E}}[\hat{\mathring{q}}_{12}(t)\delta\hat{\sigma}(t,\Xi)\mathbf{I}_{E_\varepsilon}(t)]|\\
&\qquad + |q_0(t)\delta\sigma(t,\Xi)\mathbf{I}_{E_\varepsilon}(t)|dt\Big)^\beta\Big] + L\mathbb{E}\Big[\Big(\int_0^T [|\sigma_z(t)\Xi(t)\mathbf{I}_{E_\varepsilon}(t)|^2 + |\delta\sigma(t,\Xi)\mathbf{I}_{E_\varepsilon}(t)|^2]dt\Big)^{\frac{\beta}{2}}\Big]\\
\le& L\mathbb{E}\Big[\Big(\int_{E_\varepsilon} 1 + |X^*(t)| + |Y^*(t)| + ||X^*(t)||_{L^2} + ||Y^*(t)||_{L^2} + |v(t)|\\
&\qquad + |u^*(t)| + \mathbb{E}|v(t)| + \mathbb{E}|u^*(t)|)dt\Big)^\beta\Big]\\
&+ L\mathbb{E}\Big[\Big(\int_{E_\varepsilon} 1 + |X^*(t)|^2 + |Y^*(t)|^2 + ||X^*(t)||^2_{L^2} + ||Y^*(t)||^2_{L^2} + |v(t)|^2 + |u^*(t)|^2)dt\Big)^{\frac{\beta}{2}}\Big]\\
\le& L\varepsilon^{\frac{\beta}{2}}.
\end{aligned}$$

ii) Define $\mathcal{X}^1(t) = X^\varepsilon(t) - X^*(t), \mathcal{Y}^1(t) = Y^\varepsilon(t) - Y^*(t), \mathcal{Z}^1(t) = Z^\varepsilon(t) - Z^*(t)$. Then

$$\begin{cases}
d\mathcal{X}^1(t) = (b(t,\Pi^\varepsilon(t),\mathbb{P}_{\Lambda^\varepsilon(t)},v^\varepsilon(t)) - b(t,\Pi^*(t),\mathbb{P}_{\Lambda^*(t)},u^*(t)))dt\\
\qquad\qquad + (\sigma(t,\Pi^\varepsilon(t),\mathbb{P}_{\Lambda^\varepsilon(t)},v^\varepsilon(t)) - \sigma(t,\Pi^*(t),\mathbb{P}_{\Lambda^*(t)},u^*(t)))dW(t),\ t\in[0,T],\\
d\mathcal{Y}^1(t) = -(f(t,\Pi^\varepsilon(t),\mathbb{P}_{\Lambda^\varepsilon(t)},v^\varepsilon(t)) - f(t,\Pi^*(t),\mathbb{P}_{\Lambda^*(t)},u^*(t)))dt + \mathcal{Z}^1(t)dW(t),\ t\in[0,T],\\
\mathcal{X}^1(0) = 0,\ \mathcal{Y}^1(T) = \Phi(X^\varepsilon(T),\mathbb{P}_{X^\varepsilon(T)}) - \Phi(X^*(T),\mathbb{P}_{X^*(T)}).
\end{cases}$$

For $h = b, \sigma, f$ and $\ell = x, y, z$, define

$$\begin{aligned}
h^\varepsilon_\ell(t) =& \int_0^T h_\ell(t,\Pi^*(t)+\lambda(\Pi^\varepsilon(t)-\Pi^*(t)),\mathbb{P}_{\Lambda^*(t)+\lambda(\Lambda^\varepsilon(t)-\Lambda^*(t))},v^\varepsilon(t))d\lambda,\\
\hat{\mathring{h}}^\varepsilon_{\mu_1}(t) =& \int_0^T h_{\mu_1}(t,\Pi^*(t)+\lambda(\Pi^\varepsilon(t)-\Pi^*(t)),\mathbb{P}_{\Lambda^*(t)+\lambda(\Lambda^\varepsilon(t)-\Lambda^*(t))},v^\varepsilon(t);\\
&\qquad \hat{\Lambda}^*(t)+\lambda(\hat{\Lambda}^\varepsilon(t)-\hat{\Lambda}^*(t)))d\lambda.
\end{aligned} \tag{4.8}$$

Since

$$\begin{aligned}&b(t,\Pi^\varepsilon(t),\mathbb{P}_{\Lambda^\varepsilon(t)},v^\varepsilon(t))-b(t,\Pi^*(t),\mathbb{P}_{\Lambda^*(t)},u^*(t))\\=&b_x^\varepsilon(t)(X^\varepsilon(t)-X^*(t))+b_y^\varepsilon(t)(Y^\varepsilon(t)-Y^*(t))+b_z^\varepsilon(t)(Z^\varepsilon(t)-Z^*(t))\\&+\hat{\mathbb{E}}[\hat{\mathring{b}}^\varepsilon_{\mu_1}(t)(\hat{X}^\varepsilon(t)-\hat{X}^*(t))]+\hat{\mathbb{E}}[\hat{\mathring{b}}^\varepsilon_{\mu_2}(t)(\hat{Y}^\varepsilon(t)-\hat{Y}^*(t))]+\delta b(t;v^\varepsilon(t)),\end{aligned}$$

we have

$$\begin{cases}d\mathcal{X}^1(t)=\big(b_x^\varepsilon(t)\mathcal{X}^1(t)+b_y^\varepsilon(t)\mathcal{Y}^1(t)+b_z^\varepsilon(t)\mathcal{Z}^1(t)\\\qquad\qquad+\hat{\mathbb{E}}[\hat{\mathring{b}}^\varepsilon_{\mu_1}(t)\hat{\mathcal{X}}^1(t)]+\hat{\mathbb{E}}[\hat{\mathring{b}}^\varepsilon_{\mu_2}(t)\hat{\mathcal{Y}}^1(t)]+\delta b(t;v^\varepsilon(t))\big)dt\\\qquad\qquad+\big(\sigma_x^\varepsilon(t)\mathcal{X}^1(t)+\sigma_y^\varepsilon(t)\mathcal{Y}^1(t)+\sigma_z^\varepsilon(t)\mathcal{Z}^1(t)+\hat{\mathbb{E}}[\hat{\mathring{\sigma}}^\varepsilon_{\mu_1}(t)\hat{\mathcal{X}}^1(t)]\\\qquad\qquad+\hat{\mathbb{E}}[\hat{\mathring{\sigma}}^\varepsilon_{\mu_2}(t)\hat{\mathcal{Y}}^1(t)]+\delta\sigma(t;v^\varepsilon(t)\big)dW(t),\ t\in[0,T],\\d\mathcal{Y}^1(t)=-\big(f_x^\varepsilon(t)\mathcal{X}^1(t)+f_y^\varepsilon(t)\mathcal{Y}^1(t)+f_z^\varepsilon(t)\mathcal{Z}^1(t)+\hat{\mathbb{E}}[\hat{\mathring{f}}^\varepsilon_{\mu_1}(t)\hat{\mathcal{X}}^1(t)]\\\qquad\qquad+\hat{\mathbb{E}}[\hat{\mathring{f}}^\varepsilon_{\mu_2}(t)\hat{\mathcal{Y}}^1(t)]+\delta f(t;v^\varepsilon(t))\big)dt+\mathcal{Z}^1(t)dW(t),\ t\in[0,T],\\\mathcal{X}^1(0)=0,\ \mathcal{Y}^1(T)=\Phi_x^\varepsilon(t)\mathcal{X}^1(T)+\hat{\mathbb{E}}[\hat{\mathring{\Phi}}^\varepsilon_\nu(t)\hat{\mathcal{X}}^1(T)].\end{cases}\tag{4.9}$$

Thanks to Theorem 2.2, Assumption 3.1-i) and (4.9), it yields

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\le t\le T}\big(|\mathcal{X}^1(t)|^\beta+|\mathcal{Y}^1(t)|^\beta\big)+\Big(\int_0^T|\mathcal{Z}^1(t)|^2dt\Big)^{\frac{\beta}{2}}\Big]\\\le&L\mathbb{E}\Big[\Big(\int_0^T|\delta b(t;v^\varepsilon(t))|+|\delta f(t;v^\varepsilon(t))|dt\Big)^\beta+\Big(\int_0^T|\delta\sigma(t;v^\varepsilon(t))|^2dt\Big)^{\frac{\beta}{2}}\Big]\\\le&L\mathbb{E}\Big[\Big(\int_0^T(1+|X^*(t)|+|Y^*(t)|+||X^*(t)||_{L^2}+||Y^*(t)||_{L^2}+|v(t)|+|u^*(t)|)dt\Big)^\beta\Big]\\&+L\mathbb{E}\Big[\Big(\int_0^T(1+|X^*(t)|^2+|Y^*(t)|^2+||X^*(t)||^2_{L^2}+||Y^*(t)||^2_{L^2}+|v(t)|^2+|u^*(t)|^2)dt\Big)^{\frac{\beta}{2}}\Big]\\\le&L\varepsilon^{\frac{\beta}{2}}.\end{aligned}$$

iii) For simplicity of the redaction, we denote $\mathcal{X}^2(t)=X^\varepsilon(t)-X^*(t)-X^{1,\varepsilon}(t),\mathcal{Y}^2(t)=Y^\varepsilon(t)-Y^*(t)-Y^{1,\varepsilon}(t),\mathcal{Z}^2(t)=Z^\varepsilon(t)-Z^*(t)-Z^{1,\varepsilon}(t)$. Then it yields

$$\begin{cases}d\mathcal{X}^2(t)=\big[b_x^\varepsilon(t)\mathcal{X}^2(t)+b_y^\varepsilon(t)\mathcal{Y}^2(t)+b_z^\varepsilon(t)\mathcal{Z}^2(t)+\hat{\mathbb{E}}[\hat{\mathring{b}}^\varepsilon_{\mu_1}(t)\hat{\mathcal{X}}^2(t)]+\hat{\mathbb{E}}[\hat{\mathring{b}}^\varepsilon_{\mu_2}(t)\hat{\mathcal{Y}}^2(t)]+A_2^\varepsilon(t)\big]dt\\\qquad+\big[\sigma_x^\varepsilon(t,\Xi\mathbf{I}_{E_\varepsilon})\mathcal{X}^2(t)+\sigma_y^\varepsilon(t,\Xi\mathbf{I}_{E_\varepsilon})\mathcal{Y}^2(t)+\sigma_z^\varepsilon(t,\Xi\mathbf{I}_{E_\varepsilon})\mathcal{Z}^2(t)\\\qquad+\hat{\mathbb{E}}[\hat{\mathring{\sigma}}^\varepsilon_{\mu_1}(t,\Xi\mathbf{I}_{E_\varepsilon})\hat{\mathcal{X}}^2(t)]+\hat{\mathbb{E}}[\hat{\mathring{\sigma}}^\varepsilon_{\mu_2}(t,\Xi\mathbf{I}_{E_\varepsilon})\hat{\mathcal{Y}}^2(t)]+B_2^\varepsilon(t)\big]dW(t),\ t\in[0,T],\\d\mathcal{Y}^2(t)=-\big[f_x^\varepsilon(t)\mathcal{X}^2(t)+f_y^\varepsilon(t)\mathcal{Y}^2(t)+f_z^\varepsilon(t)\mathcal{Z}^2(t)+\hat{\mathbb{E}}[\hat{\mathring{f}}^\varepsilon_{\mu_1}(t)\hat{\mathcal{X}}^2(t)]+\hat{\mathbb{E}}[\hat{\mathring{f}}^\varepsilon_{\mu_2}(t)\hat{\mathcal{Y}}^2(t)]\\\qquad+C_2^\varepsilon(t)\big]dt+\mathcal{Z}^2(t)dW(t),\ t\in[0,T],\\\mathcal{X}^2(0)=0,\ \mathcal{Y}^2(0)=\Phi_x^\varepsilon(T)\mathcal{X}^2(T)+\hat{\mathbb{E}}[\hat{\mathring{\Phi}}^\varepsilon_\nu(T)\hat{\mathcal{X}}^2(T)]+D_2^\varepsilon(T),\end{cases}$$

where

$$\begin{aligned}\sigma_x^\varepsilon(t,\Xi\mathbf{I}_{E_\varepsilon})&=\int_0^1\sigma_x(t,\Pi^*(t,\Xi\mathbf{I}_{E_\varepsilon})+\lambda(\Pi^\varepsilon(t)-\Pi^*(t,\Xi\mathbf{I}_{E_\varepsilon})),\mathbb{P}_{\Lambda^*(t)+\lambda(\Lambda^\varepsilon(t)-\Lambda^*(t))},v^\varepsilon(t))d\lambda,\\\hat{\mathring{\sigma}}^\varepsilon_{\mu_1}(t,\Xi\mathbf{I}_{E_\varepsilon})&=\int_0^1\sigma_{\mu_1}(t,\Pi^*(t,\Xi\mathbf{I}_{E_\varepsilon})+\lambda(\Pi^\varepsilon(t)-\Pi^*(t,\Xi\mathbf{I}_{E_\varepsilon})),\mathbb{P}_{\Lambda^*(t)+\lambda(\Lambda^\varepsilon(t)-\Lambda^*(t))},v^\varepsilon(t);\end{aligned}$$

$$\hat{\Lambda}^*(t)+\lambda(\hat{\Lambda}^\varepsilon(t)-\hat{\Lambda}^*(t)))d\lambda,$$

$$\begin{aligned}
A_2^\varepsilon(t)=&(b_x^\varepsilon(t)-b_x(t))X^{1,\varepsilon}(t)+(b_y^\varepsilon(t)-b_y(t))Y^{1,\varepsilon}(t)+(b_z^\varepsilon(t)-b_z(t))Z^{1,\varepsilon}(t)+\delta b(t)\mathbf{I}_{E_\varepsilon}(t)\\
&+\hat{\mathbb{E}}[(\hat{\mathring{b}}_{\mu_1}^\varepsilon(t)-\hat{\mathring{b}}_{\mu_1}(t))\hat{X}^{1,\varepsilon}(t)]+\hat{\mathbb{E}}[(\hat{\mathring{b}}_{\mu_2}^\varepsilon(t)-\hat{\mathring{b}}_{\mu_2}(t))\hat{Y}^{1,\varepsilon}(t)]+b_z(t)\Xi(t)\mathbf{I}_{E_\varepsilon}(t),\\
B_2^\varepsilon(t)=&(\sigma_x^\varepsilon(t,\Xi\mathbf{I}_{E_\varepsilon})-\sigma_x(t))X^{1,\varepsilon}(t)+(\sigma_y^\varepsilon(t,\Xi\mathbf{I}_{E_\varepsilon})-\sigma_y(t))Y^{1,\varepsilon}(t)\\
&+(\sigma_z^\varepsilon(t,\Xi\mathbf{I}_{E_\varepsilon})-\sigma_z(t))\cdot(Z^{1,\varepsilon}(t)-\Xi(t)\mathbf{I}_{E_\varepsilon}(t))\\
&+\hat{\mathbb{E}}[(\hat{\mathring{\sigma}}_{\mu_1}^\varepsilon(t,\Xi\mathbf{I}_{E_\varepsilon})-\hat{\mathring{\sigma}}_{\mu_1}(t))\hat{X}^{1,\varepsilon}(t)]+\hat{\mathbb{E}}[(\hat{\mathring{\sigma}}_{\mu_2}^\varepsilon(t,\Xi\mathbf{I}_{E_\varepsilon})-\hat{\mathring{\sigma}}_{\mu_2}(t))\hat{Y}^{1,\varepsilon}(t)],\\
C_2^\varepsilon(t)=&(f_x^\varepsilon(t)-f_x(t))X^{1,\varepsilon}(t)+(f_y^\varepsilon(t)-f_y(t))Y^{1,\varepsilon}(t)+(f_z^\varepsilon(t)-f_z(t))Z^{1,\varepsilon}(t)\\
&+q_0(t)\delta\sigma(t,\Xi)(t)\mathbf{I}_{E_\varepsilon}(t)+\hat{\mathbb{E}}[(\hat{\mathring{f}}_{\mu_1}^\varepsilon(t)-\hat{\mathring{f}}_{\mu_1}(t))\hat{X}^{1,\varepsilon}(t)]\\
&+\hat{\mathbb{E}}[(\hat{\mathring{f}}_{\mu_2}^\varepsilon(t)-\hat{\mathring{f}}_{\mu_2}(t))\hat{Y}^{1,\varepsilon}(t)]+f_z(t)\Xi(t)\mathbf{I}_{E_\varepsilon}(t)+\delta f(t)\mathbf{I}_{E_\varepsilon}(t),\\
D_2^\varepsilon(t)=&(\Phi_x^\varepsilon(T)-\Phi_x(T))X^{1,\varepsilon}(T)+\hat{\mathbb{E}}[(\hat{\mathring{\Phi}}_\nu^\varepsilon(T)-\hat{\mathring{\Phi}}_\nu(T))\hat{X}^{1,\varepsilon}(T)].
\end{aligned}$$

Thanks to Theorem 2.2, we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\big(|\mathcal{X}^2(t)|^4+|\mathcal{Y}^2(t)|^4\big)+\Big(\int_0^T|\mathcal{Z}^2(t)|^2dt\Big)^2\Big]\\
\le& L\mathbb{E}\Big\{\Big(\int_0^T|A_2^\varepsilon(t)|dt\Big)^4+\Big(\int_0^T|C_2^\varepsilon(t)|dt\Big)^4+\Big(\int_0^T|B_2^\varepsilon(t)|^2dt\Big)^2+|D_2^\varepsilon(T)|^4\Big\}.
\end{aligned}\tag{4.10}$$

Now we analyse $C_2^\varepsilon(t)$.

a$_1$) According to the Lipschitz property of f_z and (4.7)-i), ii), we have

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^T|f_z^\varepsilon(t)-f_z(t)|Z^{1,\varepsilon}(t)dt\Big)^4\Big]\\
\le&\Big\{\mathbb{E}\Big[\int_0^T|f_z^\varepsilon(t)-f_z(t)|^8dt\Big]\Big\}^{\frac12}\Big\{\mathbb{E}\Big[\Big(\int_0^T|Z^{1,\varepsilon}(t)|^2dt\Big)^4\Big]\Big\}^{\frac12}\\
\le& L\Big\{\mathbb{E}\Big[\sup_{0\le t\le T}\big(|\mathcal{X}^1(t)|^8+|\mathcal{Y}^1(t)|^8+||\mathcal{X}^1(t)||_{L^2}^8+||\mathcal{Y}^1(t)||_{L^2}^8\big)\\
&\quad+\Big(\int_0^T|\mathcal{Z}^1(t)|^2+|\delta f_z(t)|^2\mathbf{I}_{E_\varepsilon}(t)dt\Big)^4\Big]\Big\}^{\frac12}\Big\{\mathbb{E}\Big[\Big(\int_0^T|Z^{1,\varepsilon}(t)|^2dt\Big)^4\Big]\Big\}^{\frac12}\le C\varepsilon^4.
\end{aligned}$$

a$_2$) Thanks to Assumption (A3.1), one can check

$$\begin{aligned}
\delta f(t)=&f(t,\Pi^*(t),\mathbb{P}_{\Lambda^*(t)},v(t))-f(t,\Pi^*(t),\mathbb{P}_{\Lambda^*(t)},u^*(t))\\
=&f(t,X^*(t),Y^*(t),Z^*(t),\mathbb{P}_{(X^*(t),Y^*(t))},v(t))-f(t,0,0,Z^*(t),\delta_{\mathbf{0}},v(t))\\
&-(f(t,X^*(t),Y^*(t),Z^*(t),\mathbb{P}_{(X^*(t),Y^*(t))},u^*(t))-f(t,0,0,Z^*(t),\delta_{\mathbf{0}},u^*(t)))\\
&+f(t,0,0,Z^*(t),\delta_{\mathbf{0}},v(t))-f(t,0,0,Z^*(t),\delta_{\mathbf{0}},u^*(t))\\
\le& L(1+|X^*(t)|+|Y^*(t)|+||X^*(t)||_{L^2}+||Y^*(t)||_{L^2}+|v(t)|+|u^*(t)|).
\end{aligned}$$

Hence, $\mathbb{E}\big[\big(\int_0^T|\delta f(t)|\mathbf{I}_{E_\varepsilon}(t)dt\big)^4\big]\le L\varepsilon^4$.

a$_3$) Since $|\hat{\mathring{f}}_{\mu_1}^\varepsilon(t)-\hat{\mathring{f}}_{\mu_1}(t)|\le L(|\mathcal{X}^1(t)|+|\mathcal{Y}^1(t)|+|\mathcal{Z}^1(t)|+||\mathcal{X}^1(t)||_{L^2}+||\mathcal{Y}^1(t)||_{L^2}+|\hat{\mathcal{X}}^1(t)|+|\hat{\mathcal{Y}}^1(t)|)$, then it follows from Hölder inequality and (4.7)-i), ii),

$$\mathbb{E}\Big[\Big(\int_0^T|\hat{\mathbb{E}}[(\hat{\mathring{f}}_{\mu_1}^\varepsilon(t)-\hat{\mathring{f}}_{\mu_1}(t))\hat{X}^{1,\varepsilon}(t)]|dt\Big)^4\Big]$$

$$\leq L\mathbb{E}\Big[\sup_{t\in[0,T]}|X^{1,\varepsilon}(t)|^4\Big]\mathbb{E}\Big[\Big(\int_0^T \hat{\mathbb{E}}[|\hat{\mathring{f}}^{\varepsilon}_{\mu_1}(t)-\hat{\mathring{f}}_{\mu_1}(t)|^2]dt\Big)^2\Big]\leq L\varepsilon^4.$$

Consequently, $\mathbb{E}\big[\big(\int_0^T|C_2^{\varepsilon}(t)|dt\big)^4\big]\leq L\varepsilon^4$.

As for $B_2^{\varepsilon}(t)$, we only estimate the terms of $(\sigma_z^{\varepsilon}(t,\Xi\mathbf{I}_{E_\varepsilon})-\sigma_z(t))(Z^{1,\varepsilon}(t)-\Xi(t)\mathbf{I}_{E_\varepsilon}(t))$ and $\hat{\mathbb{E}}[(\hat{\mathring{\sigma}}^{\varepsilon}_{\mu_2}(t,\Xi\mathbf{I}_{E_\varepsilon})-\hat{\mathring{\sigma}}_{\mu_2}(t))\hat{Y}^{1,\varepsilon}(t)]$.

b$_1$) Notice

$$\begin{aligned}
&|\sigma_z^{\varepsilon}(t,\Xi\mathbf{I}_{E_\varepsilon})-\sigma_z(t)|\\
\leq&|\sigma_z(t,\Pi^*(t,\Xi\mathbf{I}_{E_\varepsilon}),\mathbb{P}_{\Lambda^*(t)},v^{\varepsilon}(t))-\sigma_z(t)|+|\sigma_z^{\varepsilon}(t,\Xi\mathbf{I}_{E_\varepsilon})-\sigma_z(t,\Pi^*(t,\Xi\mathbf{I}_{E_\varepsilon}),\mathbb{P}_{\Lambda^*(t)},v^{\varepsilon}(t))|\\
\leq&L(1+|X^*(t)|+|Y^*(t)|+||X^*(t)||_{L^2}+||Y^*(t)||_{L^2}+|v(t)|+|u^*(t)|+|\Xi(t)|)\mathbf{I}_{E_\varepsilon}(t)\\
&+L(|\mathcal{X}^1(t)|+|\mathcal{Y}^1(t)|+|\mathcal{Z}^1(t)-\Xi(t)\mathbf{I}_{E_\varepsilon}(t)|+||\mathcal{X}^1(t)||_{L^2}+||\mathcal{Y}^1(t)||_{L^2}),
\end{aligned}$$

and

$$Z^{1,\varepsilon}(t)-\Xi(t)\mathbf{I}_{E_\varepsilon}(t)=k_0(t)X^{1,\varepsilon}(t)+\hat{\mathbb{E}}[\hat{\mathring{k}}_1(t)\hat{X}^{1,\varepsilon}(t)]$$

(see (4.2)). From the boundness of k_0, $\hat{\mathring{k}}_1(t)$ and (4.7)-i), ii), we can get

$$\mathbb{E}\Big(\int_0^T|(\sigma_z^{\varepsilon}(t,\Xi\mathbf{I}_{E_\varepsilon})-\sigma_z(t))(Z^{1,\varepsilon}(t)-\Xi(t)\mathbf{I}_{E_\varepsilon}(t))|^2dt\Big)^2\leq L\varepsilon^4. \tag{4.11}$$

b$_2$) With the help of the Lipschitz property of $\hat{\mathring{\sigma}}_{\mu_2}$, we obtain

$$\begin{aligned}
\hat{\mathring{\sigma}}_{\mu_2}(t,\Xi\mathbf{I}_{E_\varepsilon})-\hat{\mathring{\sigma}}_{\mu_2}(t)\leq&\delta\hat{\mathring{\sigma}}_{\mu_2}(t,\Xi)\mathbf{I}_{E_\varepsilon}(t)+|\mathcal{X}^1(t)|+|\mathcal{Y}^1(t)|+|\mathcal{Z}^1(t)-\Xi(t)\mathbf{I}_{E_\varepsilon}(t)|\\
&+||\mathcal{X}^1(t)||_{L^2}+||\mathcal{Y}^1(t)||_{L^2}+|\hat{\mathcal{X}}^1(t)|+|\hat{\mathcal{Y}}^1(t)|.
\end{aligned}$$

The boundness of $\hat{\mathring{\sigma}}_{\mu_2},p_0,\hat{\mathring{p}}_1$, the relation (4.2) and Hölder inequality can imply

$$\begin{aligned}
&\hat{\mathbb{E}}[(\hat{\mathring{\sigma}}^{\varepsilon}_{\mu_2}(t,\Xi\mathbf{I}_{E_\varepsilon})-\hat{\mathring{\sigma}}_{\mu_2}(t))\hat{Y}^{1,\varepsilon}(t)]\\
=&\hat{\mathbb{E}}[(\hat{\mathring{\sigma}}^{\varepsilon}_{\mu_2}(t,\Xi\mathbf{I}_{E_\varepsilon})-\hat{\mathring{\sigma}}_{\mu_2}(t))\hat{p}_0(t)\hat{X}^{1,\varepsilon}(t)]+\hat{\mathbb{E}}[(\hat{\mathring{\sigma}}^{\varepsilon}_{\mu_2}(t,\Xi\mathbf{I}_{E_\varepsilon})-\hat{\mathring{\sigma}}_{\mu_2}(t))\bar{E}[\bar{\hat{p}}_1(t)\bar{X}^{1,\varepsilon}(t)]]\\
\leq&L\big\{\hat{\mathbb{E}}[|\hat{\mathring{\sigma}}^{\varepsilon}_{\mu_2}(t,\Xi\mathbf{I}_{E_\varepsilon})-\hat{\mathring{\sigma}}_{\mu_2}(t)|^2]\big\}^{\frac{1}{2}}\big\{\mathbb{E}|X^{1,\varepsilon}(t)|^2\big\}^{\frac{1}{2}}\\
&+L\mathbb{E}\Big[\sup_{t\in[0,T]}|X^{1,\varepsilon}(t)|\Big]\hat{\mathbb{E}}[|\hat{\mathring{\sigma}}^{\varepsilon}_{\mu_2}(t,\Xi\mathbf{I}_{E_\varepsilon})-\hat{\mathring{\sigma}}_{\mu_2}(t)|].
\end{aligned}$$

Hence, from (4.7)-i) we obtain

$$\mathbb{E}\Big(\int_0^T|\hat{\mathbb{E}}[(\hat{\mathring{\sigma}}^{\varepsilon}_{\mu_2}(t,\Xi\mathbf{I}_{E_\varepsilon})-\hat{\mathring{\sigma}}_{\mu_2}(t))\hat{Y}^{1,\varepsilon}(t)]|^2dt\Big)^2\leq L\varepsilon^4.$$

Similar to $C_2^{\varepsilon}(t),B_2^{\varepsilon}(t)$, we also have $\mathbb{E}\big[\big(\int_0^T|A_2^{\varepsilon}(t)|dt\big)^4+|D_2^{\varepsilon}(T)|^4\big]\leq L\varepsilon^4$. The proof is complete. □

Remark 4.4. It should be point that if (4.2) does not hold true, we can not obtain (4.11). In fact, we have to calculate $\mathbb{E}\big[\int_0^T|\mathcal{Z}^1(t)Z^{1,\varepsilon}(t)|^2dt\big]=\mathbb{E}\big[\int_0^T|(Z^{\varepsilon}(t)-Z^*(t))Z^{1,\varepsilon}(t)|^2dt\big]$ when estimating (4.11). But from (4.7)-i), ii), we can not get $\mathbb{E}\big(\int_0^T|\mathcal{Z}^1(t)Z^{1,\varepsilon}(t)|^2dt\big)^2\leq L\varepsilon^4$. Hence, the relation (4.2) plays an important role in our analysis.

4.2 Second-order Variational Equation

The second-order variational equation can be read as

$$\begin{cases} dX^{2,\varepsilon} = \Big\{ b_x(t)X^{2,\varepsilon}(t) + b_y(t)Y^{2,\varepsilon}(t) + b_z(t)Z^{2,\varepsilon}(t) + \hat{\mathbb{E}}[\hat{\mathring{b}}_{\mu_1}(t)\hat{X}^{2,\varepsilon}(t)] + \hat{\mathbb{E}}[\hat{\mathring{b}}_{\mu_2}(t)\hat{Y}^{2,\varepsilon}(t)] \\ \qquad + \frac{1}{2}(X^{1,\varepsilon}(t))^2[1, p_0(t), k_0(t)]D^2 b(t)[1, p_0(t), k_0(t)]^\intercal + \frac{1}{2}\hat{\mathbb{E}}[\hat{\mathring{b}}_{\mu_1 a_1}(t)(\hat{X}^{1,\varepsilon}(t))^2] \\ \qquad + \frac{1}{2}\hat{\mathbb{E}}[\hat{\mathring{b}}_{\mu_2 a_2}(t)(\hat{X}^{1,\varepsilon}(t))^2(\hat{p}_0(t))^2] + \delta b(t,\Xi)\mathbf{I}_{E_\varepsilon}(t)\Big\}dt \\ \qquad + \Big\{ \sigma_x(t)X^{2,\varepsilon}(t) + \sigma_y(t)Y^{2,\varepsilon}(t) + \sigma_z(t)Z^{2,\varepsilon}(t) + \hat{\mathbb{E}}[\hat{\mathring{\sigma}}_{\mu_1}(t)\hat{X}^{2,\varepsilon}(t)] \\ \qquad + \frac{1}{2}(X^{1,\varepsilon}(t))^2[1, p_0(t), k_0(t)]D^2\sigma(t)[1, p_0(t), k_0(t)]^\intercal + \frac{1}{2}\hat{\mathbb{E}}[\hat{\mathring{\sigma}}_{\mu_1 a_1}(t)(\hat{X}^{1,\varepsilon}(t))^2] \\ \qquad + \frac{1}{2}\hat{\mathbb{E}}[\hat{\mathring{\sigma}}_{\mu_2 a_2}(t)(\hat{X}^{1,\varepsilon}(t))^2(\hat{p}_0(t))^2] + \delta\sigma_x(t,\Xi)\mathbf{I}_{E_\varepsilon}(t)X^{1,\varepsilon}(t) + \hat{\mathbb{E}}[\hat{\mathring{\sigma}}_{\mu_2}(t)\hat{Y}^{2,\varepsilon}(t)] \\ \qquad + \delta\sigma_y(t,\Xi)\mathbf{I}_{E_\varepsilon}(t)p_0(t)X^{1,\varepsilon}(t) + \delta\sigma_z(t,\Xi)\mathbf{I}_{E_\varepsilon}(t)k_0(t)X^{1,\varepsilon}(t)\Big\}dW(t),\ t\in[0,T], \\ X^{2,\varepsilon}(0) = 0, \end{cases}$$

$$\begin{cases} dY^{2,\varepsilon}(t) = -\Big\{ f_x(t)X^{2,\varepsilon}(t) + f_y(t)Y^{2,\varepsilon}(t) + f_z(t)Z^{2,\varepsilon}(t) + \hat{\mathbb{E}}[\hat{\mathring{f}}_{\mu_1}(t)\hat{X}^{2,\varepsilon}(t)] \\ \qquad + \hat{\mathbb{E}}[\hat{\mathring{f}}_{\mu_2}(t)\hat{Y}^{2,\varepsilon}(t)] + \frac{1}{2}(X^{1,\varepsilon}(t))^2[1, p_0(t), k_0(t)]D^2 f(t)[1, p_0(t), k_0(t)]^\intercal \\ \qquad + \frac{1}{2}\hat{\mathbb{E}}[\hat{\mathring{f}}_{\mu_1 a_1}(t)(\hat{X}^{1,\varepsilon}(t))^2] + \frac{1}{2}\hat{\mathbb{E}}[\hat{\mathring{f}}_{\mu_2 a_2}(t)(\hat{p}_0(t))^2(\hat{X}^{1,\varepsilon}(t))^2] + [q(t)\delta\sigma(t,\Xi) \\ \qquad + \delta f(t,\Xi)]\mathbf{I}_{E_\varepsilon}(t)\Big\}dt + Z^{2,\varepsilon}dW(t), t\in[0,T], \\ Y^{2,\varepsilon}(T) = \Phi_x(T)X^{2,\varepsilon}(T) + \hat{\mathbb{E}}[\Phi_\mu(T)X^{2,\varepsilon}(T)] + \frac{1}{2}\Phi_{xx}(T)(X^{1,\varepsilon}(T))^2 + \frac{1}{2}\hat{\mathbb{E}}[\Phi_{\nu a}(T)(\hat{X}^{1,\varepsilon}(T))^2]. \end{cases}$$

Proposition 4.5. *Under Assumptions (A3.1)–(A3.3), for any $2 \le \beta \le 4$, there exists a constant $L > 0$ such that*

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big(|X^{2,\varepsilon}(t)|^\beta + |Y^{2,\varepsilon}(t)|^\beta\big) + \Big(\int_0^T |Z^{2,\varepsilon}(t)|^2dt\Big)^{\frac{\beta}{2}}\Big] \le L\varepsilon^\beta.$$

Proof. Thanks to Theorem 2.2, (4.7) and Hölder inequality, one can obtain

$$\begin{aligned} &\mathbb{E}\Big[\sup_{t\in[0,T]}\big(|X^{2,\varepsilon}(t)|^\beta + |Y^{2,\varepsilon}(t)|^\beta\big) + \Big(\int_0^T |Z^{2,\varepsilon}(t)|^2dt\Big)^{\frac{\beta}{2}}\Big] \\ \le& L\mathbb{E}\Big[\Big(\int_0^T (|\delta b(t,\Xi)| + |\delta\sigma(t,\Xi)| + |\delta f(t,\Xi)|)\mathbf{I}_{E_\varepsilon}(t) + |X^{1,\varepsilon}(t)|^2 + \mathbb{E}[|X^{1,\varepsilon}(t)|^2])dt\Big)^\beta\Big] \\ &+ L\mathbb{E}\Big[\Big(\int_0^T |X^{1,\varepsilon}(t)|^4 + \mathbb{E}[|X^{1,\varepsilon}(t)|^4] + |X^{1,\varepsilon}(t)|^2\mathbf{I}_{E_\varepsilon}(t)dt\Big)^{\frac{\beta}{2}}\Big] \\ \le& L\varepsilon^{\frac{\beta}{2}}\mathbb{E}\Big[\Big(\int_{E_\varepsilon} (1 + |X^*(t)| + |Y^*(t)| + ||X^*(t)||_{L^2} + ||Y^*(t)||_{L^2} + |v(t)| + |u^*(t)|)dt\Big)^{\frac{\beta}{2}}\Big] \\ &+ L\mathbb{E}\Big[\sup_{t\in[0,T]}|X^{1,\varepsilon}(t)|^{2\beta}\Big] + L\varepsilon^{\frac{\beta}{2}}\mathbb{E}\Big[\sup_{t\in[0,T]}|X^{1,\varepsilon}(t)|^\beta\Big] \le L\varepsilon^\beta. \end{aligned}$$

□

Lemma 4.6. *Let Assumptions (H3.1) hold and let* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ *be an intermediate probability space and independent of space* $(\Omega, \mathcal{F}, \mathbb{P})$, *and let* $(\varphi_1(\omega, \tilde{\omega}, t))_{t\in[0,T]}, (\varphi_2(\tilde{\omega}, t))_{t\in[0,T]}$ *be two stochastic processes defined on the product space* $(\Omega\times\tilde{\Omega}, \mathcal{F}\times\tilde{\mathcal{F}}, \mathbb{P}\otimes\tilde{\mathbb{P}})$ *and the space* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, *respectively. Moreover, assume* $\varphi_i,\ i = 1, 2$ *satisfies the following properties:*

i) *There exists a constant* $C > 0$ *such that, for* $t \in [0, T]$, $|\varphi_1(\omega, \tilde{\omega}, t)| \leq C, \mathbb{P}\otimes\tilde{\mathbb{P}}$*-a.s.*

ii) *For* $\beta \geq 1$, *there exists a constant* C_β *depending on* β *such that* $\tilde{\mathbb{E}}[\sup_{t\in[0,T]} |\varphi_2(\tilde{\omega}, t)|^{2\beta}] \leq C_\beta$. *Then*

$$\mathbb{E}\Big[\int_0^T |\tilde{\mathbb{E}}[\varphi_1(\omega, \tilde{\omega}, t)\varphi_2(\tilde{\omega}, t)\tilde{X}^{1,\varepsilon}(t)]|^4 dt\Big] \leq \varepsilon^2\rho(\varepsilon). \tag{4.12}$$

Proof. Insert (4.2) into (4.1), we have

$$\begin{cases} dX^{1,\varepsilon}(t) = \big\{\hat{\mathbb{E}}\big[\hat{X}^{1,\varepsilon}(t)\big(b_y(t)\hat{\mathring{p}}_1(t) + b_z(t)\hat{\mathring{k}}_1(t) + \hat{\mathring{b}}_{\mu_1}(t) + \hat{\mathring{b}}_{\mu_2}(t)\hat{p}_0(t) \\ \qquad\qquad + \bar{\mathbb{E}}[\bar{\mathring{b}}_{\mu_2}(t)\bar{\hat{p}}_1(t)]\big)\big] + X^{1,\varepsilon}(t)\big(b_x(t) + b_y(t)p_0(t) + b_z(t)k_0(t)\big)\big\}dt \\ \qquad + \big\{\hat{\mathbb{E}}\big[\hat{X}^{1,\varepsilon}(t)\big(\sigma_y(t)\hat{\mathring{p}}_1(t) + \sigma_z(t)\hat{\mathring{k}}_1(t) + \hat{\mathring{\sigma}}_{\mu_1}(t) + \hat{\mathring{\sigma}}_{\mu_2}(t)\hat{p}_0(t) \\ \qquad\qquad + \bar{\mathbb{E}}[\bar{\mathring{\sigma}}_{\mu_2}(t)\bar{\hat{p}}_1(t)]\big)\big] + X^{1,\varepsilon}(t)\big(\sigma_x(t) + \sigma_y(t)p_0(t) \\ \qquad\qquad + \sigma_z(t)k_0(t)\big) + \delta\sigma(t, \Xi)\mathbf{I}_{E_\varepsilon}(t)\big\}dW(t),\ t \in [0, T], \\ X^{1,\varepsilon}(0) = 0. \end{cases} \tag{4.13}$$

Notice the coefficients of the above equation are bounded, similar to the proof of Proposition 4.3[6], we have the desired result. □

Proposition 4.7. *We make the same assumption as Proposition 4.5, then*

$$Y^\varepsilon(0) = Y^*(0) + Y^{1,\varepsilon}(0) + Y^{2,\varepsilon}(0) + o(\varepsilon).$$

Proof. Define $\mathcal{X}^3(t) = X^\varepsilon(t) - X^*(t) - X^{1,\varepsilon}(t) - X^{2,\varepsilon}(t), \mathcal{Y}^3(t) = Y^\varepsilon(t) - Y^*(t) - Y^{1,\varepsilon}(t) - Y^{2,\varepsilon}(t), \mathcal{Z}^3(t) = Z^\varepsilon(t) - Z^*(t) - Z^{1,\varepsilon}(t) - Z^{2,\varepsilon}(t)$. Then we have

$$\begin{cases} d\mathcal{X}^3(t) = \big\{b_x(t)\mathcal{X}^3(t) + b_y(t)\mathcal{Y}^3(t) + b_z(t)\mathcal{Z}^3(t) + \hat{\mathbb{E}}[\hat{\mathring{b}}_{\mu_1}(t)\hat{\mathcal{X}}^3(t)] \\ \qquad\qquad + \hat{\mathbb{E}}[\hat{\mathring{b}}_{\mu_2}(t)\hat{\mathcal{Y}}^3(t)] + A_3^\varepsilon(t)\big\}dt \\ \qquad + \big\{\sigma_x(t)\mathcal{X}^3(t) + \sigma_y(t)\mathcal{Y}^3(t) + \sigma_z(t)\mathcal{Z}^3(t) + \hat{\mathbb{E}}[\hat{\mathring{b}}_{\mu_1}(t)\hat{\mathcal{X}}^3(t)] \\ \qquad\qquad + \hat{\mathbb{E}}[\hat{\mathring{b}}_{\mu_2}(t)\hat{\mathcal{Y}}^3(t)] + B_3^\varepsilon(t)\big\}dW(t), \\ d\mathcal{Y}^3(t) = -\big\{f_x(t)\mathcal{X}^3(t) + f_y(t)\mathcal{Y}^3(t) + f_z(t)\mathcal{Z}^3(t) + \hat{\mathbb{E}}[\hat{\mathring{f}}_{\mu_1}(t)\hat{\mathcal{X}}^3(t)] \\ \qquad\qquad + \hat{\mathbb{E}}[\hat{\mathring{f}}_{\mu_2}(t)\hat{\mathcal{Y}}^3(t)] + C_3^\varepsilon(t)\big\}dt + \mathcal{Z}^3(t)dW(t),\ t \in [0, T], \\ \mathcal{X}^3(0) = 0, \quad \mathcal{Y}^3(T) = \Phi_x(T)\mathcal{X}^3(T) + \hat{\mathbb{E}}[\hat{\mathring{\Phi}}_\nu(T)\hat{\mathcal{X}}^3(T)] + D_3^\varepsilon(T), \end{cases} \tag{4.14}$$

where

$$\begin{aligned} A_3^\varepsilon(t) = &\big\{\delta b_x(t, \Xi)\mathcal{X}^1(t) + \delta b_y(t, \Xi)\mathcal{Y}^1(t) + \delta b_z(t, \Xi)\big(\mathcal{Z}^1(t) - \Xi(t)\mathbf{I}_{E_\varepsilon}(t)\big) \\ &\quad + \hat{\mathbb{E}}[\delta\hat{\mathring{b}}_{\mu_1}(t, \Xi)\hat{\mathcal{X}}^1(t)] + \hat{\mathbb{E}}[\delta\hat{\mathring{\sigma}}_{\mu_2}(t, \Xi)\hat{\mathcal{Y}}^1(t)]\big\}\mathbf{I}_{E_\varepsilon}(t) \\ &+ \frac{1}{2}[\mathcal{X}^1(t), \mathcal{Y}^1(t), (\mathcal{Z}^1(t) - \Xi(t)\mathbf{I}_{E_\varepsilon}(t))]D^2b^\varepsilon(t, \Xi\mathbf{I}_{E_\varepsilon}) \\ &[\mathcal{X}^1(t), \mathcal{Y}^1(t), (\mathcal{Z}^1(t) - \Xi(t)\mathbf{I}_{E_\varepsilon}(t))]^\intercal \end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{1}{2}(X_1(t))^2[1, p_0(t), k_0(t)]D^2 b(t)[1, p_0(t), k_0(t)]^\intercal \\
&\quad + \frac{1}{2}\hat{\mathbb{E}}[\hat{\mathring{b}}^\varepsilon_{\mu_1 a_1}(t)(\hat{\mathcal{X}}^1(t))^2 - \hat{\mathring{b}}_{\mu_1 a_1}(t)(\hat{X}^{1,\varepsilon}(t))^2] \\
&\quad + \frac{1}{2}\hat{\mathbb{E}}[\hat{\mathring{b}}^\varepsilon_{\mu_2 a_2}(t)(\hat{\mathcal{Y}}^1(t))^2 - \hat{\mathring{b}}_{\mu_2 a_2}(t)(\hat{Y}^{1,\varepsilon}(t))^2], \\
B_3^\varepsilon(t) &= \big\{\delta\sigma_x(t,\Xi)\mathcal{X}^2(t) + \delta\sigma_y(t,\Xi)\mathcal{Y}^2(t) + \delta\sigma_z(t,\Xi)\mathcal{Z}^2(t) \\
&\quad + \hat{\mathbb{E}}[\delta\hat{\mathring{\sigma}}_{\mu_1}(t,\Xi)\hat{\mathcal{X}}^2(t)] + \hat{\mathbb{E}}[\delta\hat{\mathring{\sigma}}_{\mu_2}(t,\Xi)\hat{\mathcal{Y}}^2(t)]\big\}\mathbf{I}_{E_\varepsilon}(t) \\
&\quad + \frac{1}{2}[\mathcal{X}^1(t), \mathcal{Y}^1(t), (\mathcal{Z}^1(t) - \Xi(t)\mathbf{I}_{E_\varepsilon}(t))]D^2\sigma^\varepsilon(t, \Xi\mathbf{I}_{E_\varepsilon}) \\
&\quad [\mathcal{X}^1(t), \mathcal{Y}^1(t), (\mathcal{Z}^1(t) - \Xi(t)\mathbf{I}_{E_\varepsilon}(t))]^\intercal \\
&\quad - \frac{1}{2}(X^{1,\varepsilon}(t))^2[1, p_0(t), k_0(t)]D^2\sigma(t)[1, p_0(t), k_0(t)]^\intercal \qquad (4.15) \\
&\quad + \frac{1}{2}\hat{\mathbb{E}}[\hat{\mathring{\sigma}}^\varepsilon_{\mu_1 a_1}(t)(\hat{\mathcal{X}}^1(t))^2 - \hat{\mathring{\sigma}}_{\mu_1 a_1}(t)(\hat{X}^{1,\varepsilon}(t))^2] \\
&\quad + \frac{1}{2}\hat{\mathbb{E}}[\hat{\mathring{\sigma}}^\varepsilon_{\mu_2 a_2}(t)(\hat{\mathcal{Y}}^1(t))^2 - \hat{\mathring{\sigma}}_{\mu_2 a_2}(t)(\hat{Y}^{1,\varepsilon}(t))^2], \\
C_3^\varepsilon(t) &= \big\{\delta f_x(t,\Xi)\mathcal{X}^1(t) + \delta f_y(t,\Xi)\mathcal{Y}^1(t) + \delta f_z(t,\Xi)\big(\mathcal{Z}^1(t) - \Xi(t)\mathbf{I}_{E_\varepsilon}(t)\big) \\
&\quad + \hat{\mathbb{E}}[\delta\hat{\mathring{f}}_{\mu_1}(t,\Xi)\hat{\mathcal{X}}^1(t)] + \hat{\mathbb{E}}[\delta\hat{\mathring{f}}_{\mu_2}(t,\Xi)\hat{\mathcal{Y}}^1(t)]\big\}\mathbf{I}_{E_\varepsilon}(t) \\
&\quad + \frac{1}{2}[\mathcal{X}^1(t), \mathcal{Y}^1(t), (\mathcal{Z}^1(t) - \Xi(t)\mathbf{I}_{E_\varepsilon}(t))]D^2 f^\varepsilon(t, \Xi\mathbf{I}_{E_\varepsilon}) \\
&\quad [\mathcal{X}^1(t), \mathcal{Y}^1(t), (\mathcal{Z}^1(t) - \Xi(t)\mathbf{I}_{E_\varepsilon}(t))]^\intercal \\
&\quad - \frac{1}{2}(X^{1,\varepsilon}(t))^2[1, p_0(t), k_0(t)]D^2 f(t)[1, p_0(t), k_0(t)]^\intercal \\
&\quad + \frac{1}{2}\hat{\mathbb{E}}[\hat{\mathring{f}}^\varepsilon_{\mu_1 a_1}(t)(\hat{\mathcal{X}}^1(t))^2 - \hat{\mathring{f}}_{\mu_1 a_1}(t)(\hat{X}^{1,\varepsilon}(t))^2] \\
&\quad + \frac{1}{2}\hat{\mathbb{E}}[\hat{\mathring{f}}^\varepsilon_{\mu_2 a_2}(t)(\hat{\mathcal{Y}}^1(t))^2 - \hat{\mathring{f}}_{\mu_2 a_2}(t)(\hat{Y}^{1,\varepsilon}(t))^2], \\
D_3^\varepsilon(T) &= \frac{1}{2}\Big(\Phi^\varepsilon_{xx}(T)(\mathcal{X}^1(T))^2 - \Phi_{xx}(T)(X^1(T))^2\Big) \\
&\quad + \frac{1}{2}\hat{\mathbb{E}}\Big[\hat{\mathring{\Phi}}^\varepsilon_{\nu a}(T)(\hat{\mathcal{X}}^1(T))^2 - \hat{\mathring{\Phi}}_{\nu a}(T)(\hat{X}^1(T))^2\Big].
\end{aligned}
$$

Let us consider a fully coupled mean-field linear FBSDE:

$$
\begin{cases}
d\mathcal{R}(t) = \big\{f_y(t)\mathcal{R}(t) + b_y(t)\mathcal{S}(t) + \sigma_y(t)\mathcal{L}(t) + \hat{\mathbb{E}}[\hat{\mathring{f}}_{\mu_2}(t)\hat{\mathcal{R}}(t)] + \hat{\mathbb{E}}[\hat{\mathring{b}}_{\mu_2}(t)\hat{\mathcal{S}}(t)] \\
\qquad\qquad + \hat{\mathbb{E}}[\hat{\mathring{\sigma}}_{\mu_2}(t)\hat{\mathcal{L}}(t)]\big\}dt + \big\{f_z(t)\mathcal{R}(t) + b_z(t)\mathcal{S}(t) + \sigma_z(t)\mathcal{L}(t)\big\}dW(t), \\
d\mathcal{S}(t) = -\big\{f_x(t)\mathcal{R}(t) + b_x(t)\mathcal{S}(t) + \sigma_x(t)\mathcal{L}(t) + \hat{\mathbb{E}}[\hat{\mathring{f}}_{\mu_1}(t)\hat{\mathcal{R}}(t)] + \hat{\mathbb{E}}[\hat{\mathring{b}}_{\mu_1}(t)\hat{\mathcal{S}}(t)] \\
\qquad\qquad + \hat{\mathbb{E}}[\hat{\mathring{\sigma}}_{\mu_1}(t)\hat{\mathcal{L}}(t)]\big\}dt + \mathcal{L}(t)dW(t), \\
\mathcal{R}(0) = 1, \quad \mathcal{S}(T) = \Phi_x(T)\mathcal{R}(T) + \hat{\mathbb{E}}[\hat{\mathring{\Phi}}_\nu(T)\hat{\mathcal{R}}(T)].
\end{cases}
\qquad (4.16)
$$

Applying Itô formula to $\mathcal{S}(t)\mathcal{X}^3(t) - \mathcal{R}(t)\mathcal{Y}^3(t)$, we have

$$
\begin{aligned}
&d\big(\mathcal{S}(t)\mathcal{X}^3(t) - \mathcal{R}(t)\mathcal{Y}^3(t)\big) \\
=&\big\{\mathcal{S}(t)A_3^\varepsilon(t) + \mathcal{L}(t)B_3^\varepsilon(t) + \mathcal{R}(t)C_3^\varepsilon(t)\big\}dt + \big\{\mathcal{S}(t)\hat{\mathbb{E}}[\hat{\mathring{b}}_{\mu_1}(t)\hat{\mathcal{X}}^3(t)] - \mathcal{X}^3(t)\hat{\mathbb{E}}[\hat{\mathring{b}}_{\mu_1}(t)\hat{\mathcal{S}}(t)] \\
&+ \mathcal{R}(t)\hat{\mathbb{E}}[\hat{\mathring{f}}_{\mu_1}(t)\hat{\mathcal{X}}^3(t)] - \mathcal{X}^3(t)\hat{\mathbb{E}}[\hat{\mathring{f}}_{\mu_1}(t)\hat{\mathcal{R}}(t)] + \mathcal{L}(t)\hat{\mathbb{E}}[\hat{\mathring{\sigma}}_{\mu_1}(t)\hat{\mathcal{X}}^3(t)] - \mathcal{X}^3(t)\hat{\mathbb{E}}[\hat{\mathring{\sigma}}_{\mu_1}(t)\hat{\mathcal{L}}(t)]
\end{aligned}
$$

$$+\mathcal{S}(t)\hat{\mathbb{E}}[\hat{\mathring{b}}_{\mu_2}(t)\hat{\mathcal{Y}}^3(t)]-\mathcal{Y}^3(t)\hat{\mathbb{E}}[\mathring{\hat{b}}_{\mu_2}(t)\hat{\mathcal{S}}(t)]+\mathcal{R}(t)\hat{\mathbb{E}}[\hat{\mathring{f}}_{\mu_2}(t)\hat{\mathcal{Y}}^3(t)]-\mathcal{Y}^3(t)\hat{\mathbb{E}}[\mathring{\hat{f}}_{\mu_2}(t)\hat{\mathcal{R}}(t)]$$
$$+\mathcal{L}(t)\hat{\mathbb{E}}[\hat{\mathring{\sigma}}_{\mu_2}(t)\hat{\mathcal{Y}}^3(t)]-\mathcal{Y}^3(t)\hat{\mathbb{E}}[\mathring{\hat{\sigma}}_{\mu_2}(t)\hat{\mathcal{L}}(t)]\big\}dt+\{\cdots\}dW(t).$$

Integrating from 0 to T and taking expectation as well as notice

$$\mathbb{E}\big[\mathcal{S}(t)\hat{\mathbb{E}}[\hat{\mathring{b}}_{\mu_1}(t)\hat{\mathcal{X}}^3(t)]-\mathcal{X}^3(t)\hat{\mathbb{E}}[\mathring{\hat{b}}_{\mu_1}(t)\hat{\mathcal{S}}(t)]\big]=0,$$

we obtain

$$\mathcal{Y}^3(0)=\mathbb{E}[\mathcal{Y}^3(0)]=\mathbb{E}\Big[\mathcal{R}(T)D_3^\varepsilon(T)+\int_0^T\big(\mathcal{S}(t)A_3^\varepsilon(t)+\mathcal{L}(t)B_3^\varepsilon(t)+\mathcal{R}(t)C_3^\varepsilon(t)\big)dt\Big].$$

Now let us estimate $\mathcal{Y}^3(0)$ one by one.

a) First, from Hölder inequality, Assumption (A3.1) and Proposition 4.2 and notice $\mathcal{X}^1(t)=X^{1,\varepsilon}(t)+o(\varepsilon)$, it yields

$$\begin{aligned}&\mathbb{E}[|\mathcal{R}(T)D_3^\varepsilon(T)|]\\ \le&L\Big\{\mathbb{E}\Big[\Big|\frac{1}{2}\Big(\Phi_{xx}^\varepsilon(T)(\mathcal{X}^1(T))^2-\Phi_{xx}(T)(X^{1,\varepsilon}(T))^2\Big)\\ &\qquad+\frac{1}{2}\hat{\mathbb{E}}[(\hat{\mathring{\Phi}}_{\nu a}^\varepsilon(T)(\hat{\mathcal{X}}^1(T))^2-\hat{\mathring{\Phi}}_{\nu a}(T)(\hat{X}^{1,\varepsilon}(T))^2)]\Big|^2\Big]\Big\}^{\frac{1}{2}}\\ \le&L\Big\{\mathbb{E}\Big[(\Phi_{xx}^\varepsilon(T)-\Phi_{xx}(T))|^2|X^{1,\varepsilon}(T)|^4+\hat{\mathbb{E}}[|\hat{\mathring{\Phi}}_{\nu a}^\varepsilon(T)-\hat{\mathring{\Phi}}_{\nu a}(T)|^2|\hat{X}^{1,\varepsilon}(T)|^4]\Big]\Big\}^{\frac{1}{2}}+o(\varepsilon^2)\\ =&o(\varepsilon^2).\end{aligned}$$

b) Let us now analyse $\mathbb{E}\big[\int_0^T\mathcal{S}(t)A_3^\varepsilon(t)dt\big]$. Since $\mathbb{E}\big[\sup\limits_{0\le t\le T}|\mathcal{S}(t)|^2\big]<+\infty$, it is enough to prove $\mathbb{E}\big[\big(\int_0^T|A_3^\varepsilon(t)|dt\big)^2\big]=o(\varepsilon^2)$. We only prove the following two estimates:

$$\begin{aligned}&\text{i) }\mathbb{E}\Big[\Big(\int_0^T b_{zz}^\varepsilon(t,\Xi\mathbf{I}_{E_\varepsilon})(\mathcal{Z}^1(t)-\Xi(t)\mathbf{I}_{E_\varepsilon}(t))-b_{zz}(t)(k_0(t))^2(X^{1,\varepsilon}(t))^2dt\Big)^2\Big]=o(\varepsilon^2),\\ &\text{ii) }\mathbb{E}\Big[\Big(\int_0^T\hat{\mathbb{E}}[\hat{\mathring{\sigma}}_{\mu_2a_2}^\varepsilon(t)(\hat{\mathcal{Y}}^1(t))^2-\hat{\mathring{\sigma}}_{\mu_2a_2}(t)(\hat{Y}^{1,\varepsilon}(t))^2]dt\Big)^2\Big]=o(\varepsilon^2),\end{aligned}\tag{4.17}$$

because the other terms can be calculated similarly.

As for i), since $\mathcal{Z}^1(t)-\Xi(t)\mathbf{I}_{E_\varepsilon}(t)=\mathcal{Z}^2(t)+k_0(t)X^{1,\varepsilon}(t)+\hat{\mathbb{E}}[\hat{\mathring{k}}_1(t)\hat{X}^{1,\varepsilon}(t)]$, we have

$$\begin{aligned}&\mathbb{E}\Big[\Big(\int_0^T b_{zz}^\varepsilon(t,\Xi\mathbf{I}_{E_\varepsilon})\big(\mathcal{Z}^1(t)-\Xi(t)\mathbf{I}_{E_\varepsilon}(t)\big)-b_{zz}(t)(k_0(t))^2(X^{1,\varepsilon}(t))^2dt\Big)^2\Big]\\ \le&L\mathbb{E}\Big[\Big(\int_0^T b_{zz}^\varepsilon(t,\Xi\mathbf{I}_{E_\varepsilon})|\mathcal{Z}^2(t)|+(b_{zz}^\varepsilon(t,\Xi\mathbf{I}_{E_\varepsilon})-b_{zz}(t))(k_0(t))^2(X^{1,\varepsilon}(t))^2\\ &\qquad+b_{zz}^\varepsilon(t,\Xi\mathbf{I}_{E_\varepsilon})|\hat{\mathbb{E}}[\hat{\mathring{k}}_1(t)\hat{X}^{1,\varepsilon}(t)]|^2dt\Big)^2\Big]\\ \le&L\mathbb{E}\Big[\Big(\int_0^T|\mathcal{Z}^2(t)|^2dt\Big)^2\Big]+L\mathbb{E}\Big[\int_0^T|b_{zz}^\varepsilon(t,\Xi\mathbf{I}_{E_\varepsilon})-b_{zz}(t)|^2|X^{1,\varepsilon}(t)|^4dt\Big]\\ &+L\mathbb{E}[\int_0^T|\hat{\mathbb{E}}[\hat{\mathring{k}}_1(t)\hat{X}^{1,\varepsilon}(t)]|^4dt].\end{aligned}\tag{4.18}$$

According to (4.7)-iii), Lemma 4.6 and the continuity property of b_{zz}, we obtain (4.17)-i).

On the other hand, due to $\mathcal{Y}^1(t)=\mathcal{Y}^2(t)+Y^{1,\varepsilon}(t)$, the boundness of $\hat{\mathring{\sigma}}_{\mu_2a_2}$ and Hölder inequality allow to show

$$\begin{aligned}&\mathbb{E}\Big[\Big(\int_0^T\hat{\mathbb{E}}[\hat{\mathring{\sigma}}^{\varepsilon}_{\mu_2a_2}(t)(\hat{\mathcal{Y}}^1(t))^2-\hat{\mathring{\sigma}}_{\mu_2a_2}(t)(\hat{Y}^{1,\varepsilon}(t))^2]dt\Big)^2\Big]\\ \le& L\mathbb{E}\Big[\sup_{t\in[0,T]}|\mathcal{Y}^2(t)|^4\Big]+L\Big\{\mathbb{E}\Big[\sup_{t\in[0,T]}|\mathcal{Y}^2(t)|^4\Big]\Big\}^{\frac12}\Big\{\mathbb{E}\Big[\sup_{t\in[0,T]}|Y^{1,\varepsilon}(t)|^4\Big]\Big\}^{\frac12}\\ &+L\mathbb{E}\hat{\mathbb{E}}\Big[\int_0^T|\hat{\mathring{\sigma}}^{\varepsilon}_{\mu_2a_2}(t)-\hat{\mathring{\sigma}}_{\mu_2a_2}(t)||Y^{1,\varepsilon}(t)|^4dt\Big].\end{aligned}$$

Then (4.17)-ii) comes from Proposition 4.3 and the continuity property of $\hat{\mathring{\sigma}}_{\mu_2a_2}(t)$.

c) In order to estimate $\mathbb{E}[\int_0^T\mathcal{L}(t)B_3^{\varepsilon}(t)dt]$, we need to calculate the following four estimates:

$$\begin{aligned}&\text{i) }\mathbb{E}\Big[\int_0^T\mathcal{L}(t)\delta\sigma_z(t,\Xi)\big(\mathcal{Z}^1(t)-\Xi(t)\mathbf{I}_{E_\varepsilon}(t)\big)\mathbf{I}_{E_\varepsilon}(t)dt\Big]=o(\varepsilon);\\ &\text{ii) }\mathbb{E}\Big[\int_0^T\mathcal{L}(t)\hat{\mathbb{E}}[\delta\hat{\mathring{\sigma}}_{\mu_1}(t,\Xi)\hat{\mathcal{X}}^1(t)]\mathbf{I}_{E_\varepsilon}(t)dt\Big]=o(\varepsilon);\\ &\text{iii) }\mathbb{E}\Big[\int_0^T\mathcal{L}(t)\big(\sigma^{\varepsilon}_{zz}(t,\Xi\mathbf{I}_{E_\varepsilon})(\mathcal{Z}^1(t)-\Xi\mathbf{I}_{E_\varepsilon}(t))^2-\sigma_{zz}(t)(k_0(t))^2(X^{1,\varepsilon}(t))^2\big)dt\Big]=o(\varepsilon);\\ &\text{iv) }\mathbb{E}\Big[\int_0^T\mathcal{L}(t)\hat{\mathbb{E}}[\hat{\mathring{\sigma}}^{\varepsilon}_{\mu_2a_2}(t,\Xi)(\hat{\mathcal{Y}}^1(t))^2-\hat{\mathring{\sigma}}_{\mu_2a_2}(t)(\hat{Y}^{1,\varepsilon}(t))^2]dt\Big]=o(\varepsilon).\end{aligned}\tag{4.19}$$

For i), recall

$$\mathcal{Z}^1(t)-\Xi(t)\mathbf{I}_{E_\varepsilon}(t)=\mathcal{Z}^2(t)+Z^{1,\varepsilon}(t)-\Xi(t)\mathbf{I}_{E_\varepsilon}(t)=\mathcal{Z}^2(t)+k_0(t)X^{1,\varepsilon}(t)+\hat{\mathbb{E}}[\hat{\mathring{k}}_1(t)\hat{X}^{1,\varepsilon}(t)],$$

then we get from the boundness of σ_z and Hölder inequality,

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T\mathcal{L}(t)\delta\sigma_z(t,\Xi)\big(\mathcal{Z}^1(t)-\Xi(t)\mathbf{I}_{E_\varepsilon}(t)\big)\mathbf{I}_{E_\varepsilon}(t)dt\Big]\\ \le& L\mathbb{E}\Big[\int_0^T|\mathcal{L}(t)||\mathcal{Z}^2(t)+k_0(t)X^{1,\varepsilon}(t)+\hat{\mathbb{E}}[\hat{\mathring{k}}_1(t)\hat{X}^{1,\varepsilon}(t)]|\mathbf{I}_{E_\varepsilon}(t)dt\Big]\\ \le& L\Big\{\mathbb{E}\Big[\int_0^T|\mathcal{L}(t)|^2\mathbf{I}_{E_\varepsilon}(t)dt\Big]\Big\}^{\frac12}\Big\{\mathbb{E}\Big[\int_0^T|\mathcal{Z}^2(t)|^2dt\Big]\Big\}^{\frac12}\\ &+L\varepsilon^{\frac12}\Big\{\mathbb{E}\Big[\int_0^T|\mathcal{L}(t)|^2\mathbf{I}_{E_\varepsilon}(t)dt\Big]\Big\}^{\frac12}\Big\{\mathbb{E}\Big[\sup_{t\in[0,T]}|X^{1,\varepsilon}(t)|^2\Big]\Big\}^{\frac12}.\end{aligned}$$

Proposition 4.3 can show $\mathbb{E}\big[\int_0^T\mathcal{L}(t)\delta\sigma_z(t,\Xi)\big(\mathcal{Z}^1(t)-\Xi(t)\mathbf{I}_{E_\varepsilon}(t)\big)\mathbf{I}_{E_\varepsilon}(t)dt\big]\le L\varepsilon\rho(\varepsilon)$, where $\rho_1(\varepsilon):=L\big\{\mathbb{E}\big[\int_0^T|\mathcal{L}(t)|^2\mathbf{I}_{E_\varepsilon}(t)dt\big]\big\}^{\frac12}$. From Dominated Convergence Theorem, $\rho_1(\varepsilon)$ tends to 0, as $\varepsilon\to0$. (4.19)-ii) can be proved by Proposition 4.3, Hölder inequality and Dominated Convergence Theorem. Let us focus on (4.19)-iii). First,

$$\mathbb{E}\Big[\int_0^T\mathcal{L}(t)\big(\sigma^{\varepsilon}_{zz}(t,\Xi\mathbf{I}_{E_\varepsilon})(\mathcal{Z}^1(t)-\Xi(t)\mathbf{I}_{E_\varepsilon}(t))^2-\sigma_{zz}(t)(k_0(t))^2(X^{1,\varepsilon}(t))^2\big)dt\Big]=I_1(\varepsilon)+I_2(\varepsilon),$$

where

$$I_1(\varepsilon):=\mathbb{E}\Big[\int_0^T\mathcal{L}(t)\big(\sigma^{\varepsilon}_{zz}(t,\Xi\mathbf{I}_{E_\varepsilon})(\mathcal{Z}^1(t)-\Xi(t)\mathbf{I}_{E_\varepsilon}(t))^2-\sigma_{zz}(t)\big(k_0(t)X^{1,\varepsilon}(t)$$

$$+\hat{\mathbb{E}}[\mathring{\hat{k}}_1(t)\hat{X}^{1,\varepsilon}(t)]\big)^2\big)dt\Big];$$

$$I_2(\varepsilon):=\mathbb{E}\Big[\int_0^T\mathcal{L}(t)\sigma_{zz}(t)\big[\big(k_0(t)X^{1,\varepsilon}(t)+\hat{\mathbb{E}}[\mathring{\hat{k}}_1(t)\hat{X}^{1,\varepsilon}(t)]\big)^2-(k_0(t))^2\big(X^{1,\varepsilon}(t)\big)^2\big]dt\Big].$$

For $I_2(\varepsilon)$, the boundness of σ_{zz} and Lemma 4.6 can show

$$I_2(\varepsilon)\leq 2\mathbb{E}\Big[\int_0^T|\mathcal{L}(t)||\sigma_{zz}(t)||\hat{\mathbb{E}}[\mathring{\hat{k}}_1(t)\hat{X}^{1,\varepsilon}(t)]|^2dt\Big]\leq\varepsilon\rho(\varepsilon). \tag{4.20}$$

Let us now estimate $I_1(\varepsilon)$. Notice $I_1(\varepsilon)\leq I_{11}(\varepsilon)+I_{12}(\varepsilon)$, where

$$I_{11}(\varepsilon):=\mathbb{E}\Big[\int_0^T|\mathcal{L}(t)||\sigma^\varepsilon_{zz}(t,\Xi\mathbf{I}_{E_\varepsilon})-\sigma_{zz}(t)|(k_0(t)X^{1,\varepsilon}(t)+\hat{\mathbb{E}}[\mathring{\hat{k}}_1(t)\hat{X}^{1,\varepsilon}(t)])^2dt\Big];$$

$$I_{12}(\varepsilon):=\mathbb{E}\Big[\int_0^T|\mathcal{L}(t)||\sigma^\varepsilon_{zz}(t,\Xi\mathbf{I}_{E_\varepsilon})||\mathcal{Z}^2(t)||Z^1(t)-\Xi(t)\mathbf{I}_{E_\varepsilon}(t)+k_0(t)X^{1,\varepsilon}(t)$$
$$+\hat{\mathbb{E}}[\mathring{\hat{k}}_1(t)\hat{X}^{1,\varepsilon}(t)]|dt\Big].$$

Due to

$$I_{11}(\varepsilon)\leq L\mathbb{E}\Big[\int_0^T|\mathcal{L}(t)||\sigma^\varepsilon_{zz}(t,\Xi\mathbf{I}_{E_\varepsilon})-\sigma_{zz}(t)|\big(\sup_{0\leq t\leq T}|X^{1,\varepsilon}(t)|+E[\sup_{0\leq t\leq T}|X^{1,\varepsilon}(t)|]\big)^2dt\Big]$$
$$\leq L\Big\{E\Big[\sup_{0\leq t\leq T}|X^{1,\varepsilon}(t)|^4\Big]\Big\}^{\frac12}\Big\{\mathbb{E}\Big(\int_0^T|\mathcal{L}(t)||\sigma^\varepsilon_{zz}(t,\Xi\mathbf{I}_{E_\varepsilon})-\sigma_{zz}(t)|dt\Big)^2\Big\}^{\frac12},$$

it is easy from Dominated Convergence Theorem and (4.7) to get $I_{11}(\varepsilon)\leq\varepsilon\rho(\varepsilon)$. The boundness of σ_{zz} implies

$$\begin{aligned}
&I_{12}(\varepsilon)\\
\leq&\mathbb{E}\Big[\int_0^T|\mathcal{L}(t)||\mathcal{Z}^2(t)||\sigma^\varepsilon_{zz}(t,\Xi\mathbf{I}_{E_\varepsilon})||k_0(t)X^{1,\varepsilon}(t)+\hat{\mathbb{E}}[\mathring{\hat{k}}_1(t)\hat{X}^{1,\varepsilon}(t)]|dt\Big]\\
&+\mathbb{E}\Big[\int_0^T|\mathcal{L}(t)||\mathcal{Z}^2(t)||\sigma^\varepsilon_{zz}(t,\Xi\mathbf{I}_{E_\varepsilon})(Z^1(t)-\Xi(t)\mathbf{I}_{E_\varepsilon}(t))|dt\Big]\\
\leq&L\Big\{E\Big[\sup_{0\leq t\leq T}|X^{1,\varepsilon}(t)|^2\Big]\Big\}^{\frac12}\Big\{E\Big[\int_0^T|\mathcal{L}(t)|^2dt\Big]\Big\}^{\frac12}\Big\{E\Big[\int_0^T|\mathcal{Z}^2(t)|^2dt\Big]\Big\}^{\frac12}\\
&+\mathbb{E}\Big[\int_0^T|\mathcal{L}(t)||\mathcal{Z}^2(t)||2\int_0^1\sigma_z(t,\Pi^*(t,\Xi\mathbf{I}_{E_\varepsilon})+\lambda(\Pi^\varepsilon(t)-\Pi^*(t,\Xi\mathbf{I}_{E_\varepsilon})),\\
&\qquad\mathbb{P}_{\Lambda^*(t)+\lambda(\Lambda^\varepsilon(t)-\Lambda^*(t))},v^\varepsilon(t))-\sigma_z(t,\Pi^*(t,\Xi\mathbf{I}_{E_\varepsilon}),\mathbb{P}_{\Lambda^*(t)},v^\varepsilon(t))d\lambda|dt\Big]\\
&+\mathbb{E}\Big[\int_0^T|\mathcal{L}(t)||\mathcal{Z}^2(t)||\sigma_{zx}(t)\mathcal{X}^1(t)+\sigma_{zy}(t)\mathcal{Y}^1(t)+\hat{\mathbb{E}}[\mathring{\hat{\sigma}}_{z\mu_1}(t)\hat{\mathcal{X}}^1(t)+\hat{\mathbb{E}}[\mathring{\hat{\sigma}}_{z\mu_2}(t)\hat{\mathcal{Y}}^1(t)]|dt\Big].
\end{aligned}$$

Then, according to (4.7) and Dominated Convergence Theorem again, it follows $I_{12}(\varepsilon)\leq\varepsilon\rho(\varepsilon)$.

Let us now prove (4.19)-iv). Notice that

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T\mathcal{L}(t)\hat{\mathbb{E}}[\mathring{\hat{\sigma}}^\varepsilon_{\mu_2a_2}(t,\Xi)(\hat{\mathcal{Y}}^1(t))^2-\mathring{\hat{\sigma}}_{\mu_2a_2}(t)(\hat{Y}^{1,\varepsilon}(t))^2]dt\Big]\\
\leq&\mathbb{E}\Big[\int_0^T|\mathcal{L}(t)|\hat{\mathbb{E}}[|\mathring{\hat{\sigma}}^\varepsilon_{\mu_2a_2}(t,\Xi)-\mathring{\hat{\sigma}}_{\mu_2a_2}(t)||\hat{Y}^{1,\varepsilon}(t)|^2]dt\Big]\\
&+\mathbb{E}\Big[\int_0^T|\mathcal{L}(t)|\hat{\mathbb{E}}[|\mathring{\hat{\sigma}}^\varepsilon_{\mu_2a_2}(t,\Xi)|(|\hat{\mathcal{Y}}^1(t)|^2-|\hat{Y}^{1,\varepsilon}(t)|^2)]dt\Big].
\end{aligned}$$

According to Hölder inequality and the relation $Y^{1,\varepsilon}(t) = p_0(t)X^{1,\varepsilon}(t) + \hat{\mathbb{E}}[\hat{\mathring{p}}_1(t)\hat{X}^{1,\varepsilon}(t)]$, it follows

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^T |\mathcal{L}(t)|\hat{\mathbb{E}}[|\hat{\mathring{\sigma}}^{\varepsilon}_{\mu_2 a_2}(t,\Xi) - \hat{\mathring{\sigma}}_{\mu_2 a_2}(t)||\hat{Y}^{1,\varepsilon}(t)|^2]dt\Big]\\
\leq&\Big\{\mathbb{E}\int_0^T |\mathcal{L}(t)|^2 dt\Big\}^{\frac{1}{2}}\Big\{\mathbb{E}\hat{\mathbb{E}}\int_0^T |\hat{\mathring{\sigma}}^{\varepsilon}_{\mu_2 a_2}(t,\Xi) - \hat{\mathring{\sigma}}_{\mu_2 a_2}(t)|^2|k_0(t)|^4|X^{1,\varepsilon}(t)|^4 dt\Big\}^{\frac{1}{2}}\\
&+\Big\{\mathbb{E}\int_0^T |\mathcal{L}(t)|^2 dt\Big\}^{\frac{1}{2}}\Big\{\mathbb{E}\hat{\mathbb{E}}\int_0^T |\hat{\mathring{\sigma}}^{\varepsilon}_{\mu_2 a_2}(t,\Xi) - \hat{\mathring{\sigma}}_{\mu_2 a_2}(t)|^2|\tilde{\mathbb{E}}[\tilde{\hat{k}}_1(t)\tilde{X}^{1,\varepsilon}(t)]|^4 dt\Big\}^{\frac{1}{2}}.
\end{aligned}
$$

On the one hand, Dominated Convergence Theorem allows to show

$$
\Big\{\mathbb{E}\hat{\mathbb{E}}\int_0^T |\hat{\mathring{\sigma}}^{\varepsilon}_{\mu_2 a_2}(t,\Xi) - \hat{\mathring{\sigma}}_{\mu_2 a_2}(t)|^2|k_0(t)|^4|X^{1,\varepsilon}(t)|^4 dt\Big\}^{\frac{1}{2}} \leq \varepsilon\rho(\varepsilon).
$$

On the other hand, according to Lemma 4.6 and the boundness of $\hat{\mathring{\sigma}}_{\mu_2 a_2}$, we obtain

$$
\begin{aligned}
&\Big\{\mathbb{E}\hat{\mathbb{E}}\int_0^T |\hat{\mathring{\sigma}}^{\varepsilon}_{\mu_2 a_2}(t,\Xi) - \hat{\mathring{\sigma}}_{\mu_2 a_2}(t)|^2|\tilde{\mathbb{E}}[\tilde{\hat{k}}_1(t)\tilde{X}^{1,\varepsilon}(t)]|^4 dt\Big\}^{\frac{1}{2}}\\
\leq& L\Big\{\hat{\mathbb{E}}\Big[\int_0^T \tilde{\mathbb{E}}[\tilde{\hat{k}}_1(t)\tilde{X}^{1,\varepsilon}(t)]|^4 dt\Big]\Big\}^{\frac{1}{2}} \leq \varepsilon\rho(\varepsilon).
\end{aligned}
$$

Besides, thanks to the boundness of $\hat{\mathring{\sigma}}_{\mu_2 a_2}$ and Proposition 4.3, one can check

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^T |\mathcal{L}(t)|\hat{\mathbb{E}}\Big[|\hat{\mathring{\sigma}}^{\varepsilon}_{\mu_2 a_2}(t,\Xi)|(|\hat{\mathcal{Y}}^1(t)|^2 - |\hat{Y}^{1,\varepsilon}(t)|^2)\Big]dt\Big]\\
\leq& L\mathbb{E}\Big[\int_0^T |\mathcal{L}(t)|\hat{\mathbb{E}}[|\hat{\mathcal{Y}}^1(t) + \hat{Y}^{1,\varepsilon}(t)||\hat{\mathcal{Y}}^2(t)|]dt\Big]\\
\leq& L\hat{\mathbb{E}}\Big\{\sup_{t\in[0,T]}|\hat{\mathcal{Y}}^1(t) + \hat{Y}^{1,\varepsilon}(t)|^2\Big\}^{\frac{1}{2}}\hat{\mathbb{E}}\Big\{\sup_{t\in[0,T]}|\hat{\mathcal{Y}}^2(t)|^2\Big\}^{\frac{1}{2}}\Big\{\mathbb{E}\int_0^T |\mathcal{L}(t)|^2 dt\Big\}^{\frac{1}{2}} \leq L\varepsilon^{\frac{3}{2}}.
\end{aligned}
$$

Finally, $\mathbb{E}\big[\int_0^T \mathcal{R}(t)C_3^{\varepsilon}(t)dt\big]$ can be calculated similar to $\mathbb{E}\big[\int_0^T \mathcal{S}(t)A_3^{\varepsilon}(t)dt\big]$. □

In order to prove the SMP, we show the following relation of $Y^{1,\varepsilon}, X^{1,\varepsilon}$ and $X^{2,\varepsilon}$ with the help of the first- and second-order adjoint equation.

Proposition 4.8. *Let Assumptions (A3.1)-(A3.3) hold true, then*

$$
\begin{aligned}
Y^{2,\varepsilon}(t) =& p_0(t)X^{2,\varepsilon}(t) + \hat{\mathbb{E}}[\hat{\mathring{p}}_1(t)\hat{X}^{2,\varepsilon}(t)] + \frac{1}{2}P_0(t)(X^{1,\varepsilon}(t))^2\\
&+\frac{1}{2}\hat{\mathbb{E}}[\hat{\mathring{P}}_1(t)(\hat{X}^{1,\varepsilon}(t))^2] + \mathcal{M}(t),\\
Z^{2,\varepsilon}(t) =& k_0(t)X^{2,\varepsilon}(t) + \hat{\mathbb{E}}[\hat{\mathring{k}}_1(t)\hat{X}^{2,\varepsilon}(t)] + \frac{1}{2}K_0(t)(X^{1,\varepsilon}(t))^2\\
&+\frac{1}{2}\hat{\mathbb{E}}[\hat{\mathring{K}}_1(t)(\hat{X}^{1,\varepsilon}(t))^2] + J(t) + \mathcal{K}(t),
\end{aligned}
\tag{4.21}
$$

where $P_0, \hat{\mathring{P}}_1, K_0, \hat{\mathring{K}}_1$ are given in (3.8) and (3.9);

$$
\begin{aligned}
J(t) =& \big(1 - p_0(t)\sigma_z(t)\big)^{-1}p_0(t)\big\{\sigma_y(t)\mathcal{M}(t) + \sigma_z(t)\mathcal{K}(t) + \hat{\mathbb{E}}[\hat{\mathring{\sigma}}_{\mu_2}(t)\hat{\mathcal{M}}(t)]\big\}\\
&+P_0(t)\delta\sigma(t,\Xi)X^{1,\varepsilon}(t)\mathbf{I}_{E_\varepsilon}(t) + \big(1 - p_0(t)\sigma_z(t)\big)^{-1}p_0(t)X^{1,\varepsilon}(t)\mathbf{I}_{E_\varepsilon}(t)\\
&\cdot\big\{\delta\sigma_x(t,\Xi) + p_0(t)\delta\sigma_y(t,\Xi) + k_0(t)\delta\sigma_z(t,\Xi)\big\};
\end{aligned}
$$

$(\mathcal{M},\mathcal{K})$ *satisfies*

$$\begin{cases} d\mathcal{M}(t) = -\Big\{\big[H_y^0(t)+\sigma_y(t)f_z(t)p_0(t)(1-p_0(t)\sigma_z(t))^{-1}\big]\mathcal{M}(t) \\ \qquad + \hat{\mathbb{E}}\big[\big(\hat{\mathring{H}}_{\mu_2}^0(t)+\hat{\mathring{H}}_y^1(t)+\bar{\mathbb{E}}[\bar{\hat{\mathring{H}}}_{\mu_2}^1(t)] \\ \qquad + \sigma_y(t)f_z(t)p_0(t)(1-p_0(t)\sigma_z(t))^{-1}\hat{\mathring{\sigma}}_{\mu_2}(t)\big)\hat{\mathcal{M}}(t)\big] \\ \qquad + \big[H_z^0(t)+\sigma_z(t)f_z(t)p_0(t)(1-p_0(t)\sigma_z(t))^{-1}\big]\mathcal{K}(t)+\hat{\mathbb{E}}[\hat{\mathring{H}}_z^1(t)\hat{\mathcal{K}}(t)] \\ \qquad + \big[\delta H^0(t,\Xi)+\hat{\mathbb{E}}[\delta\hat{\mathring{H}}^1(t,\Xi)] \\ \qquad + \dfrac{1}{2}(P_0(t)+\hat{\mathbb{E}}[\hat{\mathring{P}}_1(t)])(\delta\sigma(t,\Xi))^2\big]\mathbf{I}_{E_\varepsilon}(t)\Big\}dt+\mathcal{K}(t)dW(t),\ t\in[0,T], \\ \mathcal{M}(T)=0, \end{cases} \tag{4.22}$$

with

$$\delta H^0(t,\Xi)=p_0(t)\delta b(t,\Xi)+q_0(t)\delta\sigma(t,\Xi)+\delta f(t,\Xi),$$
$$\hat{\mathbb{E}}[\delta\hat{\mathring{H}}^1(t,\Xi)]=\hat{\mathbb{E}}[\hat{\mathring{p}}_1(t)]\delta b(t,\Xi)+\hat{\mathbb{E}}[\hat{\mathring{q}}_{12}(t)]\delta\sigma(t,\Xi).$$

Proof. Easy (but lengthy) calculations similar to Proposition 4.1 can yield (4.21). Hence, we omit it. □

Note that under Assumptions (A3.1)–(A3.3), (4.22) is a linear mean-field BSDE. According to Theorem A.1[20], (4.22) possesses a unique solution $(\mathcal{M}(\cdot),\mathcal{K}(\cdot))\in\mathcal{S}_{\mathbb{F}}^p(0,T)\times\mathcal{H}_{\mathbb{F}}^{2,p}(0,T)$.

In order to prove our SMP, let us first study the comparison theorem of mean-field SDEs. By two examples we show that the comparison theorem of mean-field SDEs does not hold true any more, if the diffusion coefficient σ depend on mean field term, or the derivative of drift coefficient b with respect to mean-field term is negative.

Example 2. Consider

$$X^1(t)=1+\int_0^t\mathbb{E}[X^1(s)]dW(s),\ s\in[0,T],\ X^2(t)=\int_0^t\mathbb{E}[X^2(s)]dW(s),\ s\in[0,T].$$

Obviously, $X^1(t)=1+W(t),\ X^2(t)=0,\ t\in[0,T]$. It is clear that $\mathbb{P}(W(t)+1<0)>0$ and $X^1(t)<X^2(t),\ t\in[0,T]$ on set $\{W(t)+1<0\}$.

Example 3. Let us consider two mean-field SDEs over $[1,2]$:

$$X^1(t)=(W(1))^2+\int_1^t-\mathbb{E}[X^1(s)]ds+\int_1^t dW(s),$$
$$X^2(t)=\int_1^t-\mathbb{E}[X^2(s)]ds+\int_1^t dW(s).$$

It is easy to check $X^1(t)=(W(1))^2+e^{1-t}-1+W(t)-W(1),\ t\in[1,2]$ and $X^2(t)=W(t)-W(1),\ t\in[1,2]$ are the solutions of the above equations, respectively. Obviously, $X^1(2)<X^2(2)$ on $\{(W(1))^2<1-e^{-1}\}$, which is of strictly positive probability.

Lemma 4.9. *Assume $b^i,\ i=1,2$ are Lipschitz and linear growth, and moreover, there exists a constant $L>0$, such that, for $t\in[0,T],x\in\mathbb{R}^n,\xi_1,\xi_2\in L^2(\mathcal{F}_t;\mathbb{R}^n)$,*

$$b^1(t,x,\mathbb{P}_{\xi_1})-b^1(t,x,\mathbb{P}_{\xi_2})\le L\big\{\mathbb{E}\big((\xi_1-\xi_2)^+\big)^2\big\}^{\frac{1}{2}}.$$

Let $C(\cdot)$ be a given adapted bounded process and x_0^i, $i = 1, 2$ initial value. By X^1 and X^2 we denote the solution of the following mean-field SDE with data (x_0^1, b^1, C) and (x_0^2, b^2, C), respectively,

$$X^i(t) = x_0^i + \int_0^t b^i(s, X^i(s), \mathbb{P}_{X^i(s)})ds + \int_0^t C(s)X^i(s)dW(s), \qquad s \in [0, T].$$

If $x_0^1 \le x_0^2$ and $b^1(t, X^2(t), \mathbb{P}_{X^2(t)}) \le b^2(t, X^2(t), \mathbb{P}_{X^2(t)})$, $\mathbb{P}$-a.s., then $X^1(t) \le X^2(t)$, $t \in [0, T]$, $\mathbb{P}$-a.s.

Proof. Denote $\Delta X(t) = X^1(t) - X^2(t), \Delta x = x_0^1 - x_0^2$. Then

$$\Delta X(t) = \Delta x + \int_0^t (b^1(s, X^1(s), \mathbb{P}_{X^1(s)}) - b^2(s, X^2(s), \mathbb{P}_{X^2(s)})ds + \int_0^t C(s)\Delta X(s)dW(s).$$

Applying Itô's formula to $((\Delta X(t))^+)^2$, it follows

$$\begin{aligned} d((\Delta X(t))^+)^2 = &2(\Delta X(t))^+\big(b^1(t, X^1(t), \mathbb{P}_{X^1(t)}) - b^2(t, X^2(t), \mathbb{P}_{X^2(t)}\big) \\ &+ \mathbf{I}_{\{\Delta X(t)>0\}}(C(t)\Delta X(t))dt + 2(\Delta X(t))^+ C(t)\Delta X(t)dW(t). \end{aligned}$$

Recall $x_0^1 \le x_0^2$ and $b^1(t, X^2(t), \mathbb{P}_{X^2(t)}) \le b^2(t, X^2(t), \mathbb{P}_{X^2(t)})$, $\mathbb{P}$-a.s., we obtain

$$\begin{aligned} \mathbb{E}\big[((\Delta X(t))^+)^2\big] \le &\mathbb{E}\Big[\int_0^t 2(\Delta X(s))^+(b^1(s, X^1(s), \mathbb{P}_{X^1(s)}) - b^1(s, X^2(s), \mathbb{P}_{X^2(s)})ds\Big] \\ &+ \mathbb{E}\Big[\int_0^t \mathbf{I}_{\{\Delta X(s)>0\}}(C(s))^2(\Delta X(s))^2 ds\Big]. \end{aligned}$$

Thanks to the Lipschitz property of b^1 and the boundness of $C(\cdot)$, one has

$$\begin{aligned} \mathbb{E}\Big[((\Delta X(t))^+)^2\Big] \le &L\mathbb{E}\Big[\int_0^t (\Delta X(s))^+(|\Delta X(s)| + \{\mathbb{E}((\Delta X(s))^+)^2\}^{\frac{1}{2}})ds\Big] \\ &+ L\mathbb{E}\Big[\int_0^t ((\Delta X(s))^+)^2 ds\Big] \le L\mathbb{E}\Big[\int_0^t ((\Delta X(s))^+)^2 ds\Big]. \end{aligned}$$

Then the desired result comes from Gronwall inequality. □

Remark 4.10. If $b(s, x, \cdot)$ is differentiable on $\mathcal{P}_2(\mathbb{R}^n)$ and there exists a constant $L > 0$ such that, for $(s, x) \in [0, T] \times \mathbb{R}^n$, $\xi \in L^2(\mathcal{F}_s; \mathbb{R}^n)$,

$$0 \le (\partial_\nu b)(s, x, \mathbb{P}_\xi; a) \le L, \qquad a \in \mathbb{R}^n.$$

Then we have $b(t, x, \mathbb{P}_{\xi_1}) - b(t, x, \mathbb{P}_{\xi_2}) \le L\big\{E\big((\xi_1 - \xi_2)^+\big)^2\big\}^{\frac{1}{2}}$. In fact, the above inequality comes from the observation:

$$\begin{aligned} b(s, x, \mathbb{P}_{\xi_1}) - b(s, x, \mathbb{P}_{\xi_2}) = &\int_0^1 \mathbb{E}\Big[\partial_\nu b(s, x, \mathbb{P}_{\xi_2+\lambda(\xi_1-\xi_2)}; \xi_2 + \lambda(\xi_1 - \xi_2))(\xi_1 - \xi_2)\Big]d\lambda \\ &\le L\mathbb{E}[(\xi_1 - \xi_2)^+] \le L\big\{\mathbb{E}\big((\xi_1 - \xi_2)^+\big)^2\big\}^{\frac{1}{2}}. \end{aligned}$$

The reader can refer to[22] for more detail.

Corollary 4.11. *Let $A(\cdot), C(\cdot)$ and $\hat{\mathring{B}}(\cdot)$ be three adapted bounded processes defined on Ω and $\Omega\times\hat{\Omega}$, respectively. By $X(\cdot)$ we denote the solution of the following linear mean-field SDE:*

$$\begin{cases} dX(t) = \big(A(t)X(t) + \hat{\mathbb{E}}[\hat{\mathring{B}}(t)\hat{X}(t)]\big)dt + C(t)X(t)dW(t), \qquad t\in[0,T],\\ X(0)=1. \end{cases}$$

If $0\le \hat{\mathring{B}}(t)\le L, t\in[0,T]$, $\mathbb{P}\otimes\hat{\mathbb{P}}$-a.s., then $X(t)>0,\ t\in[0,T]$, $\mathbb{P}$-a.s.

Proof. Consider

$$dX^1(t) = A(t)X^1(t)dt + C(t)X^1(t)dW(t), \qquad t\in[0,T], \quad X^1(0)=1.$$

From Lemma 4.9, it follows $X(t)\ge X^1(t)>0,\ t\in[0,T],\ \mathbb{P}$-a.s. □

Now let us show the proof of Theorem 3.3.

Proof of Theorem 3.4. For simplicity, we define

$$\begin{aligned} &A(t) := H_y(t) + \sigma_y(t)f_z(t)p_0(t)(1-p_0(t)\sigma_z(t))^{-1};\\ &\hat{\mathring{B}}(t) := \hat{\mathring{H}}^0_{\mu_2}(t) + \hat{\mathring{H}}^1_y(t) + \bar{\mathbb{E}}[\bar{\hat{\mathring{H}}}^1_{\mu_2}(t)] + \sigma_y(t)f_z(t)p_0(t)(1-p_0(t)\sigma_z(t))^{-1}\hat{\mathring{\sigma}}_{\mu_2}(t);\\ &C(t) := H^0_z(t) + \sigma_z(t)f_z(t)p_0(t)(1-p_0(t)\sigma_z(t))^{-1}; \qquad \hat{\mathring{D}}(t) := \hat{\mathring{H}}^1_z(t);\\ &J(t,\Xi) := \delta H^0(t,\Xi) + \hat{\mathbb{E}}[\delta\hat{\mathring{H}}^1(t,\Xi)] + \frac{1}{2}\big(P_0(t) + \hat{\mathbb{E}}[\hat{\mathring{P}}_1(t)]\big)(\delta\sigma(t,\Xi))^2. \end{aligned}$$

From assumption $\hat{\mathring{D}}(t) = \hat{\mathring{H}}^1_z(t) = 0,\ t\in[0,T],\ \mathbb{P}\otimes\hat{\mathbb{P}}$-a.s., we can rewrite (4.22) as

$$\begin{cases} d\mathcal{M}(t) = -\big(A(t)\mathcal{M}(t) + \hat{\mathbb{E}}[\hat{\mathring{B}}(t)\hat{\mathcal{M}}(t)] + C(t)\mathcal{K}(t) + J(t,\Xi)\mathbf{I}_{E_\varepsilon}(t)\big)dt\\ \qquad\qquad + \mathcal{K}(t)dW(t), \qquad t\in[0,T],\\ \mathcal{M}(T)=0. \end{cases}$$

We now consider the dual McKean-Vlasov equation:

$$d\Gamma(t) = \big(A(t)\Gamma(t) + \hat{\mathbb{E}}[\hat{\mathring{B}}(t)\hat{\Gamma}(t)]\big)dt + C(t)\Gamma(t)dW(t),\ t\in[0,T],\ \Gamma(0)=1.$$

From Itô's formula to $\mathcal{M}(t)\Gamma(t)$, one has

$$\mathcal{M}(0) = \mathbb{E}\Big[\int_0^T -\Gamma(t)\hat{\mathbb{E}}[\hat{\mathring{B}}(t)\hat{\mathcal{M}}(t)] + \mathcal{M}(t)\hat{\mathbb{E}}[\hat{\mathring{B}}(t)\Gamma(t)] + \Gamma(t)J(t,\Xi)\mathbf{I}_{E_\varepsilon}(t)dt\Big].$$

Notice

$$\mathbb{E}[\Gamma(t)\hat{\mathbb{E}}[\hat{\mathring{B}}(t)\hat{\mathcal{M}}(t)]] = \mathbb{E}\hat{\mathbb{E}}[\Gamma(t)\hat{\mathring{B}}(t)\hat{\mathcal{M}}(t)] = \hat{\mathbb{E}}\mathbb{E}[\hat{\Gamma}(t)\mathring{\hat{B}}(t)\mathcal{M}(t)].$$

Hence, $\mathcal{M}(0) = \mathbb{E}\big[\int_0^T \Gamma(t)J(t,\Xi)\mathbf{I}_{E_\varepsilon}(t)dt\big]$, which implies for any $v\in U$, $t\in[0,T]$, P-a.s.,

$$\Gamma(t)\big(\delta H^0(t,\Xi) + \hat{\mathbb{E}}[\delta\hat{\mathring{H}}^1(t,\Xi)] + \frac{1}{2}\big(P_0(t) + \hat{\mathbb{E}}[\hat{\mathring{P}}_1(t)]\big)(\delta\sigma(t,\Xi))^2\big) \ge 0.$$

According to Corollary 4.11, we obtain the desired result. The proof is complete. □

Remark 4.12. Let us discuss two special cases:

i) If the coefficients b, σ, f, Φ are independent of mean-field term and (y, z), i.e., $b(t, x, y, z, \mu, v) = b(t, x, v), \sigma(t, x, y, z, \mu, v) = \sigma(t, x, v), f(t, x, y, z, \mu, v) = f(t, x, v)$, our case reduces to the one studied by Peng[25]. In this situation, (4.22) becomes

$$d\mathcal{M}(t) = J_1(t)\mathbf{I}_{E_\varepsilon}(t)dt + \mathcal{K}(t)dW(t), \qquad t \in [0,T], \quad \mathcal{M}(T) = 0,$$

where

$$\begin{aligned} J_1(t) = &\big(b(t, X^*(t), v(t)) - b(t, X^*(t), u^*(t)\big)p_0(t) + \big(\sigma(t, X^*(t), v(t)) - \sigma(t, X^*(t), u^*(t))\big)q_0(t) \\ &+ \big(f(t, X^*(t), v(t)) - f(t, X^*(t), u^*(t)\big) + \frac{1}{2}P_0(t)\big(\sigma(t, X^*(t), v(t)) - \sigma(t, X^*(t), u^*(t)\big)^2. \end{aligned}$$

ii) If the control system (1.1) is a fully coupled forward-backward control system without mean-field term, i.e., $b(t, x, y, z, \mu, v) = b(t, x, y, z, v), \sigma(t, x, y, z, \mu, v) = \sigma(t, x, y, z, v), f(t, x, y, z, \mu, v) = f(t, x, y, z, v)$, which is considered by Hu, Ji and Xue[17], (4.22) is of the form

$$\begin{cases} d\mathcal{M}(t) = -\big(A(t)\mathcal{M}(t) + C(t)\mathcal{K}(t) + J_2(t, \Xi)\mathbf{I}_{E_\varepsilon}(t)\big)dt + \mathcal{K}(t)dW(t), \qquad t \in [0,T], \\ \mathcal{M}(T) = 0, \end{cases}$$

where $J_2(t, \Xi) := \delta H^0(t, \Xi) + \frac{1}{2}P_0(t)\delta\sigma(t, \Xi)^2$ (see (3.41)[17]). Our SMP is just the one proved by Hu, Ji and Xue[17].

5 The Case without Assumption (A3.3)

In this section we study the case without Assumption (A3.3), i.e., $q_0 \in \mathcal{H}^{2,\beta}_{\mathbb{F}}(0,T)$, $\hat{\mathring{q}}_{11}, \hat{\mathring{q}}_{12} \in \mathcal{H}^{2,\beta}_{\mathbb{F}\otimes\hat{\mathbb{F}}}(0,T)$. From the previous section, we know that Lemma 4.6 is a critical tool in proving our SMP. The boundness of coefficients of (4.13) plays very important role in the proof of Lemma 4.6. For this, we make the following assumption:

Assumption (A5.1) b, σ are independent of z.

Clearly, under Assumptions (A3.1), (A3.2) and (A5.1), Lemma 4.6 holds. Moreover, one can check that the solution $(x(\cdot))$ of (4.3) satisfies $\mathbb{E}\big[\sup\limits_{0\le t\le T}|x(t)|^2\big] < +\infty$, which implies Proposition 4.1 also holds true. Consequently, we have

Proposition 5.1. *Suppose Assumptions (A3.1), (A3.3) and (A5.1) hold true, then*

$$\begin{aligned} &\mathbb{E}\Big[\sup_{0\le t\le T}|X^\varepsilon(t) - X^*(t) - X^{1,\varepsilon}(t) - X^{2,\varepsilon}(t)|^2\Big] \le \varepsilon^2\rho(\varepsilon), \\ &\mathbb{E}\Big[\sup_{0\le t\le T}|Y^\varepsilon(t) - Y^*(t) - Y^{1,\varepsilon}(t) - Y^{2,\varepsilon}(t)|^2 + \int_0^T |Z^\varepsilon(t) - Z^*(t) - Z^{1,\varepsilon}(t) - Z^{2,\varepsilon}(t)|^2dt\Big] \\ \le&\varepsilon^2\rho(\varepsilon). \end{aligned}$$

Proof. Thanks to Theorem 2.2 and (4.14), we can obtain

$$\begin{aligned} &\mathbb{E}\Big[\sup_{0\le t\le T}(|\mathcal{X}^3(t)|^2 + |\mathcal{Y}^3(t)|^2) + \int_0^T |\mathcal{Z}^3(t)|^2dt\Big] \\ \le&L\mathbb{E}\Big[\Big(\int_0^T |A_3^\varepsilon(t)|dt\Big)^2 + \Big(\int_0^T |C_3^\varepsilon(t)|dt\Big)^2 + \int_0^T |B_3^\varepsilon(t)|^2dt + |D_3^\varepsilon(T)|^2\Big], \end{aligned}$$

where $A_3^\varepsilon(\cdot), C_3^\varepsilon(\cdot), D_3^\varepsilon(T)$ are given (4.15), and

$$\begin{aligned}
B_3^\varepsilon(t) =&\big\{\delta\sigma_x(t,\Xi)\mathcal{X}^2(t) + \delta\sigma_y(t,\Xi)\mathcal{Y}^2(t) + \hat{\mathbb{E}}[\delta\hat{\mathring{\sigma}}_{\mu_1}(t,\Xi)\hat{\mathcal{X}}^2(t)] + \hat{\mathbb{E}}[\delta\hat{\mathring{\sigma}}_{\mu_2}(t,\Xi)\hat{\mathcal{Y}}^2(t)]\big\}\mathbf{I}_{E_\varepsilon}(t)\\
&+\frac{1}{2}[\mathcal{X}^1(t),\mathcal{Y}^1(t)]D^2\sigma^\varepsilon(t,\Xi\mathbf{I}_{E_\varepsilon})[\mathcal{X}^1(t),\mathcal{Y}^1(t)]^\intercal\\
&-\frac{1}{2}(X^{1,\varepsilon}(t))^2[1,p_0(t)]D^2\sigma(t)[1,p_0(t)]^\intercal\\
&+\frac{1}{2}\hat{\mathbb{E}}[\hat{\mathring{\sigma}}^\varepsilon_{\mu_1 a_1}(t)(\hat{\mathcal{X}}^1(t))^2 - \hat{\mathring{\sigma}}_{\mu_1 a_1}(t)(\hat{X}^{1,\varepsilon}(t))^2]\\
&+\frac{1}{2}\hat{\mathbb{E}}[\hat{\mathring{\sigma}}^\varepsilon_{\mu_2 a_2}(t)(\hat{\mathcal{Y}}^1(t))^2 - \hat{\mathring{\sigma}}_{\mu_2 a_2}(t)(\hat{Y}^{1,\varepsilon}(t))^2].
\end{aligned}$$

Next we prove $\mathbb{E}\big[\int_0^T|B_3^\varepsilon(t)|^2dt\big] \le \varepsilon^2\rho(\varepsilon)$. We just analyse the mean-field terms. The other terms can be estimated similarly. First, from Hölder inequality it follows

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T\Big|\hat{\mathbb{E}}[\delta\hat{\mathring{\sigma}}_{\mu_1}(t,\Xi)\hat{\mathcal{X}}^2(t)]\mathbf{I}_{E_\varepsilon}(t)\Big|^2dt\Big]\\
\le&\mathbb{E}\Big\{\int_{E_\varepsilon}\hat{\mathbb{E}}\Big[\sup_{0\le t\le T}|\hat{\mathcal{X}}^2(t)|^2\Big]\cdot\hat{\mathbb{E}}\big[|\delta\hat{\mathring{\sigma}}_{\mu_1}(t,\Xi)|^2\big]dt\Big\}\\
\le&\mathbb{E}\Big[\sup_{0\le t\le T}|\mathcal{X}^2(t)|^2\Big]\cdot\mathbb{E}\hat{\mathbb{E}}\Big[\int_{E_\varepsilon}|\delta\hat{\mathring{\sigma}}_{\mu_1}(t,\Xi)|^2dt\Big] \le \varepsilon^2\rho_1(\varepsilon),
\end{aligned}$$

where $\rho_2(\varepsilon) := \mathbb{E}\hat{\mathbb{E}}\big[\int_{E_\varepsilon}|\delta\hat{\mathring{\sigma}}_{\mu_1}(t,\Xi)|^2dt\big]$. Dominated Convergence Theorem allows to show $\rho_2(\varepsilon)\to 0$ as $\varepsilon\to 0$. Second, according to the boundness of $\hat{\mathring{\sigma}}_{\mu_2 a_2}(\cdot)$ and Proposition 4.3 one can check

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T\big|\hat{\mathbb{E}}[\hat{\mathring{\sigma}}^\varepsilon_{\mu_2 a_2}(t)(\hat{\mathcal{Y}}^1(t))^2 - \hat{\mathring{\sigma}}_{\mu_2 a_2}(t)(\hat{Y}^{1,\varepsilon}(t))^2]\big|^2dt\Big]\\
\le&L\hat{\mathbb{E}}\Big[\int_0^T|\hat{\mathcal{Y}}^2(t)|^2|\hat{\mathcal{Y}}^1(t)+\hat{Y}^{1,\varepsilon}(t)|^2dt\Big] + 2\mathbb{E}\hat{\mathbb{E}}\Big[\int_0^T|\hat{\mathring{\sigma}}^\varepsilon_{\mu_2 a_2}(t)-\hat{\mathring{\sigma}}_{\mu_2 a_2}(t)|^2|\hat{Y}^{1,\varepsilon}(t)|^4dt\Big]\\
\le&L\varepsilon^3 + 2\mathbb{E}\Big[\sup_{0\le t\le T}|Y^{1,\varepsilon}(t)|^4\Big]\mathbb{E}\hat{\mathbb{E}}\Big[\int_0^T|\hat{\mathring{\sigma}}^\varepsilon_{\mu_2 a_2}(t)-\hat{\mathring{\sigma}}_{\mu_2 a_2}(t)|^2dt\Big] \le \varepsilon^2\rho_3(\varepsilon),
\end{aligned}$$

where $\rho_3(\varepsilon) := L\varepsilon + 2\mathbb{E}\hat{\mathbb{E}}\big[\int_0^T|\hat{\mathring{\sigma}}^\varepsilon_{\mu_2 a_2}(t)-\hat{\mathring{\sigma}}_{\mu_2 a_2}(t)|^2dt\big]$ satisfies $\rho_3(\varepsilon)\to 0$ as $\varepsilon\downarrow 0$. □

Define for $x, y, z\in\mathbb{R}, \mu\in\mathcal{P}(\mathbb{R}^2), p_0, q_0, p_1, q_{12}\in\mathbb{R}$,

$$\begin{aligned}
&H^0(t,x,y,z,\mu,u,p_0,q_0) = b(t,x,y,\mu,u)p_0 + \sigma(t,x,y,\mu,u)q_0 + f(t,x,y,z,\mu,u),\\
&H^1(t,x,y,z,\mu,u,p_1,q_{12}) = b(t,x,y,\mu,u)p_1 + \sigma(t,x,y,\mu,u)q_{12}.
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{H}(t,x,y,z,\mu,v,p_0(t),\hat{\mathbb{E}}[\hat{\mathring{p}}_1(t)],q_0(t),\hat{\mathbb{E}}[\hat{\mathring{q}}_{12}(t)],P_0(t),\hat{\mathbb{E}}[\hat{\mathring{P}}_1(t)])\\
=&\big(p_0(t)+\hat{\mathbb{E}}[\hat{\mathring{p}}_1(t)]\big)b(t,x,y,\mu,v) + \big(q_0(t)+\hat{\mathbb{E}}[\hat{\mathring{q}}_{12}(t)]\big)\sigma(t,x,y,\mu,v)\\
&+\frac{1}{2}\big(P_0(t)+\hat{\mathbb{E}}[\hat{\mathring{P}}_1(t)]\big)\big(\sigma(t,x,y,\mu,v)-\sigma(t,X^*(t),Y^*(t),\mathbb{P}_{(X^*(t),Y^*(t))},u^*(t))\big)^2\\
&+\big(t,x,y,z+p_0(t)(\sigma(t,x,y,\mu,v)-\sigma(t,X^*(t),Y^*(t),\mathbb{P}_{(X^*(t),Y^*(t))},u^*(t))),\mu,v\big).
\end{aligned}$$

Theorem 5.2. *Under Assumption (A3.1), (A3.2) and (A5.1), let u^* be the optimal control. By (X^*,Y^*,Z^*) we denote the optimal trajectory. Let $\big((p_0(\cdot),q_0(\cdot)),(\hat{\mathring{p}}_1(\cdot),\hat{\mathring{q}}_{11}(\cdot),\hat{\mathring{q}}_{12}(\cdot))\big)$ and*

$\big((P_0(\cdot),Q_0(\cdot)),(\mathring{\hat{P}}_1(\cdot),\mathring{\hat{Q}}_{11}(\cdot),\mathring{\hat{Q}}_{12}(\cdot))\big)$ *be the solutions of the first- and second-order adjoint equations, respectively. Moreover, we assume*

$$\mathring{\hat{H}}^0_{\mu_2}(t)+\mathring{\hat{H}}^1_y(t)+\bar{\mathbb{E}}[\bar{\mathring{\hat{H}}}^1_{\mu_2}(t)]+p_0(t)f_z(t)\mathring{\hat{\sigma}}_{\mu_2}(t)\sigma_y(t)\geq 0,\ \ t\in[0,T],\ \ \mathbb{P}\otimes\hat{\mathbb{P}}\text{-}a.s.,$$

where $\mathring{\hat{H}}^0_{\mu_2}(t),\mathring{\hat{H}}^1_y(t),\bar{\mathring{\hat{H}}}^1_{\mu_2}(t)$ *is introduced in (3.2). Then for* $v\in U$, *we have a.e., a.s,*

$$\begin{aligned}&\mathcal{H}(t,X^*(t),Y^*(t),Z^*(t),\mathbb{P}_{(X^*(t),Y^*(t))},v,p_0(t),\hat{\mathbb{E}}[\mathring{\hat{p}}_1(t)],q_0(t),\hat{\mathbb{E}}[\mathring{\hat{q}}_{12}(t)],P_0(t),\hat{\mathbb{E}}[\mathring{\hat{P}}_1(t)])\\ \geq&\mathcal{H}(t,X^*(t),Y^*(t),Z^*(t),\mathbb{P}_{(X^*(t),Y^*(t))},u^*(t),p_0(t),\hat{\mathbb{E}}[\mathring{\hat{p}}_1(t)],q_0(t),\hat{\mathbb{E}}[\mathring{\hat{q}}_{12}(t)],P_0(t),\hat{\mathbb{E}}[\mathring{\hat{P}}_1(t)]).\end{aligned}$$

6 The Comparison with Buckdahn et al.'s SMP

In this section, let us consider **Problem (BLM)** (see Remark 3.1), and show the relation between the solutions of adjoint equations in [6] (see (3.11), (3.13)) and that of our adjoint equations.

If our system reduces to the system (3.3), (3.1) can be written as

$$\begin{cases}dp_0(t)=-\big\{b_x(t)p_0(t)+\sigma_x(t)q_0(t)+f_x(t)\big\}dt+q_0(t)dW(t),\ \ t\in[0,T],\\ d\mathring{\hat{p}}_1(t)=-\big\{\mathring{\hat{b}}_\mu(t)p_0(t)+\mathring{\hat{\sigma}}_\mu(t)q_0(t)+\mathring{\hat{f}}_\mu(t)+\hat{b}_x(t)\mathring{\hat{p}}_1(t)+\hat{\sigma}_x(t)\mathring{\hat{q}}_{12}(t)\\ \qquad\qquad +\bar{\mathbb{E}}[\bar{\hat{\mathring{b}}}_\mu(t)\bar{\mathring{p}}_1(t)+\bar{\hat{\mathring{\sigma}}}_\mu(t)\bar{\mathring{q}}_{12}(t)]\big\}dt+\mathring{\hat{q}}_{11}(t)dW(t)+\mathring{\hat{q}}_{12}(t)d\hat{W}(t),\\ \qquad\qquad t\in[0,T],\\ p_0(T)=\Phi_x(T),\ \ \mathring{\hat{p}}_1(T)=\mathring{\hat{\Phi}}_\mu(T).\end{cases}\tag{6.1}$$

Obviously, Assumption (A5.1) hold true. The boundness of the first-order derivatives of b,σ,f,Φ allows to show that Assumption (A3.2) also hold. Besides, it is easy to see that condition (3.13) is satisfied. As for the second-order adjoint system, (3.12) is of the form

$$\begin{cases}dP_0(t)=-\big\{(\sigma_x(t))^2P_0(t)+2b_x(t)P_0(t)+2\sigma_x(t)Q_0(t)+H^0_{xx}(t)\big\}dt\\ \qquad\qquad +Q_{12}(t)dW(t),\ \ t\in[0,T],\\ P_0(T)=\Phi_{xx}(T),\end{cases}$$

$$\begin{cases}d\mathring{\hat{P}}_1(t)=-\big\{(\hat{\sigma}_x(t))^2\mathring{\hat{P}}_1(t)+2\hat{b}_x(t)\mathring{\hat{P}}_1(t)+2\hat{\sigma}_x(t)\mathring{\hat{Q}}_{12}(t)+\mathring{\hat{H}}^1_{xx}(t)+\mathring{\hat{H}}^0_{\mu_1a_1}(t)\\ \qquad\qquad +\bar{\mathbb{E}}[\bar{\mathring{\hat{H}}}^1_{\mu_1a_1}(t)]\big\}dt+\mathring{\hat{Q}}_{11}(t)dW(t)+\mathring{\hat{Q}}_{12}(t)d\hat{W}(t),\ \ t\in[0,T],\\ \mathring{\hat{P}}_1(T)=\mathring{\hat{\Phi}}_{\nu a}(T).\end{cases}$$

Thanks to Theorem 3.3, we have

Corollary 6.1 (Buckdahn et al.'s SMP)**.** *Under Assumption (A3.1), let* u^* *be the optimal control and* X^* *be the optimal trajectory. Then*

$$\begin{aligned}&\mathcal{H}(t,X^*(t),\mathbb{P}_{X^*(t)},v,p_0(t),\hat{\mathbb{E}}[\mathring{\hat{p}}_1(t)],q_0(t),\hat{\mathbb{E}}[\mathring{\hat{q}}_{12}(t)],P_0(t),\hat{\mathbb{E}}[\mathring{\hat{P}}_1(t)])\\ \geq&\mathcal{H}(t,X^*(t),\mathbb{P}_{X^*(t)},u^*(t),p_0(t),\hat{\mathbb{E}}[\mathring{\hat{p}}_1(t)],q_0(t),\hat{\mathbb{E}}[\mathring{\hat{q}}_{12}(t)],P_0(t),\hat{\mathbb{E}}[\mathring{\hat{P}}_1(t)]),\\ &v\in U,\ a.e.,\ a.s.,\end{aligned}\tag{6.2}$$

where

$$\mathcal{H}(t,x,\mu,v,p_0(t),\hat{\mathbb{E}}[\mathring{\hat{p}}_1(t)],q_0(t),\hat{\mathbb{E}}[\mathring{\hat{q}}_{12}(t)],P_0(t),\hat{\mathbb{E}}[\mathring{\hat{P}}_1(t)])$$

$$
\begin{aligned}
&=\big(p_0(t)+\hat{\mathbb{E}}[\mathring{\hat{p}}_1(t)]\big)b(t,x,\mu,v)+\big(q_0(t)+\hat{\mathbb{E}}[\mathring{\hat{q}}_{12}(t)]\big)\sigma(t,x,\mu,v)+f(t,x,\mu,v)\\
&\quad+\frac{1}{2}\big(P_0(t)+\hat{\mathbb{E}}[\mathring{\hat{P}}_1(t)]\big)\big(\sigma(t,x,\mu,v)-\sigma(t,X^*(t),\mathbb{P}_{X^*(t)},u^*(t))\big)^2. \qquad (6.3)
\end{aligned}
$$

Next let us show that the above SMP is the same as that given by Buckdahn, Li, Ma (see Theorem 3.5[6]). In fact, from (6.1) it follows

$$
\begin{cases}
d\hat{\mathbb{E}}[\mathring{\hat{p}}_1(t)]=-\big\{\hat{\mathbb{E}}\big[\mathring{\hat{b}}_\mu(t)\hat{p}_0(t)+\mathring{\hat{\sigma}}_\mu(t)\hat{q}_0(t)+\mathring{\hat{f}}_\mu(t)\big]+b_x(t)\hat{\mathbb{E}}[\mathring{\hat{p}}_1(t)]+\sigma_x(t)\hat{\mathbb{E}}[\mathring{\hat{q}}_{12}(t)]\\
\qquad\qquad+\hat{\mathbb{E}}\bar{\mathbb{E}}[\mathring{\bar{b}}_\mu(t)\bar{\hat{p}}_1(t)+\mathring{\bar{\sigma}}_\mu(t)\bar{\hat{q}}_{12}(t)]\big\}dt+\hat{\mathbb{E}}[\mathring{\hat{q}}_{12}(t)]dW(t),\ t\in[0,T],\\
\hat{\mathbb{E}}[\mathring{\hat{p}}_1(T)]=\hat{\mathbb{E}}[\mathring{\hat{\Phi}}_\mu(T)].
\end{cases}
$$

Notice the fact

$$
\hat{\mathbb{E}}\bar{\mathbb{E}}[\mathring{\bar{b}}_\mu(t)\bar{\hat{p}}_1(t)]=\bar{\mathbb{E}}\hat{\mathbb{E}}[\mathring{\hat{b}}_\mu(t)\bar{\hat{p}}_1(t)]=\hat{\mathbb{E}}[\mathring{\hat{b}}_\mu(t)\bar{\mathbb{E}}[\bar{\hat{p}}_1(t)]],
$$

we obtain

$$
\begin{cases}
d\{p_0(t)+\hat{\mathbb{E}}[\mathring{\hat{p}}_1(t)]\}=-\big\{b_x(t)\{p_0(t)+\hat{\mathbb{E}}[\mathring{\hat{p}}_1(t)]\}+\sigma_x(t)\{q_0(t)+\hat{\mathbb{E}}[\mathring{\hat{q}}_{12}(t)]\}+f_x(t)\\
\qquad\qquad+\hat{\mathbb{E}}[\mathring{\hat{f}}_\mu(t)]+\hat{\mathbb{E}}[\mathring{\hat{b}}_\mu(t)(\hat{p}_0(t)+\bar{\mathbb{E}}[\bar{\hat{p}}_1(t))]+\hat{\mathbb{E}}[\mathring{\hat{\sigma}}_\mu(t)(\hat{q}_0(t)\\
\qquad\qquad+\bar{\mathbb{E}}[\bar{\hat{q}}_{12}(t))]]\big\}dt+\big\{q_0(t)+\hat{\mathbb{E}}[\mathring{\hat{q}}_{12}(t)]\big\}dW(t),\ t\in[0,T],\\
p_0(T)+\hat{\mathbb{E}}[\mathring{\hat{p}}_1(T)]=\Phi_x(T)+\hat{\mathbb{E}}[\mathring{\hat{\Phi}}_\mu(T)].
\end{cases}
$$

According to the uniqueness of the solution of mean-field BSDE (see Theorem 3.1[7]), we have

$$
p(t)=p_0(t)+\hat{\mathbb{E}}[\mathring{\hat{p}}_1(t)],\quad q(t)=q_0(t)+\hat{\mathbb{E}}[\mathring{\hat{q}}_{12}(t)],\ t\in[0,T], \qquad (6.4)
$$

where $(p(\cdot),q(\cdot))$ is the solution of (3.4). Similar to the above analysis, we can also get

$$
\begin{cases}
d\big(P_0(t)+\hat{\mathbb{E}}[\mathring{\hat{P}}_1(t)]\big)=-\big\{(\sigma_x(t))^2\big(P_0(t)+\hat{\mathbb{E}}[\mathring{\hat{P}}_1(t)]\big)+2b_x(t)\big(P_0(t)+\hat{\mathbb{E}}[\mathring{\hat{P}}_1(t)]\big)\\
\qquad\qquad+2\sigma_x(t)(Q_0(t)+\hat{\mathbb{E}}[\mathring{\hat{Q}}_{12}(t)]\big)+H^0_{xx}(t)+\hat{\mathbb{E}}[\mathring{\hat{H}}^1_{xx}(t)]\\
\qquad\qquad+\hat{\mathbb{E}}[\mathring{\hat{H}}^0_{xx}(t)+\bar{\mathbb{E}}[\mathring{\bar{\hat{H}}}^1_{xx}(t)]]\big\}dt+\big(Q_0(t)+\hat{\mathbb{E}}[\mathring{\hat{Q}}_{12}(t)]\big)dW(t),\\
\qquad t\in[0,T],\\
P_0(T)+\hat{\mathbb{E}}[\mathring{\hat{P}}_1(T)]=\Phi_{xx}(T)+\hat{\mathbb{E}}[\mathring{\hat{\Phi}}_{\nu a}(T)].
\end{cases}
$$

Since

$$
\begin{aligned}
&H^0_{xx}(t)+\hat{\mathbb{E}}[\mathring{\hat{H}}_{xx}(t)]=b_{xx}(t)\big(p_0(t)+\hat{\mathbb{E}}[\mathring{\hat{p}}_1(t)]\big)+\sigma_{xx}(t)\big(q_0(t)+\hat{\mathbb{E}}[\mathring{\hat{q}}_{12}(t)]\big)+f_{xx}(t),\\
&\hat{\mathbb{E}}\big[\mathring{\hat{H}}^0_{\mu_1a_1}+\bar{\mathbb{E}}[\mathring{\bar{\hat{H}}}_{\mu_1a_1}(t)]\big]=\hat{\mathbb{E}}\big[\mathring{\hat{b}}_{\mu_1a_1}(t)\big(p_0(t)+\bar{\mathbb{E}}[\bar{\hat{p}}_1(t)]\big)\\
&\qquad\qquad+\mathring{\hat{\sigma}}_{\mu_1a_1}(t)\big(q_0(t)+\bar{\mathbb{E}}[\bar{\hat{q}}_{12}(t)]\big)+\mathring{\hat{f}}_{\mu_1a_1}(t)\big],
\end{aligned}
$$

then according to Theorem 3.1[7] again, we have

$$
P(t)=P_0(t)+\hat{\mathbb{E}}[\mathring{\hat{P}}_1(t)],\quad Q(t)=Q_0(t)+\hat{\mathbb{E}}[\mathring{\hat{Q}}_{12}(t)],\qquad t\in[0,T], \qquad (6.5)
$$

where $(P(\cdot),Q(\cdot))$ is the solution of the following BSDE:

$$
\begin{cases}
dP(t)=-\big\{(\sigma_x(t))^2P(t)+2b_x(t)P(t)+2\sigma_x(t)Q(t)+H_{xx}(t)+\hat{\mathbb{E}}[\hat{H}_{\mu_1a_1}(t)]\big\}dt\\
\qquad\qquad+Q(t)dW(t),\ t\in[0,T],\\
P(T)=\Phi_{xx}(T)+\hat{\mathbb{E}}[\hat{\Phi}_{\nu a}(T)].
\end{cases}
$$

and

$$H_{xx}(t) = b_{xx}(t)p(t) + \sigma_{xx}(t)q(t) + f_{xx}(t),$$
$$H_{\mu_1 a_1}(t) = \mathring{\hat{b}}_{\mu_1 a_1}(t)p(t) + \mathring{\hat{\sigma}}_{\mu_1 a_1}(t)q(t) + \mathring{\hat{f}}_{\mu_1 a_1}(t);$$

$(p(\cdot), q(\cdot))$ is the solution of (3.4). Clearly, from (6.2), (6.3), (6.4) and (6.5) we can see that our SMP is consistent with Buckdahn et al.'s SMP.

7 Appendix

Theorem 7.1. *Suppose* $\mathring{\hat{A}}_i(t) : [0,T]\times\Omega\times\hat{\Omega} \to \mathbb{R}^m, \mathring{\hat{B}}_i(t), \mathring{\hat{C}}_i(t) : [0,T]\times\Omega\times\hat{\Omega} \to \mathbb{R}^{m\times d},\ i = 1,2$ *are bounded stochastic processes. Let* $\mathring{\hat{\xi}} : \Omega \times \hat{\Omega} \to \mathbb{R}^m$ *and* $\mathring{\hat{D}}(t) : [0,T] \times \Omega \times \hat{\Omega} \to \mathbb{R}^m$ *satisfy*

$$\hat{\mathbb{E}}\mathbb{E}\big[|\mathring{\hat{\xi}}|^\beta\big] < +\infty, \quad \hat{\mathbb{E}}\mathbb{E}\Big[\Big(\int_0^T |\mathring{\hat{D}}(t)|^2 dt\Big)^{\frac{\beta}{2}}\Big] < +\infty, \qquad \beta \geq 2.$$

Then the following mean-field BSDE

$$\begin{cases} d\mathring{\hat{Y}}(t) = -\big\{\mathring{\hat{A}}_1(t)\mathring{\hat{Y}}(t) + \bar{\mathbb{E}}[\bar{\mathring{\hat{A}}}_2(t)\bar{\mathring{\hat{Y}}}(t)] + \mathring{\hat{B}}_1(t)\mathring{\hat{Z}}_{11}(t) + \bar{\mathbb{E}}[\bar{\mathring{\hat{B}}}_2(t)\bar{\mathring{\hat{Z}}}_{11}(t)] \\ \qquad\qquad + \mathring{\hat{C}}_1(t)\mathring{\hat{Z}}_{12}(t) + \bar{\mathbb{E}}[\bar{\mathring{\hat{C}}}_2(t)\bar{\mathring{\hat{Z}}}_{12}(t)] + \mathring{\hat{D}}(t)\big\}dt \\ \qquad\qquad + \mathring{\hat{Z}}_{11}(t)dW(t) + \mathring{\hat{Z}}_{12}(t)d\hat{W}(t), t \in [0,T], \\ \mathring{\hat{Y}}(T) = \xi \end{cases}$$

exists a unique solution $(\mathring{\hat{Y}}, \mathring{\hat{Z}}_{11}, \mathring{\hat{Z}}_{12}) \in \mathcal{S}^\beta_{\mathbb{F}\otimes\hat{\mathbb{F}}}(0,T;\mathbb{R}^m) \times \mathcal{H}^{2,\beta}_{\mathbb{F}\otimes\hat{\mathbb{F}}}(0,T;\mathbb{R}^{m\times d}) \times \mathcal{H}^{2,\beta}_{\mathbb{F}\otimes\hat{\mathbb{F}}}(0,T;\mathbb{R}^{m\times d})$.

The proof is similar to the proof of Proposition 3.2[4].

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Conflict of Interest

The authors declare no conflict of interest.

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