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The Perturbed Compound Poisson Risk Model with Proportional Investment

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Abstract In this paper, the insurance company considers venture capital and risk-free investment in a constant proportion. The surplus process is perturbed by diffusion. At first, the integro-differential equations satisfied by the expected discounted dividend payments and the Gerber-Shiu function are derived. Then, the approximate solutions of the integro-differential equations are obtained through the sinc method. Finally, the numerical examples are given when the claim sizes follow different distributions. Furthermore, the errors between the explicit solution and the numerical solution are discussed in a special case.

Keywords expected discounted dividend payments; lognormal distribution; proportional investment; perturbed risk model; sinc numerical method
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1 Introduction

In recent years, the study of the compound Poisson risk model with diffusion has become a new branch of the financial risk model. Gerber[12] introduced the compound Poisson risk model perturbed by diffusion and obtained the asymptotic estimate of the ruin probability. After that, many scholars have extensively studied more complex perturbed risk models, see references [8–11, 14, 15, 21, 22, 25], etc., for details.

We begin to introduce the perturbed compound Poisson risk model mentioned in [12]. The surplus at time t of the company is

$$X_t = x + ct - S_t + \sigma_1 B_{1t}, \qquad t \geq 0, \tag{1.1}$$

where $x \geq 0$ is the initial capital, $c > 0$ represents the premium received per unit time. $S_t = \sum_{i=1}^{N_t} Y_i$ is the total claims until t, $\{N_t\}_{t\geq 0}$ is a homogeneous Poisson process with parameter $\lambda > 0$ and $N_t = \sup\{j : T_1 + T_2 + \cdots + T_j \leq t\}$ is the claim number during the time interval $[0, t]$, and inter-claim times $\{T_i\}_{i=1}^{\infty}$ is a sequence of mutually i.i.d. following exponential distribution. $\{Y_i\}_{i=1}^{\infty}$ is a set of nonnegative i.i.d. random variables with common distribution function $F_Y(\cdot)$ and probability density function $f_Y(\cdot)$. $\{B_{1t}\}_{t\geq 0}$ is a standard Brownian motion, and $\sigma_1 > 0$ is the diffusion coefficient representing an additional uncertainty of the surplus process. Denote the time of ruin by $T = \inf\{t : X_t \leq 0\}$. Define the ruin probability with the initial surplus x by $\psi(x) = P\{T < \infty | X_0 = x\}$.

As we all know, investment refers to that investors invest a certain amount of funds in the current period and expect to obtain certain income in the future; insurers also hope to provide insurance strategies to avoid property risks in the investment process. The various risk models

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with investment have been studied by many researchers. Jostein Paulsen[17] considered the risk model under stochastic return on investments as well as a stochastic level of inflation. It was concluded that the random environment has a great influence on the ruin probability in [17]. Cai[2] studied the ruin problems of the classical risk model with stochastic interest rates. Yang et al.[23] studied a renewal risk model with stochastic investment return process. For further study on the problem of stochastic return, see references [3, 4, 7, 26, 27].

The actual operation of insurance companies is usually portfolio investment, including both risk-free investment and risk-based investment. Where the risk-free asset $\{P_t\}_{t\geq 0}$ satisfies

$$dP_t = rP_t dt, \tag{1.2}$$

where $r(r>0)$ is interest rate of risk-free asset. The risky asset process $\{Q_t\}_{t\geq 0}$ following a Lévy process defined as

$$Q_t = e^{Z_t}, \tag{1.3}$$
$$Z_t = a^* t + \sigma_2 B_{2t}, \tag{1.4}$$

where $a^*(a^*>0)$ represents the instant rate of the expected return of risky asset and $\sigma_2(\sigma_2>0)$ denotes the volatility rate of risky asset. $\{B_{2t}\}_{t\geq 0}$ represents the uncertainty related to the return on investment, which is a standard Brownian motion. We assume that $\{Y_i\}_{i=1}^{\infty}$, $\{N_t\}_{t\geq 0}$, $\{B_{1t}\}_{t\geq 0}$ and $\{B_{2t}\}_{t\geq 0}$ are mutually independent. The risky asset process $\{Q_t\}_{t\geq 0}$ satisfies

$$\frac{dQ_t}{Q_t} = \Big(a^* + \frac{1}{2}\sigma_2^2\Big)dt + \sigma_2 dB_{2t}. \tag{1.5}$$

Let p represent the proportion of the capital invested in the risky asset, where $0<p<1$. Obviously, the rest $1-p$ means the proportion of risk-free investment. Thus, the surplus process under the two kinds of investments satisfies

$$dX_t = pX_{t-}\frac{dQ_t}{Q_t} + (1-p)X_{t-}\frac{dP_t}{P_t} + cdt - dS_t + \sigma_1 dB_{1t}, \tag{1.6}$$

where X_{t-} indicates the left limit when the surplus approaches t from the left.

De Finetti[5] proposed a dividend strategy in the insurance risk model to optimize the surplus in the insurance portfolio. Since then, scholars have studied the dividend problem on various risk models, including [1, 6, 20]. Wan[21] studied the dividend payments and ruin problems in the perturbed compound Poisson process with a dividend barrier strategy, but the investment was not considered in that document. Chen and Ou[3] applied a dividend barrier strategy to the stochastic return risk model; they used the sinc numerical method to obtain the approximate solutions of the expected discounted dividend function and the expected penalty function. Based on their research model[3], we consider the influence of the disturbance of the classical model.

In the risk model (1.6), we consider dividend payments will be distributed to shareholders at a constant rate α $(0<\alpha\leq c)$ when the surplus is above a constant barrier $b(b>0)$; however, no dividend when the surplus is below b. The surplus process with dividend payments is denoted by $\{X_{bt}\}_{t\geq 0}$, then, we have

$$\begin{aligned} dX_{bt} &= \begin{cases} pX_{bt-}\dfrac{dQ_t}{Q_t} + (1-p)X_{bt-}\dfrac{dP_t}{P_t} + cdt - dS_t + \sigma_1 dB_{1t}, & X_{bt-}<b, \\ pX_{bt-}\dfrac{dQ_t}{Q_t} + (1-p)X_{bt-}\dfrac{dP_t}{P_t} + (c-\alpha)dt - dS_t + \sigma_1 dB_{1t}, & X_{bt-}\geq b \end{cases} \\ &= \begin{cases} \sigma_1 dB_{1t} + p\sigma_2 X_{bt-}dB_{2t} + (\beta X_{bt-}+c)dt - dS_t, & X_{bt-}<b, \\ \sigma_1 dB_{1t} + p\sigma_2 X_{bt-}dB_{2t} + (\beta X_{bt-}+c-\alpha)dt - dS_t, & X_{bt-}\geq b, \end{cases} \end{aligned} \tag{1.7}$$

where $\beta = (a^* + \frac{1}{2}\sigma_2^2)p + (1-p)r$, and the positive security-loading condition is $c - \alpha > \lambda E[Y_1]$.

The present value of the total discounted dividend under the threshold dividend policy controlled by boundary b is

$$D_{x,b} = \alpha \int_0^{T_b} e^{-\delta t} I(X_{bt} > b)dt$$

representing the discounted dividends add up to T_b, where $\delta > 0$ is the discount factor, $T_b = \inf\{t : X_{bt} \leq 0\}$ is the moment of ruin. It is clear that $D_{x,b} \in (0, \alpha/\delta)$. For $x \geq 0$, $V(x;b)$ is the expectation of $D_{x,b}$, $V(x;b) = E[D_{x,b}|X_0 = x]$. Denoting the set of all dividend strategies by $\mathscr{D}$, and we find the optimal dividend threshold b^*, satisfying $V(x;b^*) = \sup_{b\in\mathscr{D}} V(x;b)$.

The Gerber-Shiu function of the model (1.7) is defined by

$$\Phi(x;b) = E[e^{-\gamma T_b}\omega(X_{T_{b-}}, |X_{T_b}|)I(T_b < +\infty)|X_0 = x], \tag{1.8}$$

where $\omega(x_0, y_0)$, ($x_0 \geq 0$, $y_0 \geq 0$), is a nonnegative penalty function, $X_{T_{b-}}$ represents the instantaneous surplus before ruin, and X_{T_b} represents the deficit at ruin time, $\gamma > 0$ is the discounted factor, and $I(\cdot)$ is the indicative function. In particular, if $\gamma = 0$ and $\omega(0,0) = 1$, $\Phi(x;b)$ is converted to the ruin probability $\psi(x;b) = P\{T_b < \infty|X_0 = x\}$. Let $(\Omega, \mathscr{F}, \mathbf{F}, \mathbf{P})$ be a filtered probability space containing all processes and random variables, satisfy the usual conditions, i.e., $\mathscr{F}_t$ is right continuous and $\mathbf{P}$-complete. In this paper, $V(x;b)$ and $\Phi(x;b)$ are fully smooth.

The rest of this paper is arranged as follows. In Section 2, we derive the equations and the boundary conditions satisfied by $V(x;b)$ and $\Phi(x;b)$. In Section 3, we use sinc numerical method to find the approximate solutions of the equations. In Section 4, we give some examples to describe the influence of investment proportion p on the dividend payments $V(x;b)$ and ruin probability $\psi(x;b)$.

2 Integro-differential Equations

2.1 The Expected Discounted Dividend Payments

In this section, we will get the integro-differential equations satisfied by the expected discounted dividend payments $V(x;b)$. Clearly, $V(x;b)$ behaves differently when the value range of x is different. Hence, for convenience, we write $V_1(x;b)$ for $0 \leq x \leq b$, and $V_2(x;b)$ for $x > b$. The following theorems are two cases when the expression of the function $V(x;b)$ is different.

Theorem 2.1. *For $0 \leq x \leq b$, $V(x;b)$ satisfies the integro-differential equation*

$$\begin{aligned}&\frac{1}{2}(\sigma_1^2 + p^2x^2\sigma_2^2)V_1^{''}(x;b) + (\beta x + c)V_1^{'}(x;b) - (\delta + \lambda)V_1(x;b)\\&+ \lambda \int_0^x V_1(x-y;b)dF_Y(y) = 0,\end{aligned} \tag{2.1}$$

and for $b < x < +\infty$, $V(x;b)$ satisfies the integro-differential equation

$$\begin{aligned}&\frac{1}{2}(\sigma_1^2 + p^2x^2\sigma_2^2)V_2^{''}(x;b) + (\beta x + c - \alpha)V_2^{'}(x;b) - (\delta + \lambda)V_2(x;b)\\&+ \lambda\Big[\int_0^{x-b} V_2(x-y;b)dF_Y(y) + \int_{x-b}^{x} V_1(x-y;b)dF_Y(y)\Big] + \alpha = 0,\end{aligned} \tag{2.2}$$

the boundary conditions are

$$V_1(0;b) = 0; \tag{2.3}$$

$$\lim_{x\to+\infty} V_2(x;b)=\frac{\alpha}{\delta}. \tag{2.4}$$

Proof. For convenience, let

$$\begin{aligned} h_{1t} =&\sigma_1 dB_{1t} + px\sigma_2 dB_{2t} + (\beta x + c)dt, \\ h_{2t} =&\sigma_1 dB_{1t} + px\sigma_2 dB_{2t} + (\beta x + c - \alpha)dt. \end{aligned}$$

Furthermore, $P(T_1 > dt) = e^{-\lambda dt} = 1 - \lambda dt + o(dt)$, $P(T_1 \le dt) = \lambda dt + o(dt)$, considering a small interval $(0, dt]$, and discussing the time of the first claim, we get the formula for $0 \le x \le b$

$$\begin{aligned} V_1(x;b) &= e^{-\delta dt}\{e^{-\lambda dt}E[V_1(x+h_{1t};b)] + (1-e^{-\lambda dt})E[V_1(x+h_{1t}-Y_1;b)]\} \\ &= (1-\delta t)(1-\lambda dt)E[V_1(x+h_{1t};b)] \\ &\quad + (1-\delta t)(\lambda dt)E\Big[\int_0^{x+h_{1t}-b} V_1(x+h_{1t}-y;b)dF_Y(y)\Big] + o(t). \end{aligned} \tag{2.5}$$

By Itô formula, we get

$$E[V_1(x+h_{1t};b)] = E[V_1(x;b) + V_1^{'}(x;b)h_{1t} + \frac{1}{2}V_1^{''}(x;b)(h_{1t})^2] + o(t).$$

If $b < x < +\infty$,

$$\begin{aligned} V_2(x;b) =&e^{-\delta dt}\{\alpha dt + e^{-\lambda dt}E[V_2(x+h_{2t};b)]\} \\ &+ e^{-\delta dt}(1-e^{-\lambda dt})E[E[V_2(x+h_{2t}-Y_1;b)|Y_1\in(0,x+h_{2t}-b)] \\ &+ E[V_1(x+h_{2t}-Y_1;b)|Y_1\in(x+h_{2t}-b,+\infty)]] \\ =&(1-\delta t)\{\alpha dt + (1-\lambda dt)E[V_1(x+h_{2t};b)]\} \\ &+ (1-\delta t)\lambda dt E\Big[\int_0^{x+h_{2t}-b} V_2(x+h_{2t}-y;b)dF_Y(y)\Big] \\ &+ (1-\delta t)\lambda dt E\Big[\int_{x+h_{2t}-b}^{+\infty} V_1(x+h_{2t}-y;b)dF_Y(y)\Big] + o(t). \end{aligned} \tag{2.6}$$

By Itô formula, we have

$$E[V_2(x+h_{2t};b)] = V_2(x;b) + E\Big[V_2^{'}(x;b)h_{2t} + \frac{1}{2}V_2^{''}(x;b)(h_{2t})^2\Big] + o(t).$$

Subtracting $V_1(x;b)$ and $V_2(x;b)$ on both sides of (2.5) and (2.6) respectively, dividing dt and then letting $dt \to 0$, we obtain the integro-differential equations (2.1) and (2.2).

Moreover, when $X_0 = 0$, ruin is immediate and no dividend is paid. When X_0 tends to infinity, ruin will never happen, and dividends are always paid at the rate α per unit time. So we obtain (2.3) and (2.4). The proof of Theorem 2.1 is completed. □

Remark 2.1. Because of the smoothness of the discounted aggregate penalty function, we obtain $V_1(b-;b) = V_2(b+;b); V_1^{'}(b-;b) = V_2^{'}(b+;b)$. A detailed discussion can be seen in [21].

2.2 The Expected Discounted Penalty Function

Obviously, $\Phi(x;b)$ also behaves differently, depending on the value between its initial surplus x and the barrier level b. Hence, for convenience, we denote $\Phi(x;b) = \Phi_1(x;b)$ if $0 \le x \le b$ and $\Phi(x;b) = \Phi_2(x;b)$ if $x > b$. Using the similar proof method in equations (2.1) and (2.2), we get the following theorems.

Theorem 2.2. *For $0 \leq x \leq b$, $\Phi_1(x;b)$ satisfies*

$$\frac{1}{2}(\sigma_1^2 + p^2x^2\sigma_2^2)\Phi_1^{''}(x;b) + (\beta x + c)\Phi_1^{'}(x;b) - (\lambda+\gamma)\Phi_1(x;b) + \lambda\Big[\int_0^x \Phi_1(x-y;b)dF_Y(y) + \int_x^{+\infty}\omega(x,y-x)dF_Y(y)\Big] = 0, \tag{2.7}$$

and for $b < x < +\infty$, $\Phi_2(x;b)$ satisfies integro-differential equation

$$\frac{1}{2}(\sigma_1^2 + p^2x^2\sigma_2^2)\Phi_2^{''}(x;b) + (\beta x + c - \alpha)\Phi_2^{'}(x;b) - (\lambda+\gamma)\Phi_2(x;b) + \lambda\Big[\int_0^{x-b} \Phi_2(x-y;b)dF_Y(y) + \int_{x-b}^{x}\Phi_1(x-y;b)dF_Y(y) + \int_x^{+\infty}\omega(x,y-x)dF_Y(y)\Big] = 0, \tag{2.8}$$

the boundary conditions are

$$\Phi_1(0;b) = \omega(0,0); \tag{2.9}$$

$$\lim_{x\to+\infty}\Phi_2(x;b) = 0. \tag{2.10}$$

Proof . Since the proof process of equations (2.7) and (2.8) is similar to equations (2.1) and (2.2), we will not repeat it here. When $x = 0$, it goes to ruin immediately, then, $T_b = 0$. In this case, both of the instantaneous surplus before ruin and deficit at ruin are zero, thus condition (2.9) is met; when $x \to +\infty$, ruin does not happen at all, hence $T_b = \infty$ and condition (2.10) is also satisfied. □

Remark 2.2. Because of the smoothness of the discounted aggregate penalty function, we obtain $\Phi_1(b-;b) = \Phi_2(b+;b)$ and $\Phi_1^{'}(b-;b) = \Phi_2^{'}(b+;b)$.

3 Sinc Asymptotic Numerical Analysis

Since the exact solutions of the integro-differential equations (2.1)–(2.2) and (2.7)–(2.8) are not easy to obtain, we will provide a numerical approximation method by using the sinc function in this section.

3.1 Sinc Function Preliminaries

Since Frank Stenger[19] developed the sinc numerical method, subsequently, this method has been widely used in the field of numerical analysis, see, e.g. [4, 13, 18]. Because the explicit solutions of the equations are difficult to obtain, we will discuss the numerical solution. Numerical analysis is widely used in various fields, such as the approximate transformation of orthogonal transformation, the approximate solutions of ordinary differential equations and partial differential equations[24].

We use the Cardinal function $C(g,h)$ to characterize sinc methods, which is the sinc expansion of function g, defined as

$$C(g,h)(u) = \sum_{k\in n} g(kh)\mathrm{sinc}\Big\{\frac{u}{h} - k\Big\}, \qquad -\infty < u < +\infty. \tag{3.1}$$

where $h > 0$ is the step size, and the sinc function is defined on the real field $\mathbf{R}$, by

$$\operatorname{sinc}(u) = \begin{cases} \dfrac{\operatorname{sinc}(\pi u)}{(\pi u)}, & u \neq 0, \\ 1, & u = 0. \end{cases}$$

For any $h > 0$, the translation sinc functions with equally spaced nodes are represented as

$$S(j,h)(u) = \operatorname{sinc}\Big(\frac{u - jh}{h}\Big), \qquad j = 0, \pm 1, \pm 2 \cdots .$$

When u takes the interpolating points kh, the above formula is converted to

$$S(j,h)(kh) = \delta_{jk}^{(0)} = \begin{cases} 0, & k \neq j, \\ 1, & k = j. \end{cases}$$

Definition 3.1 (Definition 1.5.2[18]). *On the real number field* $\boldsymbol{R}$*, let* ν *represents a smooth one-to-one mapping from* $\Gamma (\in C)$ *to* $\boldsymbol{R}$*, with end-point* s_1 *and* s_2 *onto* $\boldsymbol{R}$*, such that* $\nu(s_1) = -\infty$ *and* $\nu(s_2) = +\infty$*. Let* $\kappa = (\nu)^{-1}$ *represents the inverse map, so that*

$$\Gamma = \{u \in C : u = \kappa(x),\ x \in R\}.$$

Based on ν*,* κ *and a positive number* h*, we define the sinc points* u_k

$$u_k = u_k(h) = \kappa(kh), \qquad k = 0, \pm 1, \pm 2 \cdots ,$$

and a function ζ*, by* $\zeta(u) = e^{\nu(u)}$*.*

Let $\hat{\alpha}, \hat{\beta}$ *and* $\hat{d}$ *be in* R^+*, and* $L_{\hat{\alpha},\hat{\beta},\hat{d}}(\nu)$ *is the set of all functions* g *defined on* Γ*, here*

$$g(u) = \begin{cases} O(|\zeta(u)|^{\hat{\alpha}}), & u \to s_1, \\ O(|\zeta(u)|^{-\hat{\beta}}), & u \to s_2, \end{cases}$$

and the Fourier transform $\{g \circ \nu^{-1}\}^{\sim}$ *satisfies the relation*

$$\{g \circ \nu^{-1}\}^{\sim}(\xi) = o(e^{-\hat{d}|\xi|})$$

for all $\xi \in \boldsymbol{R}$*, where* $\hat{\alpha},\ \hat{\beta} \in (0,1]$*, and* $\hat{d} \in (0,\pi)$*. Another family of functions is* $M_{\hat{\alpha},\hat{\beta},\hat{d}}(\nu)$ *defined on* Γ*, such that* $\vartheta = g - Lg \in L_{\hat{\alpha},\hat{\beta},\hat{d}}(\nu)$ *and where* Lg *is defined by*

$$Lg(u) = \frac{g(s_1) + \zeta(u) g(s_2)}{1 + \zeta(u)}.$$

Writing $N^*(N^* > 0)$ *as a integer, and integers* M^**,* m^* *are defined as*

$$m^* = \Big[\frac{\hat{\beta} N^*}{\hat{\alpha}}\Big], \qquad m^* = M^* + N^* + 1.$$

A diagonal matrix $D_{m^*}(g)$ *and a computational operator* V_{m^*} *are defined as*

$$D_{m^*}(g) = \operatorname{diag}[g(u_{-M^*}), \cdots, g(u_{N^*})],$$
$$V_{m^*}(g) = (g(u_{-M^*}), \cdots, g(u_{N^*}))^T,$$

where $[\cdot]$ *represents the maximum integer,* g *is a function defined on* $(0,+\infty)$*, and* T *means transpose. Set*

$$
\begin{aligned}
&h=\Big(\frac{\pi\hat{d}}{\hat{\beta}N^*}\Big)^{\frac{1}{2}},\\
&\gamma_j=S(j,h)\circ\nu,\ \ j=-M^*,\cdots,N^*,\\
&\omega_j=\gamma_j,\ \ j=-M^*+1,\cdots,N^*-1,\\
&\omega_{-M^*}=\frac{1}{1+\zeta}-\sum_{j=-M^*+1}^{N^*}\frac{\gamma_j}{1+e^{jh}},\\
&\omega_{N^*}=\frac{\zeta}{1+\zeta}-\sum_{j=-M^*}^{N^*-1}\frac{e^{jh}\gamma_j}{1+e^{jh}},\\
&\omega^*_{-M^*}=(1+e^{-M^*h})\Big[\frac{1}{1+\zeta}-\sum_{j=-M^*+1}^{N^*}\frac{\gamma_j}{1+e^{jh}}\Big],\\
&\omega^*_{N^*}=(1+e^{-Nh})\Big[\frac{\zeta}{1+\zeta}-\sum_{j=-M^*}^{N^*-1}\frac{e^{jh}\gamma_j}{1+e^{jh}}\Big],\\
&\Omega_{m^*}=(\omega_{-M^*},\cdots,\omega_{N^*}),\\
&\Omega^*_{m^*}=(\omega^*_{-M^*},\omega_{-M^*+1},\cdots,\omega_{N^*-1},\omega^*_{N^*}).
\end{aligned}
$$

Let

$$
\delta^{(-1)}_{kj}=\frac{1}{2}+\int_0^{k-j}\frac{\sin(\pi t)}{\pi t}dt,
$$

then we denote matrix $I^{(-1)}=[\delta^{(-1)}_{kj}]$ *whose elements in row* k *and column* j *are given by* $\delta^{(-1)}_{kj}$*.*

Theorem 3.1[16]**.** *Let* ν *be a one-to-one conformal transformation defined on* Γ*. Then*

$$
\delta^{(0)}_{jk}=[S(j,h)\circ\nu(u)]|_{u=u_k}=\begin{cases}0, & k\neq j,\\ 1, & k=j,\end{cases}
$$

$$
\delta^{(1)}_{jk}=h\frac{d}{d\nu}[S(j,h)\circ\nu(u)]|_{u=u_k}=\begin{cases}\dfrac{(-1)^{k-j}}{k-j}, & k\neq j,\\ 0, & k=j,\end{cases}
$$

and

$$
\delta^{(2)}_{jk}=h^2\frac{d^2}{d\nu^2}[S(j,h)\circ\nu(u)]|_{u=u_k}=\begin{cases}\dfrac{-2(-1)^{k-j}}{(k-j)^2}, & k\neq j,\\ -\dfrac{\pi^2}{3}, & k=j.\end{cases}\tag{3.2}
$$

3.2 Numerical Approximate Solution of $V(x;b)$

To construct an approximate estimate on the interval $(0,+\infty)$, let $\nu(x)=\log x$, then we define the one to one mapping of $R_+\to R$, thus $\zeta(x)=e^{\nu(x)}=x$. For all $h>0$, the sinc grid points x_k $(k=0,\pm1,\pm2,\cdots)$ take the form

$$
x_k=\nu^{-1}(kh)=e^{kh}.
$$

Based on the sinc method, we get the composite translated sinc functions

$$S_j(x) = S(j,h) \circ \nu(x) = \operatorname{sinc}\Big(\frac{\nu(x) - jh}{h}\Big)$$

on the interval $(0, +\infty)$ for $u \in \Gamma$.

We apply the sinc method steps to rearrange the integro-differential equations (2.1)–(2.2) into

$$\begin{aligned} &\frac{1}{2}(\sigma_1^2 + p^2x^2\sigma_2^2)V''(x;b) + (\beta x + c - \alpha I(x > b))V'(x;b) - (\delta + \lambda)V(x;b) \\ &+ \lambda \int_0^x V(x-y;b)f_Y(y)dy + \alpha I(x > b) = 0, \end{aligned} \tag{3.3}$$

by applying convolution formula, the above equation can further be written as

$$\begin{aligned} &\frac{1}{2}(\sigma_1^2 + p^2x^2\sigma_2^2)V''(x;b) + (\beta x + c - \alpha I(x > b))V'(x;b) - (\delta + \lambda)V(x;b) \\ &+ \lambda \int_0^x V(y;b)f_Y(x-y)dy + \alpha I(x > b) = 0, \end{aligned} \tag{3.4}$$

with boundary conditions

$$V(0;b) = 0, \qquad \lim_{x\to\infty} V(x;b) = \frac{\alpha}{\delta}.$$

It can be seen from the Definition 3.1 of sinc function that

$$LV(x;b) = \frac{V(x_1;b) + \zeta(x)V(x_2;b)}{1+\zeta(x)}.$$

When $x_1 = 0$, $x_2 \to \infty$, set

$$W(x) = V(x;b) - LV(x;b) = V(x;b) - \frac{x}{1+x}\frac{\alpha}{\delta}, \tag{3.5}$$

then $W(x) \in L_{\hat{\alpha},\hat{\beta},\hat{d}}(\nu)$, so

$$V(x;b) = W(x) + \frac{x}{1+x}\frac{\alpha}{\delta}, \tag{3.6}$$

$$V'(x;b) = W'(x) + \frac{1}{(1+x)^2}\frac{\alpha}{\delta}, \tag{3.7}$$

$$V''(x;b) = W''(x) - \frac{2}{(1+x)^3}\frac{\alpha}{\delta}. \tag{3.8}$$

Substituting (3.6)–(3.8) into (3.4), and each side of the above equation is divided by $\frac{1}{2}(\sigma_1^2 + p^2x^2\sigma_2^2)$, the following equation is obtained

$$W''(x) + P_1(x)W'(x) + P_2(x)W(x) + \lambda P_3(x)\int_0^x f_Y(x-y)W(y)dy + R(x) = 0, \tag{3.9}$$

where

$$\begin{aligned} P_1(x) =& \frac{2(\beta x + c - \alpha I(x>b))}{\sigma_1^2 + p^2x^2\sigma_2^2}, \qquad P_2(x) = -\frac{2(\delta+\lambda)}{\sigma_1^2 + p^2x^2\sigma_2^2}, \qquad P_3(x) = \frac{2}{\sigma_1^2 + p^2x^2\sigma_2^2}, \\ R(x) =& \frac{2\alpha I(x>b)}{\sigma_1^2 + p^2x^2\sigma_2^2} - \frac{2\alpha}{\delta}\frac{1}{(1+x)^3} + \frac{\alpha}{\delta}\frac{1}{(1+x)^2}P_1(x) + \frac{\alpha}{\delta}\frac{x}{1+x}P_2(x) \end{aligned}$$

$$+ \lambda \int_0^x \frac{\alpha}{\delta} \frac{y}{1+y} P_3(x) f_Y(x-y) dy.$$

When x equals 0 or x goes to ∞,

$$W(0) = 0, \qquad \lim_{x\to\infty} W(x) = 0,$$

Then by using Theorems 1.5.13, 1.5.14 and 1.5.20 of reference [18], we obtain

$$\int_0^x f_Y(x-y) W(y) dy \approx \sum_{j=-M^*}^{N^*} \sum_{i=-M^*}^{N^*} \omega_i A_i W_j, \tag{3.10}$$

$$W(x) \approx \tilde{W}(x) = \sum_{j=-M^*}^{N^*} W_j S(j,h) \circ \nu(x), \tag{3.11}$$

where

$$A = XSX^{-1} = hI^{-1} D_{m^*} \Big(\frac{1}{(\nu)'} \Big),$$

and S is a diagonal matrix. $A = [A_{ij}]$ is an $M^* + N^* + 1$-dimensional square matrix, W_j denotes approximate estimate of $W(x_j)$, and $\nu(x) = \ln x$.

Substituting the integral terms of (3.10) into equation (3.9), and using sinc grid points x_k ($k = -M^*, \cdots, N^*$) to approach x, and substituting (3.11) into (3.9), we obtain

$$\begin{aligned} &\tilde{W}''(x_k) + P_1(x_k)\tilde{W}'(x_k) + P_2(x_k)\tilde{W}(x_k) \\ &+ \lambda P_3(x_k) \sum_{j=-M^*}^{N^*} \sum_{i=-M^*}^{N^*} \omega_i(x_k) A_i W_j + R(x_k) = 0, \end{aligned} \tag{3.12}$$

where

$$\tilde{W}(x_k) = \sum_{j=-M^*}^{N^*} W_j [S(j,h) \circ \nu(x_k)] = \sum_{W_j=-M^*}^{N^*} W_j \delta_{jk}^{(0)}, \tag{3.13}$$

$$\tilde{W}'(x_k) = \sum_{j=-M^*}^{N^*} W_j [S(j,h) \circ \nu(x_k)]' = \sum_{j=-M^*}^{N^*} W_j \nu'(x_k) h^{-1} \delta_{jk}^{(1)}, \tag{3.14}$$

$$\begin{aligned} \tilde{W}''(x_k) &= \sum_{j=-M^*}^{N^*} W_j [S(j,h) \circ \nu(x_k)]'' \\ &= \sum_{j=-M^*}^{N^*} W_j [\nu''(x_k) h^{-1} \delta_{jk}^{(1)} + (\nu'(x_k))^2 h^{-2} \delta_{jk}^{(2)}]. \end{aligned} \tag{3.15}$$

By replacing (3.13)–(3.15) in (3.12), the following equation is obtained

$$\begin{aligned} &\sum_{j=-M^*}^{N^*} \Big\{ \nu''(x_k)\delta_{jk}^{(1)} h^{-1} + \nu'(x_k)^2 \delta_{jk}^{(2)} h^{-2} + P_1(x_k)\nu'(x_k)\delta_{jk}^{(1)} h^{-1} \\ &+ P_2(x_k)\delta_{jk}^{(0)} + \lambda P_3(x_k) \sum_{j=-M^*}^{N^*} \sum_{i=-M^*}^{N^*} \omega_i(x_k) A_{ij} \Big\} W_j = -R(x_k). \end{aligned} \tag{3.16}$$

Multiplying both ends of the above equation by $\frac{h^2}{(\nu'(x_k))^2}$, we have

$$\sum_{j=-M^*}^{N^*}\Big\{\delta_{jk}^{(2)}+h\Big[\frac{\nu''(x_k)}{(\nu'(x_k))^2}+\frac{P_1(x_k)}{\nu'(x_k)}\Big]\delta_{jk}^{(1)}+h^2\frac{P_2(x_k)}{(\nu'(x_k))^2}\delta_{jk}^{(0)}$$
$$+\lambda\frac{P_3(x_k)h^2}{(\nu'(x_k))^2}\sum_{i=-M^*}^{N^*}\omega_i(x_k)A_{ij}\Big\}W_j=-\frac{h^2R(x_k)}{(\nu'(x_k))^2}. \tag{3.17}$$

Since

$$\delta_{jk}^{(0)}=\delta_{kj}^{(0)},\qquad \delta_{jk}^{(1)}=-\delta_{kj}^{(1)},\qquad \delta_{jk}^{(2)}=\delta_{kj}^{(2)},\qquad \frac{\nu''(x_k)}{(\nu'(x_k))^2}=-\Big(\frac{1}{\nu'(x_k)}\Big)',$$

thus the results after transformation are as follows

$$\sum_{j=-M^*}^{N^*}\Big\{\delta_{kj}^{(2)}+h\Big[\Big(\frac{1}{\nu'(x_k)}\Big)'-\frac{P_1(x_k)}{\nu'(x_k)}\Big]\delta_{kj}^{(1)}+h^2\frac{P_2(x_k)}{(\nu'(x_k))^2}\delta_{kj}^{(0)}$$
$$+\lambda\frac{P_3(x_k)h^2}{(\nu'(x_k))^2}\sum_{i=-M^*}^{N^*}\omega_i(x_k)A_{ij}\Big\}W_j=-\frac{h^2R(x_k)}{(\nu'(x_k))^2},\qquad k=-M^*,\cdots,N^*. \tag{3.18}$$

Set $I^{(m)}=[\delta_{kj}^{(m)}]$, $m=-1,0,1,2$, where $\delta_{kj}^{(m)}$ is the element in row k and column j, here, $I^{(m)}$ is a square matrix of order M^*+N^*+1. Equation (3.18) can be rewritten in matrix form, such as

$$BW=R, \tag{3.19}$$

where $W=[W_j]^T,\ j=-M^*,\cdots,N^*$,

$$R=\Big[-h^2\frac{R(x_{-M^*})}{(\nu'(x_{-M^*}))^2},\cdots,-h^2\frac{R(x_{N^*})}{(\nu'(x_{N^*}))^2}\Big],$$
$$B=I^{(2)}-hD_{m^*}\left(\left(\frac{1}{\nu}\right)'-\frac{P_1}{\nu'}\right)I^{(1)}+h^2D_{m^*}\left(\frac{P_2}{(\nu')^2}\right)I^{(0)}+h^2D_{m^*}\left(\frac{P_3}{(\nu')^2}\right)\Omega_{m^*}^*A.$$

Equation (3.19) is M^*+N^*+1-dimensional, where W_j are unknown parameters, $j=-M^*,\cdots,N^*$, so W_j can be obtained by solving equation (3.19). Thus, the approximate solution of $W(x)$ is obtained from equation (3.11), and according to (3.5), the expression of the numerical solution of $V(x;b)$ is

$$V(x;b)\approx\tilde{W}(x)+\frac{x}{1+x}\frac{\alpha}{\delta}=\sum_{j=-M^*}^{N^*}W_jS(j,h)\circ\nu(x)+\frac{x}{1+x}\frac{\alpha}{\delta}. \tag{3.20}$$

We denote the sinc approximation error of the expected discounted dividend payments by

$$ERR_V=V(x;b)-\Big\{\sum_{j=-M^*}^{N^*}W_jS(j,h)\circ\nu(x)+\frac{x}{1+x}\frac{\alpha}{\delta}\Big\}.$$

3.3 Numerical Approximate Solution of $\Phi(x;b)$

Similarly, rearranging the integro-differential equations (2.7)–(2.8), we get the following equation

$$\frac{1}{2}(\sigma_1^2+p^2x^2\sigma_2^2)\Phi''(x;b)+(\beta x+c-\alpha I(x>b))\Phi'(x;b)-(\lambda+\gamma)\Phi(x;b)$$

$$+\lambda\int_0^x \Phi(x-y;b)f_Y(y)dy+\lambda\int_x^{+\infty}\omega(x,y-x)f_Y(y)dy=0, \tag{3.21}$$

by applying convolution formula, the above equation can be written as

$$\begin{aligned}&\frac{1}{2}(\sigma_1^2+p^2x^2\sigma_2^2)\Phi''(x;b)+(\beta x+c-\alpha I(x>b))\Phi'(x;b)-(\lambda+\gamma)\Phi(x;b)\\&+\lambda\int_0^x \Phi(y;b)f_Y(x-y)dy+\lambda\int_x^{+\infty}\omega(x,y-x)f_Y(y)dy=0.\end{aligned} \tag{3.22}$$

with boundary conditions

$$\Phi(0;b)=1,\qquad \lim_{x\to\infty}\Phi(x;b)=0.$$

It can be seen from the Definition 3.1 of sinc function preliminaries

$$L\Phi(x;b)=\frac{\Phi(x_1;b)+\zeta(x)\Phi(x_2;b)}{1+\zeta(x)},$$

where $\zeta(x)=e^{\nu(x)}$. When $x_1=0$, $x_2\to\infty$, setting

$$\bar{W}(x)=\Phi(x;b)-L\Phi(x;b)=\Phi(x;b)-\frac{1}{1+x},$$

then $\bar{W}(x)\in L_{\hat{\alpha},\hat{\beta},\hat{d}}(\nu)$, and $\bar{W}(x)$ satisfies

$$\bar{W}''(x)+P_1(x)\bar{W}'(x)+P_2(x)\bar{W}(x)+\lambda\int_0^x f_Y(x-y)\bar{W}(y)dy+\bar{R}(x)=0, \tag{3.23}$$

the boundary conditions are

$$\bar{W}(0)=0,\qquad \lim_{x\to+\infty}\bar{W}(x)=0,$$

where

$$\begin{aligned}\bar{R}(x)=&\frac{2}{(1+x)^3}-\frac{P_1(x)}{(1+x)^2}+\frac{P_2(x)}{(1+x)}+\lambda P_3(x)\int_0^x\frac{1}{1+y}f_Y(x-y)dy\\&+\frac{2\lambda}{(\sigma_1^2+p^2x^2\sigma_2^2)}\int_0^x\omega(x,y-x)dF_Y(y).\end{aligned} \tag{3.24}$$

Calculation method similar to $V(x;b)$, we get

$$B\bar{W}=\bar{R}, \tag{3.25}$$

where $\bar{W}=[\bar{W}_{-M^*},\cdots,\bar{W}_{N^*}]^T$, $\bar{W}_l$ represents the estimated value of $\bar{W}_l(x_l)$, we have

$$\bar{R}=\Big[-h^2\frac{\bar{R}(x_{-M^*})}{(\nu'(x_{-M^*}))^2},\cdots,-h^2\frac{\bar{R}(x_{N^*})}{(\nu'(x_{N^*}))^2}\Big]. \tag{3.26}$$

Therefore, the expression of the approximate solution of $\Phi(x;b)$ is

$$\Phi(x;b)\approx\bar{W}(x)+\frac{1}{1+x}=\sum_{l=-M^*}^{N^*}\bar{W}_lS(j,h)\circ\nu(x)+\frac{1}{1+x}. \tag{3.27}$$

We denote the sinc approximation error of the expected discounted penalty function by

$$ERR_\Phi=\Phi(x;b)-\Big\{\sum_{l=-M^*}^{N^*}\bar{W}_lS(j,h)\circ\nu(x)+\frac{1}{1+x}\Big\}.$$

4 Numerical Illustrations

4.1 Error Analysis for a Particular Case

Gao and Yin[9] discussed the perturbed risk model with a constant dividend barrier, and obtained explicit solutions (ES) of the expected discounted dividend payments and the ruin probability when the claim sizes are exponentially distributed, and the corresponding solutions of the equation can be found in Section 6 (P.464). In the special case where $\sigma_2 = 0, r = 0, p = 0$ and $w(x, y) = 1$, we can obtain integral-differential expressions consistent with Gao and Yin. In this paper, the corresponding error is obtained by comparing the explicited solution with the sinc approximation solution (SA). The values of parameters are the same in the simulation process, and the specific situation of $V(x; 8)$ and $\psi(x; 8)$ can be directly listed in Tables 4.1 and 4.2.

Table 4.1. The values of $V(x; 8)$ when $\sigma_1 = 0.8,\ \lambda = 1,\ \delta = 0.06$

	$x = 7.8$	$x = 7.85$	$x = 7.9$	$x = 7.95$	$x = 8.0$	$x = 8.05$	$x = 8.1$	$x = 8.15$
SA	5.7366	5.7416	5.7467	5.7517	5.7567	5.7617	5.7666	5.7715
ES	5.5325	5.5936	5.6554	5.7177	5.7807	5.8444	5.9087	5.9737
error	0.2040	0.1480	0.0913	0.0340	0.0240	0.0827	0.1421	0.2022

Table 4.2. The values of $\psi(x; 8)$ when $\sigma_1 = 0.8, \lambda = 1, \delta = 0.06$

	$x = 7.8$	$x = 7.85$	$x = 7.9$	$x = 7.95$	$x = 8.0$	$x = 8.05$	$x = 8.1$	$x = 8.15$
SA	0.1047	0.1042	0.1037	0.1032	0.1027	0.1022	0.1018	0.1013
ES	0.0105	0.0105	0.0106	0.0107	0.0107	0.0108	0.0109	0.0110
error	0.0942	0.0936	0.0931	0.0925	0.0920	0.0914	0.0909	0.0903

It can be shown that the numerical solution obtained by sinc method is very close to the analytical solution, and the difference between the results of the two methods is in a controllable range.

4.2 Numerical Examples

In this section, we use the sinc approximation method to calculate the expected discounted dividend payments and the ruin probability. When the claim sizes are exponential, mixture of two exponential or lognormal distributions. Next, we analyze the numerical solution of $V(x; b)$ or $\psi(x; b)$ in the case of exponential distribution, mixed exponential distribution or lognormal distribution.

4.2.1 The Exponential Distribution Case

We assume that $f_Y(y)$ follows an exponential distribution, in this subsection, it is assumed that the specific expression of $f_Y(y)$ is

$$f_Y(y) = \begin{cases} \eta e^{-\eta y}, & y > 0, \\ 0, & y \leq 0. \end{cases}$$

Thus

$$f_Y(x - y) = \begin{cases} \eta e^{-\eta(x-y)}, & y < x, \\ 0, & y \geq x. \end{cases}$$

The following examples are discussed under the parameter values $\lambda = 1,\ \sigma_2 = 0.2,\ r = \delta = \gamma = 0.06,\ a^* = 0.5,\ c = 0.4, \alpha = 0.1,\ \eta = 5$ and the barrier level $b = 0.5$ or $b = 5$.

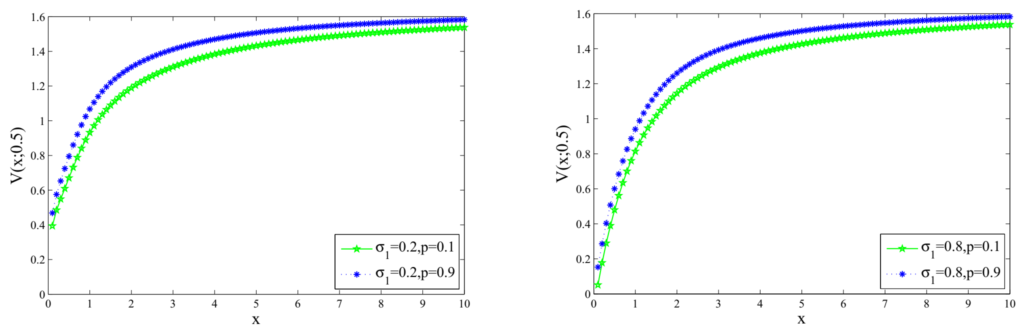

Figure 4.1. Curves of $V(x; 0.5)$ when $N^* = 10,\ \hat{\alpha} = \frac{1}{4},\ \hat{\beta} = \frac{1}{2},\ \hat{d} = \frac{\pi}{4}$, $\sigma_1 = 0.2$(left) and $\sigma_1 = 0.8$(right).

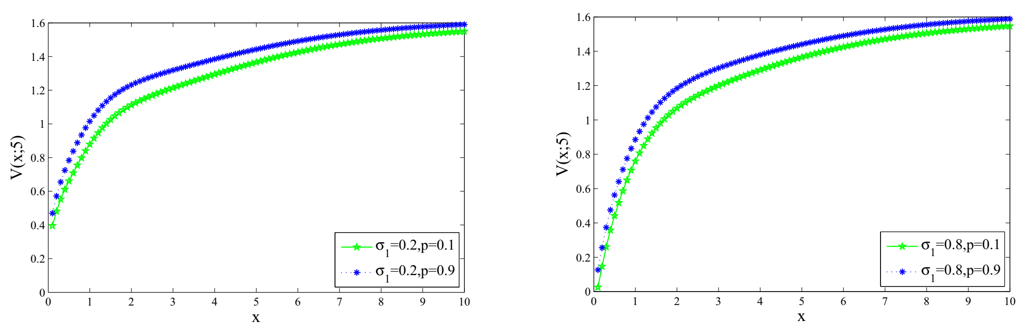

Figure 4.2. Curves of $V(x; 5)$ when $N^* = 10,\ \hat{\alpha} = \frac{1}{4},\ \hat{\beta} = \frac{1}{2},\ \hat{d} = \frac{\pi}{4}$, $\sigma_1 = 0.2$(left) and $\sigma_1 = 0.8$(right).

Example A1. Figures 4.1 and 4.2 show the impact of investment proportion p on the results of $V(x; 0.5)$ and $V(x; 5)$ for $x \in (0, 10)$ under $\sigma_1 = 0.2$ and $\sigma_1 = 0.8$. When the diffusion coefficient σ_1 increases, the overall fluctuation trend of $V(x; b)$ function is large, and when p increases, it is more affected by investment risk. Combining the two groups of figures, we can observe that with the increase of b, the fluctuation curve of dividend payments $V(x; b)$ tends to be stable faster. Let $\sigma_1 = 0.8$ and $p = 0.9$, we fix the initial surplus x to seek the optimal dividend level b^*, and obtained the following results. It can be seen that $V(x; b)$ take the maximum value at $b = 0.4$ in Table 4.3, so the optimal b^* is around 0.4.

Table 4.3. The values of $V(5; b)$ when $b \in (0, 5)$

	$b = 0$	$b = 0.1$	$b = 0.2$	$b = 0.3$	$b = 0.4$
$V(5; b)$	1.29675	1.29678	1.29685	1.29688	1.29690
	$b = 0.5$	$b = 1$	$b = 2$	$b = 3$	$b = 5$
$V(5; b)$	1.29628	1.29226	1.29220	1.28115	1.25310

Example A2. Let $\delta = 0,\ \ \omega(\cdot, \cdot) = 1$, equation (1.8) is transformed into ruin probability $\psi(x; b) = P(T_b < \infty | X_0 = x)$. In this case,

$$\bar{R}(x) = \frac{2}{(1+x)^3} - \frac{P_1(x)}{(1+x)^2} + \frac{P_2(x)}{(1+x)} + \lambda P_3(x) \int_0^x \frac{1}{1+y} \eta e^{-\eta(x-y)} dy$$

$$+\frac{2\lambda}{(\sigma_1^2+p^2x^2\sigma_2^2)}e^{-\eta x}.$$

For $x \in (0,10)$, we calculate the specific numbers of ruin probability when N^* takes different numbers. Tables 4.4 and 4.5 represent the estimated values of $\psi(x;0.5)$ and $\psi(x;5)$ for different x. It can be seen that even if N^* is small, the values of ruin probability can tend to be stable.

Table 4.4. The results of $\psi(x;0.5)$ when $\sigma_1=0.8,\ p=0.1,\ \hat{\alpha}=\frac{\pi}{4},\ \hat{\beta}=\frac{\pi}{4},\ \hat{d}=\frac{\pi}{4}$ in different values of N^*

N^*	$x=0.15$	$x=0.5$	$x=0.8$	$x=1.5$	$x=3$	$x=5$	$x=7$	$x=10$
10	0.93133	0.72779	0.60107	0.41127	0.23741	0.15279	0.11332	0.09021
15	0.93299	0.72637	0.60044	0.41112	0.23736	0.15283	0.11329	0.09019
20	0.93400	0.72534	0.59981	0.41102	0.23739	0.15280	0.11331	0.09021
25	0.93474	0.72455	0.59930	0.41092	0.23740	0.15278	0.11333	0.09020

Table 4.5. The results of $\psi(x;5)$ $\sigma_1=0.8,\ p=0.1,\ \hat{\alpha}=\frac{\pi}{4},\ \hat{\beta}=\frac{\pi}{4},\ \hat{d}=\frac{\pi}{4}$ in different values of N^*

N^*	$x=0.15$	$x=0.5$	$x=0.8$	$x=1.5$	$x=3$	$x=5$	$x=7$	$x=10$
10	0.91720	0.70714	0.57984	0.39619	0.23103	0.15211	0.11366	0.08182
15	0.92022	0.70804	0.58051	0.39616	0.23061	0.15109	0.11347	0.08188
20	0.92210	0.70878	0.58071	0.39624	0.23085	0.15015	0.11303	0.08198
25	0.92347	0.70921	0.58093	0.39627	0.23076	0.15119	0.11349	0.08177

Example A3. Figures 4.3 and 4.4 show the impact of investment proportion p on the results of $\psi(x;0.5)$ and $\psi(x;5)$ for $x \in (0,10)$ under $\sigma_1=0.2$ and $\sigma_1=0.8$. As shown in Figures 4.3 and 4.4, when p is constant, the greater σ_1 is, the ruin probability fluctuates relatively large. When σ_1 is constant, the greater p is, the higher the ruin probability is. From these two sets of figures, it can be judged that when $b=5$, the inflection point of the ruin probability curve moves forward.

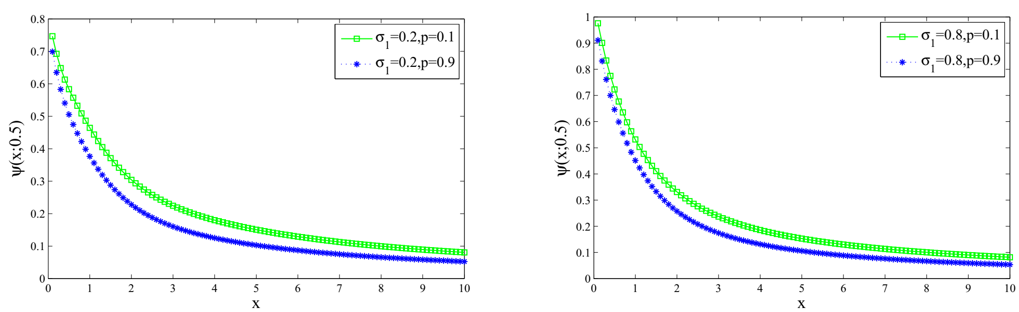

Figure 4.3. Curves of ruin probability $\psi(x;0.5)$ when $N^*=10,\ \hat{\alpha}=\frac{1}{4},\ \hat{\beta}=\frac{1}{2},\ \hat{d}=\frac{\pi}{4}$, $\sigma_1=0.2$(left) and $\sigma_1=0.8$(right).

4.2.2 A Mixture of Two Exponential Distributions Case

We assume that $f_Y(y)$ follows a mixture of two exponential distributions, in this subsection, the probability density function is

$$f_Y(y)=\begin{cases}\tau_1\theta e^{-\theta y}+\tau_2\theta^* e^{-\theta^* y}, & y>0,\\ 0, & y\le 0.\end{cases}$$

where $\tau_1, \tau_2 > 0,\ \tau_1 + \tau_2 = 1$, thus

$$f_Y(x-y) = \begin{cases} \tau_1 \theta e^{-\theta(x-y)} + \tau_2 \theta^* e^{-\theta^*(x-y)}, & y < x, \\ 0, & y \geq x. \end{cases}$$

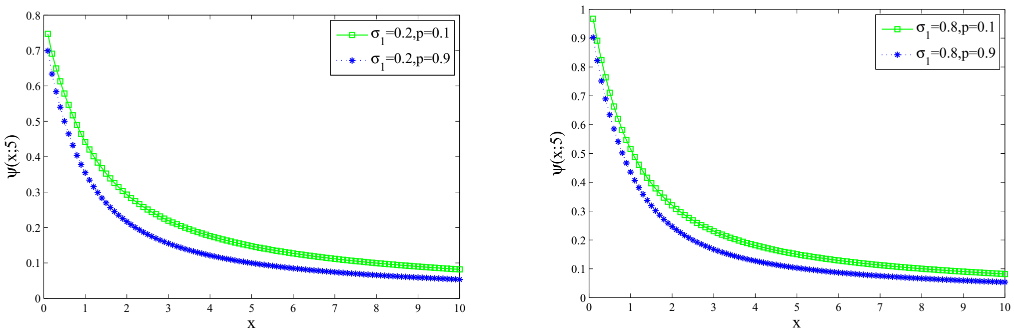

Figure 4.4. Curves of ruin probability $\psi(x;5)$ when $N^* = 10,\ \hat{\alpha} = \frac{1}{4},\ \hat{\beta} = \frac{1}{2},\ \hat{d} = \frac{\pi}{4}$, $\sigma_1 = 0.2$(left) and $\sigma_1 = 0.8$(right).

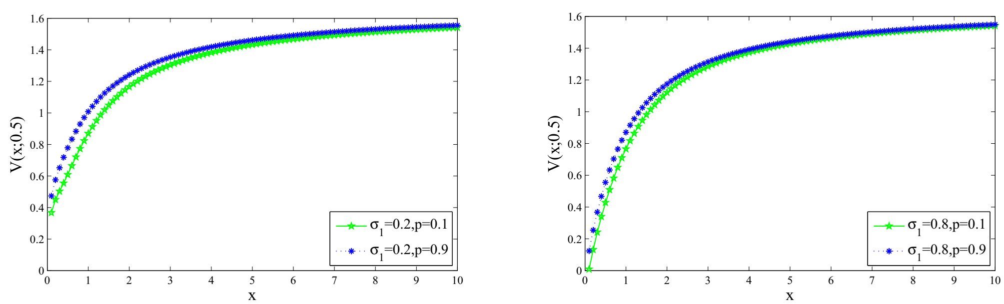

Figure 4.5. $V(x;0.5)$ based on parameters $N^* = 10,\ \hat{\alpha} = \frac{1}{4},\ \hat{\beta} = \frac{1}{2},\ \hat{d} = \frac{\pi}{4},\ \sigma_1 = 0.2$(left) and $\sigma_1 = 0.8$(right).

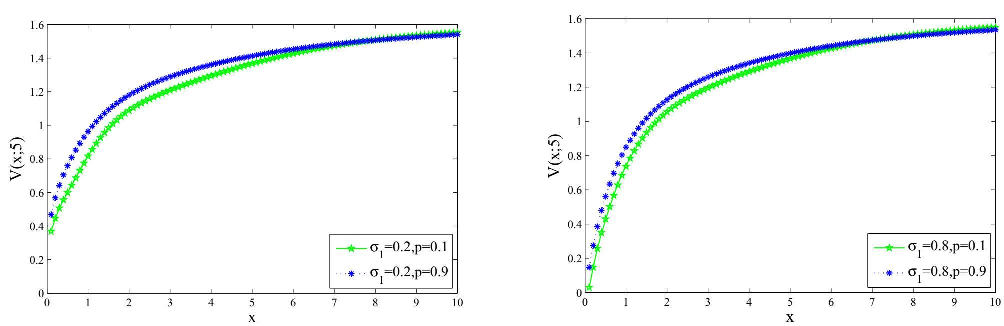

Figure 4.6. $V(x;5)$ based on parameters $N^* = 10,\ \hat{\alpha} = \frac{1}{4},\ \hat{\beta} = \frac{1}{2},\ \hat{d} = \frac{\pi}{4},\ \sigma_1 = 0.2$(left) and $\sigma_1 = 0.8$(right).

We discuss the following examples under parameter values $\lambda = 1,\ \sigma_2 = 1,\ r = \delta = \gamma = 0.06,\ a^* = 0.5,\ c = 0.4,\ \alpha = 0.1,\ \theta = 3,\ \theta^* = 3,\ \tau_1 = 0.8,\ \tau_2 = 0.2$ and the barrier level $b = 0.5$ or $b = 5$.

Example B1. Figures 4.5 and 4.6 describe how investment proportion p affects the results of $V(x;0.5)$ and $V(x;5)$ for $x \in (0,10)$ under $\sigma_1 = 0.2$ and $\sigma_1 = 0.8$. It can be seen from Figures 4.5 and 4.6, as the value of u increases, the two curves almost coincide, and the influencing factors of volatility coefficient σ_1 gradually decrease. Therefore, σ_1 has a great impact on the initial value of the expected discounted dividend payments. Comparing Figures 4.5 and 4.6, it is obvious that the curves of $V(x;5)$ almost coincide when the value of b is large enough under different p.

Table 4.6. The values of $V(5;b)$ when $b \in (0,5)$

	$b=0$	$b=0.1$	$b=0.2$	$b=0.3$	$b=0.4$
$V(5;b)$	1.32008	1.32013	1.32023	1.32029	1.32031
	$b=0.5$	$b=1$	$b=2$	$b=3$	$b=5$
$V(5;b)$	1.31936	1.31403	1.31406	1.30102	1.27796

By selecting the same parameter as the exponential distribution case, we get the results as shown in the following table. It can be seen intuitively that the discounted dividend payments reach the maximum when $b = 0.4$ in Table 4.6, so the optimal dividend level b^* is around 0.4.

Example B2. By analogy with the example of the exponential distribution, when the claim sizes follow a mixture of two exponential distributions. In this case,

$$\begin{aligned}\bar{R}(x) =& \frac{2}{(1+x)^3} + \frac{2\lambda}{(\sigma_1^2 + p^2x^2\sigma_2^2)}(\tau_1 e^{-\theta x} + \tau_2 e^{-\theta^* x}) - \frac{P_1(x)}{(1+x)^2} + \frac{P_2(x)}{(1+x)} \\ &+ \lambda P_3(x)\int_0^x \frac{1}{1+y}(\tau_1\theta e^{-\theta(x-y)} + \tau_2\theta^* e^{-\theta^*(x-y)})dy.\end{aligned}$$

For $x \in (0,10)$, we also calculate the specific numbers of ruin probability when N^* takes different results. Tables 4.7 and 4.8 show that even if N^* is small, the values of ruin probability can also stabilize. It can be seen from the tables that the probability of ruin can be reduced when the value of b is increased.

Table 4.7. The results of $\psi(x;0.5)$ based on parameters $\sigma_1 = 0.8, p = 0.1$, $\hat{\alpha} = \frac{\pi}{4}$, $\hat{\beta} = \frac{\pi}{4}$, $\hat{d} = \frac{\pi}{4}$ in different values of N^*

N^*	$x=0.15$	$x=0.5$	$x=0.8$	$x=1.5$	$x=3$	$x=5$	$x=7$	$x=10$
10	0.93986	0.73906	0.61213	0.41751	0.23624	0.14968	0.11022	0.08741
15	0.94204	0.73782	0.61158	0.41738	0.23618	0.14973	0.11020	0.08739
20	0.94337	0.73690	0.61099	0.41729	0.23621	0.14968	0.11022	0.08741
25	0.94433	0.73619	0.61052	0.41720	0.23622	0.14966	0.11024	0.08740

Table 4.8. The results of $\psi(x;5)$ based on parameters $\sigma_1 = 0.8, p = 0.1$, $\hat{\alpha} = \frac{\pi}{4}$, $\hat{\beta} = \frac{\pi}{4}$, $\hat{d} = \frac{\pi}{4}$ in different values of N^*

N^*	$x=0.15$	$x=0.5$	$x=0.8$	$x=1.5$	$x=3$	$x=5$	$x=7$	$x=10$
10	0.90355	0.69700	0.57311	0.39176	0.22656	0.14808	0.11022	0.07906
15	0.90552	0.69751	0.57354	0.39170	0.22616	0.14706	0.11000	0.07912
20	0.90673	0.69797	0.57360	0.39174	0.22639	0.14616	0.10954	0.07921
25	0.90763	0.69822	0.57373	0.39174	0.22630	0.14716	0.11002	0.07901

Example B3. Figures 4.7 and 4.8 describe how investment proportion p affects the results of ruin probability $\psi(x;0.5)$ and $\psi(x;5)$ for $x \in (0,10)$ under $\sigma_1 = 0.2$ and $\sigma_1 = 0.8$. As shown in the Figures 4.7 and 4.8, when σ_1 is constant, with the gradual increase of x value, the investment proportion has little effect on the ruin probability and the investment risk decreases.

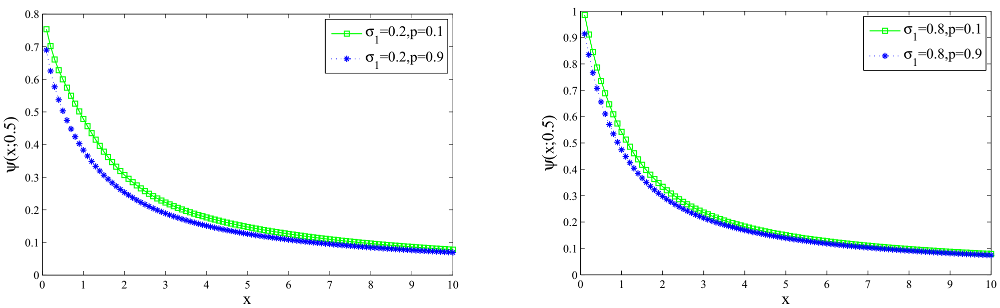

Figure 4.7. Ruin probability $\psi(x;0.5)$ based on parameters $N^* = 10,\ \hat{\alpha} = \frac{1}{4},\ \hat{\beta} = \frac{1}{2},\ \hat{d} = \frac{\pi}{4}$, $\sigma_1 = 0.2$(left) and $\sigma_1 = 0.8$(right).

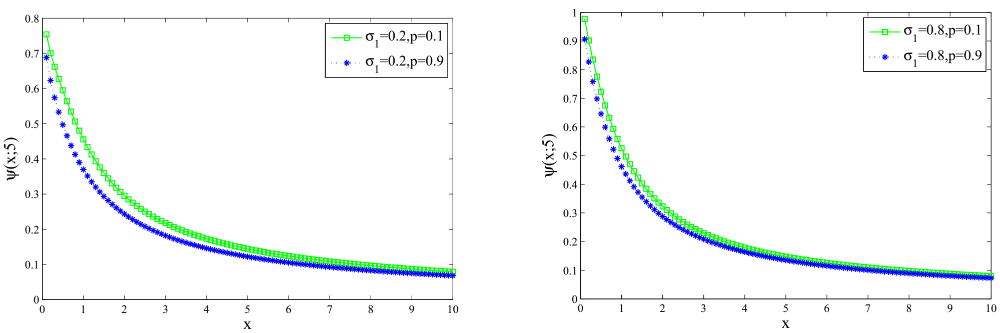

Figure 4.8. Ruin probability $\psi(x;5)$ based on parameters $N^* = 10,\ \hat{\alpha} = \frac{1}{4},\ \hat{\beta} = \frac{1}{2},\ \hat{d} = \frac{\pi}{4}$, and $\sigma_1 = 0.2$(left) and $\sigma_1 = 0.8$(right).

4.2.3 The Lognormal Distribution Case

It was found that the claim sizes in automobile insurance loss obeys lognormal distribution. In this part, we apply this distribution to our risk model. Assume that $f_Y(y)$ follows a lognormal distribution with parameter $(\mu, 2s^2)$, where $2s^2$ is the process variance parameter. In this subsection, we assume the probability density function of the claim amount is

$$f_Y(y) = \begin{cases} \dfrac{1}{2\pi s y} e^{-\frac{(\ln y - \mu)^2}{4s^2}}, & y > 0, \\ 0, & y \leq 0. \end{cases}$$

Thus

$$f_Y(x-y) = \begin{cases} \frac{1}{2\pi s(x-y)} e^{-\frac{[\ln(x-y)-\mu]^2}{4s^2}}, & y < x, \\ 0, & y \geq x. \end{cases}$$

Furthermore, let $\lambda = 1,\ \sigma_2 = 0.2, r = \delta = \gamma = 0.06, a^* = 0.5,\ c = 1.2,\ \alpha = 0.1, \mu = 0.08, s = 0.02$ and the barrier level $b = 0.5$ or $b = 5$. We discuss the influence of parameter change on ruin probability and the expected discounted dividend payments.

Example C1. By observing Figures 4.9 and 4.10, we can see that the curves fluctuation range of $V(x; 0.5)$ and $V(x; 5)$ for $x \in (0, 10)$ is large when $\sigma_1 = 0.2$, and their minimum point is around $x = 1$. When $\sigma_1 = 0.8$, the value of $V(x; b)$ increases with the increase of x. From the two groups of graphs, $p = 0.9$ pays more dividends than $p = 0.1$.

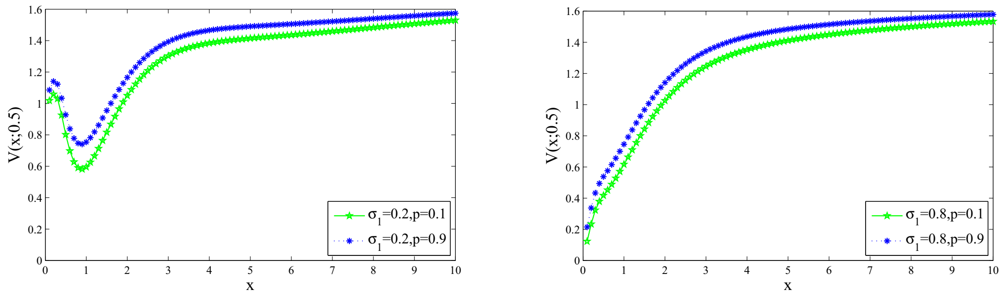

Figure 4.9. Images of $V(x;0.5)$ when $N^*=10$, $\hat{\alpha}=\frac{\pi}{4}$, $\hat{\beta}=\frac{\pi}{4}$, $\hat{d}=\frac{\pi}{2}$, $\sigma_1=0.2$(left) and $\sigma_1=0.8$(right).

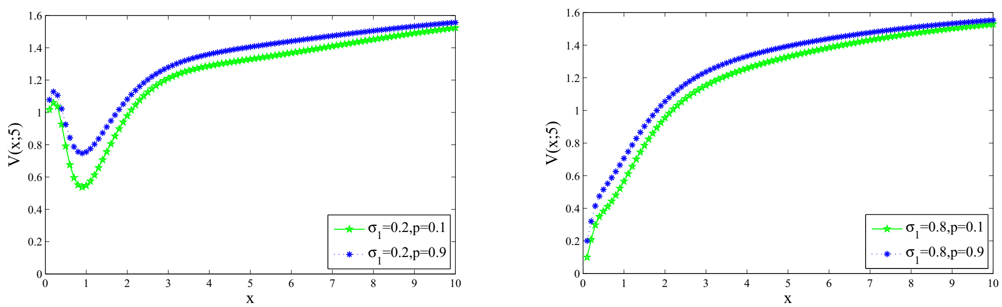

Figure 4.10. Images of $V(x;5)$ when $N^*=10$, $\hat{\alpha}=\frac{\pi}{4}$, $\hat{\beta}=\frac{\pi}{4}$, $\hat{d}=\frac{\pi}{2}$, $\sigma_1=0.2$(left) and $\sigma_1=0.8$(right).

Using the same procedure for the lognormal distribution as in the example above, we get the following presentation in Table 4.9. The optimal threshold dividend level b^* is also around 0.4.

Table 4.9. The values of $V(5;b)$ when $b\in(0,5)$

	$b=0$	$b=0.1$	$b=0.2$	$b=0.3$	$b=0.4$
$V(5;b)$	1.32454	1.32458	1.32460	1.32466	1.32469
	$b=0.5$	$b=1$	$b=2$	$b=3$	$b=5$
$V(5;b)$	1.32416	1.31913	1.31910	1.30420	1.26846

Example C2. Using a similar method to the above two cases, and bringing the density function of the claim sizes to follow lognormal distribution into the equation (3.24), the result is transformed into

$$\begin{aligned}\bar{R}(x)=&\frac{2}{1+x^3}-\frac{P_1(x)}{1+x^2}+2P_3(x)\int_x^{+\infty}\frac{1}{2\pi sy}e^{-\frac{(\ln y-\mu)^2}{4s^2}}dy+\frac{P_2(x)}{1+x}\\&+\lambda P_3(x)\int_0^x\frac{1}{1+y}\frac{1}{2\pi s(x-y)}e^{-\frac{[\ln(x-y)-\mu]^2}{4s^2}}dy.\end{aligned}$$

Under $x\in(0,10)$, an approximate estimate of the ruin probability is given when N^* takes the following four different numbers. Tables 4.10 and 4.11 list the results of $\psi(x;0.5)$ and $\psi(x;5)$ when the parameters take different values. By analyzing the two groups of tables, as the value of b increases, the corresponding ruin probability gradually decreases and then tend to a stable range eventually. Moreover, the values of N^* have little effect on the results of ruin probability. when N^* is small, it will reach the exact value.

Example C3. Comparing Figures 4.11 and 4.12, it can be seen that when $\sigma_1 = 0.2$, the changing trend of curves $\psi(x; 0.5)$ and $\psi(x; 5)$ for $x \in (0, 10)$ first increases and then decreases, and reaches the maximum value near $x = 1.2$. With the increase of σ_1, the variation range of curves decreases, which is affected by perturbed factors. In addition, with the increase of b, it can be obtained from Figure 4.12 that the two curves of $\psi(x; b)$ overlap regardless of the value of investment proportion p.

Table 4.10. The numerical values of $\psi(x; 0.5)$ when $\sigma_1 = 0.8,\ p = 0.1,\ \hat{\alpha} = \frac{\pi}{4},\ \hat{\beta} = \frac{\pi}{4},\ \hat{d} = \frac{\pi}{4}$ in different values of N^*

N^*	$x = 0.15$	$x = 0.5$	$x = 0.8$	$x = 1.5$	$x = 3$	$x = 5$	$x = 7$	$x = 10$
10	0.92760	0.74796	0.68463	0.50469	0.26161	0.15576	0.11168	0.07971
15	0.92480	0.74238	0.67825	0.50376	0.26188	0.15537	0.11223	0.07956
20	0.92321	0.73983	0.67501	0.50340	0.26173	0.15559	0.11214	0.07952
25	0.92240	0.73826	0.67305	0.50324	0.26162	0.15563	0.11206	0.07962

Table 4.11. The numerical values of $\psi(x; 5)$ when $\sigma_1 = 0.8,\ p = 0.1,\ \hat{\alpha} = \frac{\pi}{4},\ \hat{\beta} = \frac{\pi}{4},\ \hat{d} = \frac{\pi}{4}$ in different values of N^*

N^*	$x = 0.15$	$x = 0.5$	$x = 0.8$	$x = 1.5$	$x = 3$	$x = 5$	$x = 7$	$x = 10$
10	0.91561	0.72823	0.66434	0.49039	0.25559	0.15513	0.11201	0.07964
15	0.91373	0.72493	0.65923	0.48957	0.25552	0.15373	0.11241	0.07955
20	0.91280	0.72411	0.65678	0.48937	0.25558	0.15310	0.11188	0.07961
25	0.91249	0.72373	0.65553	0.48933	0.25536	0.15414	0.11221	0.07952

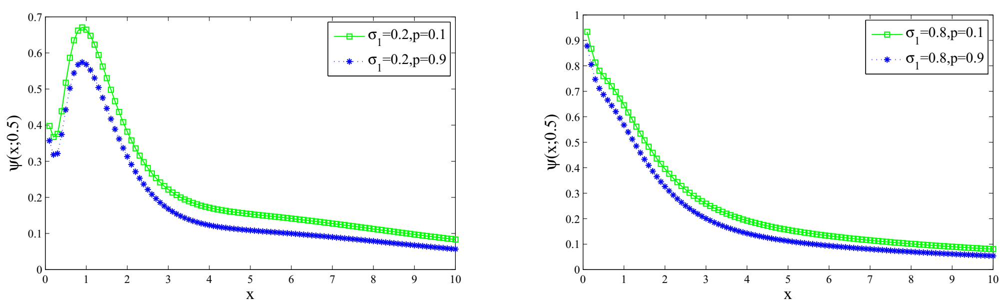

Figure 4.11. Images of ruin probability $\psi(x; 0.5)$ when $N^* = 10,\ \hat{\alpha} = \frac{\pi}{4},\ \hat{\beta} = \frac{\pi}{4},\ \hat{d} = \frac{\pi}{2}$, $\sigma_1 = 0.2$(left) and $\sigma_1 = 0.8$(right).

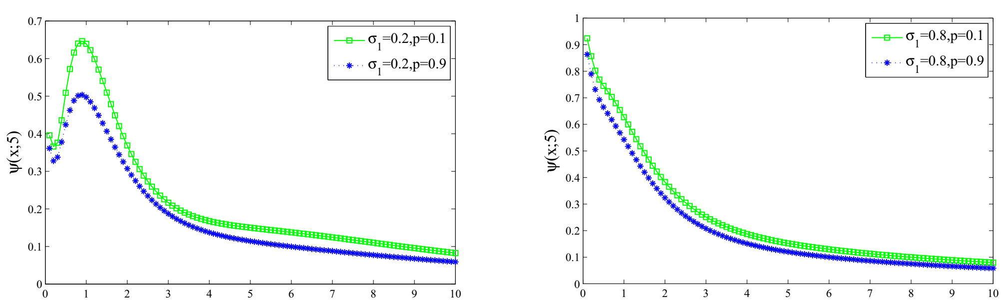

Figure 4.12. Images of ruin probability $\psi(x; 5)$ when $N^* = 10,\ \hat{\alpha} = \frac{\pi}{4},\ \hat{\beta} = \frac{\pi}{4},\ \hat{d} = \frac{\pi}{2}$, $\sigma_1 = 0.2$(left) and $\sigma_1 = 0.8$(right).

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Conflict of Interest

The authors declare no conflict of interest.

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