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# Barrier Option Pricing in Regime Switching Models with Rebates

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**Abstract** This paper is concerned with the valuation of single and double barrier knock-out call options in a Markovian regime switching model with specific rebates. The integral formulas of the rebates are derived via matrix Wiener-Hopf factorizations and Fourier transform techniques, also, the integral representations of the option prices are constructed. Moreover, the first-passage time density functions in two-state regime model are derived. As applications, several numerical algorithms and numerical examples are presented.

**Keywords** option pricing; Markovian regime switching; Wiener-Hopf factorization; Fourier transform
**2020 MR Subject Classification** 60G40; 60J60

## 1 Introduction

Barrier options, one of the most widely traded exotic options in the financial markets, are path-dependent derivatives with a payoff at maturity that depend not only on the terminal price but also on whether or not a specified asset price arrives a certain boundary. Common examples are the knock-in and knock-out call and put options which are popular in corporate finance and widely used by institutional investors because of the cheapness. For example, barrier options can be applied for purpose of hedging, traders may use these options to obtain insurance protection when the price of the underlying asset is above or below a certain level.

It is well-known that the financial system presents periodic changes under the influence of external factors, therefore the option pricing models should consider cyclical behavior in financial markets, and the regime switching models can effectively describe various randomly changing economical factors, since different regimes can be identified by macroeconomic factors like rate of return and price volatility of derivatives. Based on Hamilton's seminal work, the problem of option pricing in regime switching models has been an important topic in economic analysis and financial time series, see references [9, 11, 12, 21] for detailed discussions. Model parameters in regime switching models are assumed to depend on a continuous time and finite-state Markov chain, the states of the chain represent different states of an economy, which is a major advantage compared with other models[25].

On account of prevalence of regime switching models, increasing research efforts have been devoted to barrier option pricing. Chan et al. considered a continuous-time financial model with a money market account and a share, and presented an explicit analytic solution in infinite series form for the price of European-style barrier option in two-state regime[2]. Fall et al. discussed the analytical solution of the Black-Scholes equation and the analytical solution of the generalized Black-Scholes equation both described by the Caputo generalized fractional

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derivative[7]. Masi et al. used a partial differential equation (PDE) approach to price an European call option for a two-regime process in continuous time[19]. Eloe et al. studied a double barrier option when the underlying asset price followed a regime switching exponential mean-reverting process via the combination of analysis of a deterministic boundary value problem with probabilistic approach, and proved the existence of the smooth solution of the boundary value system[6]. Elliott et al. considered the valuation of both European-style and American-style barrier options in a regime switching model. Both the probabilistic and PDE approaches were used to price the barrier options, and a semi-analytical solution was derived[5]. Kudryavtsev et al. suggested two fast and accurate methods, the fast Wiener-Hopf method and the iterative Wiener-Hopf method, for pricing barrier options for a wide class of Lévy processes[17]. Hieber showed that, in the case of two or three-state regimes or in the case of a zero drift term, the matrix Wiener-Hopf factorization could be derived analytically[14], however, the factorization usually had to be computed numerically in the case of more than three-state regimes, one can also refer to [16, 17]. Hieber used two-state matrix Wiener-Hopf factorization to price exotic options, such as digital option, lookback option and down-and-out barrier option[15]. The lattice method and trinomial tree method were also used to price options in regime switching model[10, 26].

For holders of barrier options, the closer the initial price is to the barrier, the greater the likelihood of option failure, so more rebates are needed as compensation to hedge the losses of option failure. Therefore, it is of great importance in introducing rebates in option pricing to reduce the risk. Le et al. presented an innovative decomposition approach to price American up-and-out put options with a time-dependent rebates by using the continuous Fourier sine transform approach[18]. Park et al. derived an analytic formula for the American knock-out option with rebates by using Laplace-Carson transform method[22]. However, the specific forms of the rebates were not provided in [18, 22], in this paper, we define specific rebate functions to achieve high-risk and high rebates. In addition, the calculation of barrier options with rebates is quite complicated, especially there is no analytical solution in many cases. We decompose barrier options into a combination of several European options, and obtain closed-form expressions, specifically, it makes the calculation simpler. Furthermore, motivated by the work of Hieber[15], we construct the integral representations of barrier options pricing in regime switching models with specific rebates in the Black-Scholes framework, and investigate the first-passage time density functions by using the inverse Fourier transform and the matrix Wiener-Hopf factorization techniques. As applications, we focus our attention on two-state regime model in numerical illustration, several comparison to available numerical algorithms and classical Black-Scholes models are achieved.

The remainder of the paper is organized as follows. The dynamic models with Markovian regime switching are described in Section 2. The integral representations of the rebates, the pricing formulas of single-barrier and double-barrier options and the first-passage time density functions in two-state regime are constructed in Section 3. The numerical examples are presented in Section 4. Finally, the conclusion of the present work is given in Section 5.

## 2 The Dynamic Models with Regime Switching

In this section, we aim at the dynamic models with Markovian regime switching Black-Scholes economy. We consider a continuous-time financial model with two primitive securities, for simplicity, write $B$ as a bond and $S$ as a share, these securities can be traded continuously over time in a finite interval $[0, T]$.

In order to describe the evolution of $k$-state ($k \in \mathbb{N}^+$) of the economy with regime switching, let $X = X(t)_{0 \le t \le T}$ be a continuous-time Markov chain on $(\Omega, \mathcal{F}, \mathbb{P})$ with finite $k$-state space $\mathscr{D}$, and assume that $\mathscr{D} = \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_k\}, \mathbf{e}_i \in \mathbb{R}^k,\ i = 1, 2, \cdots, k$, is a canonical state space.

Let $\mathbf{A}_0 = (a_{ij})_{k\times k}$ be generator matrix of the chain $X$, where $a_{ij} \geq 0$ for $i \neq j$, and $\sum_{j=1}^{k} a_{ij} = 0$ for $i = 1, 2, \cdots, k$. The probability of transition of the chain $X$ from state $\mathbf{e}_i$ to state $\mathbf{e}_j$ is $-a_{ij}/a_{ii},\ i \neq j$. According to Ref. [3], it follows that

$$X(t) = X(0) + \int_0^t \mathbf{A}_0 X(s)\, ds + M(t), \tag{2.1}$$

where $M(t)$ is a martingale.

Let $r > 0$ be a constant risk-free interest rate, the price process $\{B(t), t \in [0, T]\}$ of the bond is given by

$$B(t) = e^{rt}, \qquad B(0) = 1.$$

For any $t \in [0, T]$, we define that

$$\mu_t = \boldsymbol{\mu}(X_t) = \langle \boldsymbol{\mu}, X_t \rangle, \qquad \sigma_t = \sigma(X_t) = \langle \boldsymbol{\sigma}, X_t \rangle, \tag{2.2}$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_k)'$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_k)'$ satisfy $\mu_i > r, \sigma_i > 0,\ i = 1, 2, \cdots, k$. $\langle \cdot, \cdot \rangle$ denotes the scalar product, and $\mu_i, \sigma_i$ represent the expected return rate and the volatility of the share $S$ when the economy is in the $i$-th state, respectively.

Suppose that the price $S_t$ of underlying assets at time $t$ under the real-world measure $\mathbb{P}$ follows a Markov-modulated geometric Brownian motion, which satisfies

$$dS_t = \mu_t S_t\, dt + \sigma_t S_t\, dW_t, \tag{2.3}$$

where $W_t$ is the standard Brownian motion under $\mathbb{P}$, and independent of $X_t$. Furthermore, denote by $\mathcal{F}_t = \sigma\{S_s, X_s; s \leq t\}$ the filtration. Let $\mathbb{Q}$ be the risk-neutral measure, then the discounted stock price process $e^{-rt} S_t$ is a martingale under $\mathbb{Q}$. According to Ref. [5], $\mathbb{Q}$ is given by the following Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\Big\{ \int_0^t \frac{r - \mu_s}{\sigma_s}\, dW_s - \frac{1}{2} \int_0^t \Big( \frac{r - \mu_s}{\sigma_s} \Big)^2 ds \Big\}. \tag{2.4}$$

By Girsanov's theorem, it follows that

$$W_t^{\mathbb{Q}} := W_t + \int_0^t \frac{r - \mu_s}{\sigma_s}\, dt, \tag{2.5}$$

where $W_t^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$. Therefore, the stock price model under $\mathbb{Q}$ can be written as

$$dS_t = r S_t\, dt + \sigma_t S_t\, dW_t^{\mathbb{Q}}. \tag{2.6}$$

Let $s_t = \ln S_t, t \in [0, T]$, then the characteristic function (one can refer to [14]) of $s_T$ conditional on $\mathcal{F}_t$ under $\mathbb{Q}$ in a regime switching model (2.6) is given by

$$\psi_{s_T|\mathcal{F}_t}(v) = e^{\mathrm{i} v s_t} \left\langle \exp\left\{ \big(\mathbf{A}_0 + \mathbf{\Phi}(v)\big)(T - t) \right\} X_t, \mathbf{1} \right\rangle, \tag{2.7}$$

where $\mathbf{\Phi}(v) = \mathbf{diag}\left\{ \mathrm{i}vr - \frac{1}{2} v^2 \sigma_1^2, \mathrm{i}vr - \frac{1}{2} v^2 \sigma_2^2, \cdots, \mathrm{i}vr - \frac{1}{2} v^2 \sigma_k^2 \right\},\ \mathbf{1} = (1, 1, \cdots, 1)'$ is a $k$-dimension column vector of ones.

## 3 Barrier Option Pricing with Specific Rebates

Barrier option pricing was first investigated by Merton[20] and it has been an important topic in economic market. In this section, we will investigate single-barrier and double-barrier knock-out call options with a strike price $K$, maturity $T$, upper barrier $U$ and lower barrier $L$, which satisfy $L < S_0 < K < U$. Let $\tau_U^t = \inf\{s \geq t | S_s \geq U\}$ and $\tau_L^t = \inf\{s \geq t | S_s \leq L\}$ be the first hitting time for upper barrier $U$ and lower barrier $L$, respectively. Denote $\tau_{LU}^t = \tau_U^t \wedge \tau_L^t = \inf\{s \geq t | S_s \notin (L, U)\}$, and $\mathbb{I}_{\{\cdot\}}$ is indicator function. The payoff of such single-barrier option is given by

$$g(S_T) = \begin{cases} (S_T - K)^+, \quad S_t > L \text{ (or } S_t < U), \qquad t \in [0, T], \\ R(t), \qquad \text{otherwise}, \end{cases}$$

and the payoff of such double-barrier option is given by

$$g(S_T) = \begin{cases} (S_T - K)^+, \qquad L < S_t < U, \quad t \in [0, T], \\ R(t), \text{otherwise}, \end{cases}$$

where the $R(t)$ is a time-varying rebates payment when the options are worthless. Then we derive the price of single-barrier up-and-out call option with rebate as follows:

$$\begin{aligned} & B_{\text{SUC}}(t, S_t, X_t; T, K) \\ = & \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S_T - K)^+ \mathbb{I}_{\{\tau_U^t > T\}} + e^{-r(\tau_U^t - t)} R(T-t) \mathbb{I}_{\{\tau_U^t \leq T\}} \big| \mathcal{F}_t\big] \\ = & \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S_T - K)^+ \mathbb{I}_{\{\tau_U^t > T\}} \big| \mathcal{F}_t\big] + \mathbb{E}^{\mathbb{Q}}\big[e^{-r(\tau_U^t - t)} R(T-t) \mathbb{I}_{\{\tau_U^t \leq T\}} \big| \mathcal{F}_t\big] \\ =: & B_{\text{SUC},1} + B_{\text{SUC},2}. \end{aligned} \tag{3.1}$$

Similarly, the price of single-barrier down-and-out call option with rebate is given by

$$\begin{aligned} & B_{\text{SDC}}(t, S_t, X_t; T, K) \\ = & \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S_T - K)^+ \mathbb{I}_{\{\tau_L^t > T\}} + e^{-r(\tau_L^t - t)} R(T-t) \mathbb{I}_{\{\tau_L^t \leq T\}} \big| \mathcal{F}_t\big] \\ = & \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S_T - K)^+ \mathbb{I}_{\{\tau_L^t > T\}} \big| \mathcal{F}_t\big] + \mathbb{E}^{\mathbb{Q}}\big[e^{-r(\tau_L^t - t)} R(T-t) \mathbb{I}_{\{\tau_L^t \leq T\}} \big| \mathcal{F}_t\big] \\ =: & B_{\text{SDC},1} + B_{\text{SDC},2}. \end{aligned} \tag{3.2}$$

Finally, the price of double-barrier knock-out call option with rebate is as follows:

$$\begin{aligned} & B_{\text{DKC}}(t, S_t, X_t; T, K) \\ = & \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S_T - K)^+ \mathbb{I}_{\{\tau_{LU}^t > T\}} + e^{-r(\tau_{LU}^t - t)} R(T-t) \mathbb{I}_{\{\tau_{LU}^t \leq T\}} \big| \mathcal{F}_t\big] \\ = & \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S_T - K)^+ \mathbb{I}_{\{\tau_{LU}^t > T\}} \big| \mathcal{F}_t\big] + \mathbb{E}^{\mathbb{Q}}\big[e^{-r(\tau_{LU}^t - t)} R(T-t) \mathbb{I}_{\{\tau_{LU}^t \leq T\}} \big| \mathcal{F}_t\big] \\ =: & B_{\text{DKC},1} + B_{\text{DKC},2}, \end{aligned} \tag{3.3}$$

where $B_{\text{SUC},2}, B_{\text{SDC},2}, B_{\text{DKC},2}$ are the corresponding rebates, and the $R(x)$ is defined by

$$R(x) = r_{sp} S_t (e^{rx} - 1). \tag{3.4}$$

It is easy to show that $R(x)$ is a monotonically increasing function of $x$ and satisfies $R(0) = 0$. According to [18, 22], there are two main reasons to define the rebate as (3.4): Firstly, in finance practice, the purpose of providing rebates is to partly compensate for the loss of the option in the event that the knock-out feature is activated before expiration, but not at expiration. The earlier the knock-out feature is activated, the more loss the holder suffers, thereby the more amount of rebates should be paid to the holder. Secondly, for the coefficient $r_{sp}$, we assume that there are $k$ kinds of independent risky assets whose required rates of return are $\mu_i$, risks are $\sigma_i$, $i = 1, 2, \cdots, k$, respectively. Then one can construct a portfolio via the $k$ risk assets. Denote by $\mu_p, \sigma_p$ the required rate of return and risk of the portfolio, the Sharpe ratio of the portfolio defined in [24] is as follows:

$$r_{sp} = \frac{\mu_p - r}{\sigma_p}, \tag{3.5}$$

where $\mu_p = \sum_{j=1}^{k} \omega_j \mu_j, \sigma_p^2 = \sum_{j=1}^{k} \omega_j^2 \sigma_j^2$, and $\omega_j \in [0, 1]$ is the proportion of the $j$-th risk asset in the portfolio. Then we can obtain an alternative optimal $r_{sp}$ via minimizing the portfolio's risk $\sigma_p$ in (3.5). It is significant to use the Sharp ratio, an important indicator of excess returns of portfolio, to measure the excess returns relative to bond with initial value $S_t$ and maturity $T - t$ in a switching economy.

**Remark 3.1.** One can also construct a portfolio by $k$ kinds of correlative risky assets with the required rates of return $\mu_i$ and risks $\sigma_i$, $i = 1, 2, \cdots, k$. Then $\mu_p = \sum_{j=1}^{k} \omega_j \mu_j$ and

$$\sigma_p^2 = \sum_{j=1}^{k} \omega_j^2 \sigma_j^2 + 2 \sum_{1 \le i < j \le k} \omega_i \omega_j \mathrm{Cov}(S^i, S^j),$$

where $Cov(S^i, S^j)$ denotes the covariance of $i$-th and $j$-th risky assets.

In what follows, we will price single-barrier and double-barrier options under model (2.6). Before pricing call options, let us recall some serviceable results, such as the Fourier transform of (2.6), and the first-passage time densities with respect to the matrix Wiener-Hopf factorization, the following lemmas and proposition come from references [1, 14, 15], we omit the proofs here.

**Lemma 3.2.** *Consider the regime switching model defined in* (2.6)*. Under* $\mathbb{Q}$*, the price of an European call option at time* $t$ *with strike* $K$ *and maturity* $T$ *is written as*

$$C(t, S_t, X_t; T, K) = \frac{e^{-r(T-t)}}{\pi} \int_0^\infty e^{-(\alpha + \mathrm{i}v)\ln K} \Big[ \frac{\psi_{s_T|\mathcal{F}_t}(v - \mathrm{i}(1+\alpha))}{\alpha^2 + \alpha - v^2 - \mathrm{i}(1+2\alpha)v} \Big] dv, \tag{3.6}$$

*where* $\alpha \in [1, 2]$ *is an arbitrary constant and* $\psi_{s_T|\mathcal{F}_t}(\cdot)$ *is the conditional characteristic function of* $s_T$ *given by* (2.7)*.*

**Lemma 3.3.** *Consider the regime switching model defined in* (2.6) *with initial distribution on the states* $X_0 \in \mathbb{R}^k$*. For two constant barriers* $L = e^l < S_0 = e^{s_0} < U = e^u$*, it holds that, for* $\tau \in (0, \infty)$*, one-sided first-passage time densities can be expressed respectively by*

$$f_L(\tau, X_0) = \frac{1}{\pi} \int_0^\infty e^{-\mathrm{i}v\tau} \big[ X_0' \exp\{\mathbf{A}_-(s_0 - l)\}\mathbf{1} \big] \, dv, \tag{3.7}$$

*and*

$$f_U(\tau, X_0) = \frac{1}{\pi} \int_0^\infty e^{-\mathrm{i}v\tau} \big[ X_0' \exp\{\mathbf{A}_+(u - s_0)\}\mathbf{1} \big] \, dv. \tag{3.8}$$

*For* $v > 0$*, the matrices* $\mathbf{A}_\pm$ *are the matrix Wiener-Hopf factorizations of* $(S, X)$ *defined via* $\Xi(-\mathbf{A}_+) = \Xi(\mathbf{A}_-) = \mathbf{0}$*, where*

$$\Xi(\mathbf{A}) = \frac{1}{2}\mathbf{\Sigma}^2\mathbf{A}^2 + \mathbf{V}\mathbf{A} + \mathbf{A}_0 - v\mathbf{E}_k, \tag{3.9}$$

$\mathbf{\Sigma} = \mathbf{diag}\{\sigma_1, \sigma_2, \cdots, \sigma_k\}$, $\mathbf{V} = \mathbf{diag}\left\{r - \frac{1}{2}\sigma_1^2, r - \frac{1}{2}\sigma_2^2, \cdots, r - \frac{1}{2}\sigma_k^2\right\}$, $\mathbf{E}_k$ *is identity matrix.*

**Lemma 3.4.** *Consider the regime switching model defined in (2.6) with initial distribution on the states $X_0 \in \mathbb{R}^k$. For two constant barriers $L = e^l < U = e^u$, it holds that, for $\tau \in (0,\infty)$, two-sided first-passage time density can be expressed by*

$$f_{LU}(\tau, X_0) = \frac{1}{\pi}\int_0^\infty e^{-\mathrm{i}v\tau}\big[X_0'\big(\Psi^+(s_0)+\Psi^-(s_0)\big)\mathbf{1}\big]\,dv, \tag{3.10}$$

*where*

$$\Psi^+(x) = \big(\exp\{\mathbf{A}_+(U-x)\} - \exp\{\mathbf{A}_-(x-l)\}\mathbf{Y}_+\big)(\mathbf{E}_k - \mathbf{Y}_-\mathbf{Y}_+)^{-1}, \tag{3.11}$$

$$\Psi^-(x) = \big(\exp\{\mathbf{A}_-(x-L)\} - \exp\{\mathbf{A}_+(u-x)\}\mathbf{Y}_-\big)(\mathbf{E}_k - \mathbf{Y}_+\mathbf{Y}_-)^{-1}, \tag{3.12}$$

*and* $\mathbf{Y}_\pm = \exp\{\mathbf{A}_\pm(u-l)\}$.

**Proposition 3.5.** *Consider the regime switching model defined in (2.6). For a constant barriers $L = e^l < S_0$. Under $\mathbb{Q}$, the price of a digital option with payoff $\mathbb{I}_{\{\tau_L^t\le T\}}$ and maturity $T$, at time $t$ are defined by*

$$D(t,T,X_t) = \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}\mathbb{I}_{\{\tau_L^t\le T\}}\big|\mathcal{F}_t\big],$$

*then*

$$D(t,T,X_t) = \frac{e^{-r(T-t)}}{\pi}\int_t^T\int_0^\infty e^{-\mathrm{i}v\tau}\big[X_t'\exp\{\mathbf{A}_-(s_t-l)\}\mathbf{1}\big]\,dv\,d\tau. \tag{3.13}$$

In the sequel, we give the integral formulas of single-barrier up/down-and-out and double-barrier knock-out option prices in a regime switching model with specific rebates.

**Theorem 3.6.** *Consider the regime switching model defined in (2.6). Under $\mathbb{Q}$, the price of a up-and-out barrier option at time $t$ with upper barrier $U = e^u$, strike $K$, and maturity $T$, is given by*

$$\begin{aligned}
&B_{\mathrm{SUC}}(t,S_t,X_t;T,K)\\
=&C(t,S_t,X_t;T,K)\\
&-\frac{1}{\pi}\sum_{j=1}^k\int_t^T\int_0^\infty e^{-r(\tau-t)-\mathrm{i}v\tau}C(0,U,\mathbf{e}_j;T-\tau,K)\big[X_t'\exp\{\mathbf{A}_+(u-s_t)\}\mathbf{e}_j\big]\,dv\,d\tau\\
&+\frac{1}{\pi}r_{sp}S_t(e^{r(T-t)}-1)\int_t^T\int_0^\infty e^{-r(\tau-t)-\mathrm{i}v\tau}\big[X_t'\exp\{\mathbf{A}_+(u-s_t)\}\mathbf{1}\big]\,dv\,d\tau,
\end{aligned}\tag{3.14}$$

*where $C(t,S_t,X_t;T,K)$ is the price at time $t$ of a standard European call option with strike $K$, maturity $T$, $\mathbf{e}_j$ is the $j$-th unit vector, and $\mathbf{A}_+$ is the matrix Wiener-Hopf factorization defined in Lemma 3.3.*

*Proof.* According to [15], we denote the first-passage time densities conditional on $X_{\tau_U^t} = \mathbf{e}_j$ by $f_U(\tau, X_t, \mathbf{e}_j)$, where $\mathbf{e}_j$ is the $j$-th unit vector. Firstly, we have for $B_{\mathrm{SUC},1}$,

$$B_{\mathrm{SUC},1} = \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S_T-K)^+\big|\mathcal{F}_t\big] - \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S_T-K)^+\mathbb{I}_{\{\tau_U^t\le T\}}\big|\mathcal{F}_t\big] =: \mathbf{I}_1 - \mathbf{I}_2.$$

Note that $\mathbf{I}_1 = C(t,S_t,X_t;T,K)$. For $\mathbf{I}_2$, we can obtain

$$\begin{aligned}
\mathbf{I}_2 =&\mathbb{E}^{\mathbb{Q}}\Big[\sum_{j=1}^k\big[\mathbb{E}^{\mathbb{Q}}\big(e^{-r(T-t)}(S_T-K)^+\big|S_{\tau_U^t}=U, X_{\tau_U^t}=\mathbf{e}_j\big)\big]\mathbb{I}_{\{X_{\tau_U^t}=\mathbf{e}_j\}}\Big]\\
=&\sum_{j=1}^k\mathbb{E}^{\mathbb{Q}}\big[e^{-r(\tau_U^t-t)}C(0,U,\mathbf{e}_j;T-\tau_U^t,K)\mathbb{I}_{\{\tau_U^t\le T\}}\big]
\end{aligned}$$

$$=\sum_{j=1}^{k}\int_t^T e^{-r(\tau-t)}C(0,U,\mathbf{e}_j;T-\tau,K)f_U(\tau,X_t,\mathbf{e}_j)\,d\tau. \tag{3.15}$$

Then, as for $B_{\mathrm{SUC},2}$, observe that

$$\begin{aligned}B_{\mathrm{SUC},2} =&\mathbb{E}^{\mathbb{Q}}\big[r_{sp}S_t e^{-r(\tau_U^t-t)}(e^{r(T-t)}-1)\mathbb{I}_{\{\tau_U^t\le T\}}\big|\mathcal{F}_t\big]\\ =&r_{sp}S_t\big(e^{r(T-t)}-1\big)\int_t^T e^{-r(\tau-t)}f_U(\tau,X_t)\,d\tau.\end{aligned} \tag{3.16}$$

According to (3.7), it can be concluded that

$$B_{\mathrm{SUC},2}=\frac{1}{\pi}r_{sp}S_t(e^{r(T-t)}-1)\int_t^T\int_0^\infty e^{-r(\tau-t)-\mathrm{i}v\tau}\big[X_t'\exp\{\mathbf{A}_+(u-s_t)\}\mathbf{1}\big]\,dv\,d\tau. \tag{3.17}$$

Combining (3.15), (3.17) and Lemma 3.3, we obtain (3.14). □

**Theorem 3.7.** *Consider the regime switching model defined in (2.6). Under $\mathbb{Q}$, the price of a down-and-out barrier option at time $t$ with lower barrier $L=e^l$, strike $K$, and maturity $T$, is given by*

$$\begin{aligned}&B_{\mathrm{SDC}}(t,S_t,X_t;T,K)\\ =&C(t,S_t,X_t;T,K)\\ &-\frac{1}{\pi}\sum_{j=1}^{k}\int_t^T\int_0^\infty e^{-r(\tau-t)-\mathrm{i}v\tau}C(0,L,\mathbf{e}_j;T-\tau,K)\big[X_t'\exp\{\mathbf{A}_-(s_t-l)\}\mathbf{e}_j\big]\,dv\,d\tau\\ &+\frac{1}{\pi}r_{sp}S_t(e^{r(T-t)}-1)\int_t^T\int_0^\infty e^{-r(\tau-t)-\mathrm{i}v\tau}\big[X_t'\exp\{\mathbf{A}_-(s_t-l)\}\mathbf{1}\big]\,dv\,d\tau,\end{aligned} \tag{3.18}$$

*where $C(t,S_t,X_t;T,K)$ is the price at time $t$ of a standard European call option with strike $K$, maturity $T$, $\mathbf{e}_j$ is the $j$-th unit vector, and $\mathbf{A}_-$ is the matrix Wiener-Hopf factorization defined in Lemma 3.3.*

*Proof.* The proof is analogous to that of Theorem 3.6, observe that

$$\begin{aligned}B_{\mathrm{SDC},1} =&\mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S_T-K)^+\big|\mathcal{F}_t\big]-\mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S_T-K)^+\mathbb{I}_{\{\tau_L^t\le T\}}\big|\mathcal{F}_t\big]\\ =&C(t,S_t,X_t;T,K)-\mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S_T-K)^+\mathbb{I}_{\{\tau_L^t\le T\}}\big|\mathcal{F}_t\big]=:\mathbf{I_1}-\mathbf{I_3}.\end{aligned}$$

For $\mathbf{I}_3$, it follows that

$$\begin{aligned}\mathbf{I_3} =&\mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S_T-K)^+\mathbb{I}_{\{\tau_L^t\le T\}}\big|\mathcal{F}_t\big]\\ =&\mathbb{E}^{\mathbb{Q}}\Big[\sum_{j=1}^{k}\big[\mathbb{E}^{\mathbb{Q}}\big(e^{-r(T-t)}(S_T-K)^+\big|S_{\tau_L^t}=L,X_{\tau_L^t}=\mathbf{e}_j\big)\big]\mathbb{I}_{\{X_{\tau_L^t}=\mathbf{e}_j\}}\Big]\\ =&\sum_{j=1}^{k}\mathbb{E}^{\mathbb{Q}}\big[e^{-r(\tau_L^t-t)}C(0,L,\mathbf{e}_j;T-\tau_L^t,K)\mathbb{I}_{\{\tau_L^t\le T\}}\big]\\ =&\sum_{j=1}^{k}\int_t^T e^{-r(\tau-t)}C(0,L,\mathbf{e}_j;T-\tau,K)f_L(\tau,X_t,\mathbf{e}_j)\,d\tau.\end{aligned} \tag{3.19}$$

As for $B_{\mathrm{SDC},2}$, on account of (3.8), it can be concluded that

$$B_{\mathrm{SDC},2} =\mathbb{E}^{\mathbb{Q}}\big[r_{sp}S_t e^{-r(\tau_L^t-t)}(e^{r(T-t)}-1)\mathbb{I}_{\{\tau_L^t\le T\}}\big|\mathcal{F}_t\big]$$

$$
\begin{aligned}
&= r_{sp} S_t (e^{r(T-t)} - 1) \int_t^T e^{-r(\tau - t)} f_L(\tau, X_t)\, d\tau \\
&= \frac{1}{\pi} r_{sp} S_t (e^{r(T-t)} - 1) \int_t^T \int_0^\infty e^{-r(\tau-t) - iv\tau} \big[X_t' \exp\{\mathbf{A}_-(s_t - l)\}\mathbf{1}\big]\, dv\, d\tau. \qquad (3.20)
\end{aligned}
$$

Combining (3.19), (3.20) and Lemma 3.3, the proof of (3.18) is completed. □

**Theorem 3.8.** *Consider the regime switching model defined in (2.6). Under $\mathbb{Q}$, the price of a knock-out double-barriers option at time $t$ with upper barrier $U = e^u$ and lower barrier $L = e^l$, strike $K$, and maturity $T$, is given by*

$$
\begin{aligned}
&B_{\mathrm{DKC}}(t, S_t, X_t; T, K) \\
=& C(t, S_t, X_t; T, K) \\
&- \frac{1}{\pi} \sum_{j=1}^k \int_t^T \int_0^\infty e^{-r(\tau-t) - iv\tau} C(0, L, \mathbf{e}_j; T - \tau, K) \big[X_t' \big(\Psi^+(s_t) + \Psi^-(s_t)\big)\mathbf{e}_j\big]\, dv\, d\tau \\
&- \frac{1}{\pi} \sum_{j=1}^k \int_t^T \int_0^\infty e^{-r(\tau-t) - iv\tau} C(0, U, \mathbf{e}_j; T - \tau, K) \big[X_t' \big(\Psi^+(s_t) + \Psi^-(s_t)\big)\mathbf{e}_j\big]\, dv\, d\tau \\
&+ \frac{1}{\pi} r_{sp} S_t \big(e^{r(T-t)} - 1\big) \int_t^T \int_0^\infty e^{-r(\tau-t) - iv\tau} \big[X_t' \big(\Psi^+(s_t) + \Psi^-(s_t)\big)\mathbf{1}\big]\, dv\, d\tau, \qquad (3.21)
\end{aligned}
$$

*where $C(t, S_t, X_t; T, K)$ is the price at time $t$ of a standard European call option with strike $K$, maturity $T$, $\mathbf{e}_j$ is the $j$-th unit vector, and $\Psi^+$, $\Psi^-$ are defined in Lemma 3.4.*

*Proof.* We first consider $B_{\mathrm{DKC},1}$, along a similar procedure as that in (3.15), (3.19), it turns out that

$$
\begin{aligned}
B_{\mathrm{DKC},1} =& \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S_T - K)^+ \big| \mathcal{F}_t\big] - \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S_T - K)^+ \mathbb{I}_{\{\tau_{LU}^t \le T\}} \big| \mathcal{F}_t\big] \\
=& C(t, S_t, X_t; T, K) - \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S_T - K)^+ \mathbb{I}_{\{\tau_{LU}^t \le T\}} \big| \mathcal{F}_t\big] =: \mathbf{I}_1 - \mathbf{I}_4.
\end{aligned}
$$

As for $\mathbf{I}_4$, we have

$$
\begin{aligned}
\mathbf{I}_4 =& \mathbb{E}^{\mathbb{Q}}\Big[\sum_{j=1}^k \big[\mathbb{E}^{\mathbb{Q}}\big(e^{-r(T-t)}(S_T - K)^+ \big| S_{\tau_{LU}^t} = U, X_{\tau_{LU}^t} = \mathbf{e}_j\big)\big] \mathbb{I}_{\{X_{\tau_{LU}^t} = \mathbf{e}_j\}}\Big] \\
&+ \mathbb{E}^{\mathbb{Q}}\Big[\sum_{j=1}^k \big[\mathbb{E}^{\mathbb{Q}}\big(e^{-r(T-t)}(S_T - K)^+ \big| S_{\tau_{LU}^t} = L, X_{\tau_{LU}^t} = \mathbf{e}_j\big)\big] \mathbb{I}_{\{X_{\tau_{LU}^t} = \mathbf{e}_j\}}\Big] =: \Pi_1 + \Pi_2.
\end{aligned}
$$

For $\Pi_1$, it turns out that

$$
\begin{aligned}
\Pi_1 =& \sum_{j=1}^k \mathbb{E}^{\mathbb{Q}}\big[e^{-r(\tau_{LU}^t - t)} C(0, U, \mathbf{e}_j; T - \tau_{LU}^t, K) \mathbb{I}_{\{\tau_{LU}^t \le T\}}\big] \\
=& \sum_{j=1}^k \int_t^T e^{-r(\tau - t)} C(0, U, \mathbf{e}_j; T - \tau, K) f_{LU}(\tau, X_t, \mathbf{e}_j)\, d\tau. \qquad (3.22)
\end{aligned}
$$

Similarly, we have for $\Pi_2$,

$$
\Pi_2 = \sum_{j=1}^k \mathbb{E}^{\mathbb{Q}}\big[e^{-r(\tau_{LU}^t - t)} C(0, L, \mathbf{e}_j; T - \tau_{LU}^t, K) \mathbb{I}_{\{\tau_{LU}^t \le T\}}\big]
$$

$$=\sum_{j=1}^{k}\int_t^T e^{-r(\tau-t)}C(0,L,\mathbf{e}_j;T-\tau,K)f_{LU}(\tau,X_t,\mathbf{e}_j)\,d\tau. \tag{3.23}$$

Therefore, we obtain the integral formulations of $B_{\mathrm{DKC},1}$.

Next, we turn to $B_{\mathrm{DKC},2}$, it follows that

$$\begin{aligned}B_{\mathrm{DKC},2} =&\mathbb{E}^{\mathbb{Q}}\big[r_{sp}S_t e^{-r(\tau^t_{LU}-t)}(e^{r(T-t)}-1)\mathbb{I}_{\{\tau^t_{LU}\le T\}}\big|\mathcal{F}_t\big]\\ =&r_{sp}S_t(e^{r(T-t)}-1)\int_t^T e^{-r(\tau-t)}f_{LU}(\tau,X_t)\,d\tau\\ =&\frac{1}{\pi}r_{sp}S_t(e^{r(T-t)}-1)\int_t^T\int_0^\infty e^{-r(\tau-t)-\mathrm{i}v\tau}\big[X_t'(\Psi^+(s_t)+\Psi^-(s_t))\mathbf{1}\big]\,dv\,d\tau. \end{aligned}\tag{3.24}$$

Combining (3.22)–(3.24) and Lemma 3.3, we complete the proof of (3.21). □

Therefore, we accomplish the integral representations of barrier options price in regime switching models with specific rebates. According to the theoretical results as shown above, it is obvious that the matrix Wiener-Hopf factorizations play an important role in the pricing formulas. In the following, we will give an example to illustrate how to derive the first-passage time density functions via matrix Wiener-Hopf factorization and Fourier transform in two-state regime. In this case, the solutions of the matrix Wiener-Hopf factorizations depend on the roots of the following equation:

$$\Big[\frac{1}{2}\sigma_1^2\beta^2+\mu_1\beta+a_{11}-v\Big]\Big[\frac{1}{2}\sigma_2^2\beta^2+\mu_2\beta+a_{22}-v\Big]-a_{11}a_{22}=0. \tag{3.25}$$

Equation (3.25) has four real roots, which satisfy $-\infty<\beta_{1,v}<\beta_{2,v}<0<\beta_{3,v}<\beta_{4,v}<\infty$, one can refer to [8, 15] for more details. Let $X_0=(1,0)'$, then applying Fourier transform and Lemma 3.3, we can obtain

$$\begin{aligned}f_L(\tau,\mathbf{d}_0)=&\frac{1}{\pi}\int_0^\infty\frac{e^{-\mathrm{i}v\tau}}{\beta_{1,v}-\beta_{2,v}}\Big[\frac{\beta_{1,v}-\beta_{1,v}\beta_{2,v}-\frac{2v}{\sigma_1^2}}{\beta_{1,v}+\beta_{2,v}+\frac{2\mu_1}{\sigma_1^2}}e^{\beta_{2,v}(s_0-l)}\\ &-\frac{\beta_{2,v}-\beta_{1,v}\beta_{2,v}-\frac{2v}{\sigma_1^2}}{\beta_{1,v}+\beta_{2,v}+\frac{2\mu_1}{\sigma_1^2}}e^{\beta_{1,v}(s_0-l)}\Big]dv,\end{aligned}\tag{3.26}$$

and

$$\begin{aligned}f_U(\tau,\mathbf{d}_0)=&\frac{1}{\pi}\int_0^\infty\frac{e^{-\mathrm{i}v\tau}}{\beta_{3,v}-\beta_{4,v}}\Big[\frac{\beta_{3,v}-\beta_{3,v}\beta_{4,v}-\frac{2v}{\sigma_1^2}}{\beta_{3,v}+\beta_{4,v}+\frac{2\mu_1}{\sigma_1^2}}e^{-\beta_{4,v}(u-s_0)}\\ &-\frac{\beta_{4,v}-\beta_{3,v}\beta_{4,v}-\frac{2v}{\sigma_1^2}}{\beta_{3,v}+\beta_{4,v}+\frac{2\mu_1}{\sigma_1^2}}e^{-\beta_{3,v}(u-s_0)}\Big]dv.\end{aligned}\tag{3.27}$$

For $f_{LU}(\tau,\mathbf{d}_0)$, its closed-form expression is extremely complicated, therefore, it is omitted here. Nevertheless, for given specific parameters, it is easy for numerical calculation. In the case of $k(\ge 3)$ regimes, the matrix Wiener-Hopf factorizations could be derived via eigenvalue algorithm, one can refer to [23] for details.

## 4 Numerical Analysis

In this section, we provide some results of numerical simulation of the integral representations for single/double-barrier call option pricing. We compare our option prices and CPU time-consuming in Markovian regime switching economy with rebates (MRR) to Black-Scholes approximation (see, e.g., [4]), and to Brownian bridge algorithm (see, e.g., [13]). Also, we present numerical results for single and two-state regimes in a standard (constant-volatility) Black-Scholes model.

We first present two available numerical algorithms applied in literature as follows.

**(1) Black-Scholes approximation (BSA).** Denote by $\tau_i(t,T)$ the occupation time of the chain $X$ in state $\mathbf{e}_i$ in the time interval $[t,T]$, where $\tau_i(t,T)=\int_t^T\langle X_s,\mathbf{e}_i\rangle\,ds$. Let $\mathcal{T}(t,T):=(\tau_1(t,T),\tau_2(t,T),\cdots,\tau_n(t,T))'$, then the characteristic function $\phi_t(\boldsymbol{u},X_t):=\mathbb{E}^{\mathbb{Q}}\left[e^{\mathrm{i}\langle\mathcal{T}(t,T),\boldsymbol{u}\rangle}\middle|\mathcal{F}_t\right]$ of $\mathcal{T}(t,T)$ is $\phi_t(\boldsymbol{u},X_t)=\left\langle\exp\left\{[\mathbf{A}_0+\mathrm{i}\mathbf{diag}(\boldsymbol{u})](T-t)\right\}X_t,\mathbf{1}\right\rangle$, where $\boldsymbol{u}=(u_1,u_2,\cdots,u_k)'\in\mathbb{R}^k$. The average variance $\sigma_{rw}^2=\mathbb{E}^{\mathbb{Q}}\big[\sum_{j=1}^{k}\tau_j(t,T)\sigma_j^2|\mathcal{F}_t\big]$. A natural approximation of the barrier option price is setting the volatility in the Black-Scholes price formula to $\sigma_{rw}$.

**(2) Brownian bridge algorithm (BBA).** An efficient and unbiased algorithm, which combines Monte-Carlo simulation and analytical derivation. The algorithm requires to simulate the price process at state changes of Markov chain.

Next, we put $L=80, U=160, K=120, r=0.05, T=1,\ X_0=(1,0)'$, and

$$\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix}=\begin{bmatrix}0.06\\ 0.08\end{bmatrix},\qquad \begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix}=\begin{bmatrix}0.20\\ 0.25\end{bmatrix},\qquad \mathbf{A}_0=\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}.$$

By minimizing the portfolio's risk $\sigma_p$ in (3.5), we get $\omega_1=0.6098,\ \omega_2=0.3902$, and we further obtain $\mu_p=0.0678,\ \sigma_p=0.1562,\ r_{sp}=0.1140$. Setting $\alpha=1.5$ in Lemma 3.2. Then, the numerical results are presented in Tables 1–6 and Figure 1, respectively.

First, Tables 1–3 compare the option pricing and average CPU times of MRR, BSA and BBA models. On the one hand, the results show that all three methods can obtain more accurate pricing when initial price $S_0$ is away from the barrier value. However, BSA model leads to significant deviation when $S_0$ approaches the barrier value. On the other hand, BSA model takes the least time in computation, while BBA model takes the most time. Subsequently, MRR method has advantages in both accuracy and computation time.

**Table 1.** The value of $B_{\mathrm{SUC},1}(0,S_0,X_0;T,K)$

| | Option prices | | | | | CPU time(s) |
|---|---|---|---|---|---|---|
| | $S_0=85$ | $S_0=90$ | $S_0=100$ | $S_0=110$ | $S_0=115$ | |
| MRR | 0.7942 | 1.2792 | 2.5869 | 4.1671 | 4.8778 | 4.11 |
| BSA | 0.7743 | 1.2408 | 2.4989 | 3.8504 | 4.3851 | 0.05 |
| BBA | 0.7933 | 1.2776 | 2.5835 | 4.1612 | 4.8702 | 14.94 |

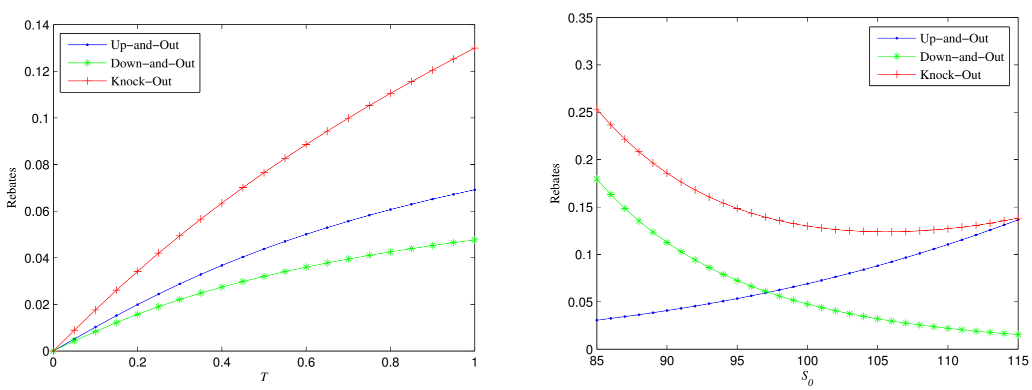
**Table 2.** The value of $B_{\mathrm{SDC},1}(0,S_0,X_0;T,K)$

| | Option prices | | | | | CPU time(s) |
|---|---|---|---|---|---|---|
| | $S_0=85$ | $S_0=90$ | $S_0=100$ | $S_0=110$ | $S_0=115$ | |
| MRR | 0.8347 | 1.7384 | 4.3424 | 8.4124 | 10.9796 | 6.34 |
| BSA | 0.7512 | 1.6115 | 4.1470 | 8.1600 | 10.7600 | 0.05 |
| BBA | 0.8335 | 1.7362 | 4.3359 | 8.3998 | 10.9664 | 16.15 |

**Table 3.** The value of $B_{\mathrm{DKC},1}(0, S_0, X_0; T, K)$

| | Option prices | | | | | CPU time(s) |
|---|---|---|---|---|---|---|
| | $S_0 = 85$ | $S_0 = 90$ | $S_0 = 100$ | $S_0 = 110$ | $S_0 = 115$ | |
| MRR | 0.5751 | 1.1863 | 2.5630 | 3.9526 | 4.4816 | 10.12 |
| BSA | 0.5187 | 1.1212 | 2.4759 | 3.8498 | 4.3875 | 0.07 |
| BBA | 0.5738 | 1.1848 | 2.5621 | 3.9481 | 4.4769 | 30.81 |

Next, Figure 1 compares the rebates related to different initial price $S_0$ and different maturity $T$. Figure 1(a) shows that the rebates for single and double barrier options increase as $T$ increases. However, Figure 1(b) shows the price of the rebate for up-and-out options increases with the increase of $S_0$, the price of the rebate for down-and-out options decreases with the increase of $S_0$, and decreases first and then increases with the increase of $S_0$ for knock-out options, that is, two segments are high, and the middle is low with increase of $S_0$. This can be understood as the closer the initial price is to the barrier value, the greater the likelihood of option expiration, and the holder will also face greater risk. This requires more rebates as compensation to hedge the loss of option expiration, which has important theoretical and practical significance for option pricing in the real financial market.




**(a)** Rebates with different $T$ ($S_0 = 100$) **(b)** Rebates with different $S_0$ ($T = 1$)

**Figure 1.** The dynamic trend of rebates

Finally, Tables 4–6 present the barrier option pricing for single and two-state regime models. For up-and-out option, Table 4 shows there are significant deviations in the pricing of Black-Scholes models with regime 1 and regime 2 when the initial price approaches and moves away from the barrier value, respectively. However, the pricing of the regime switching model with two-state is more robust and accurate. For down-and-out option, Table 5 shows the pricing of the regime switching model with two state and single state with regime 2 is relatively accurate. When the initial price approaches the lower barrier value, the pricing deviation of the two models is small, and as the initial price increases, the deviation gradually increases. For knock-out double-barriers option, Table 6 shows the pricing of the regime switching model with two-state is accurate.

**Table 4.** The value of $B_{\mathrm{SDC},1}(0, S_0, X_0; T, K)$

| | Regime switching model | Black-Scholes model | |
|---|---|---|---|
| | | Regime 1 | Regime 2 |
| $S_0 = 85$ | 0.8347 | 0.4949 | 0.9905 |
| $S_0 = 90$ | 1.7384 | 1.1154 | 2.0707 |
| $S_0 = 100$ | 4.3424 | 3.2407 | 4.9602 |
| $S_0 = 110$ | 8.4124 | 7.0045 | 9.1783 |
| $S_0 = 115$ | 10.9796 | 9.5586 | 11.8192 |

**Table 5.** The value of $B_{\mathrm{SUC},1}(0, S_0, X_0; T, K)$

| | Regime switching model | Black-Scholes model | |
|---|---|---|---|
| | | Regime 1 | Regime 2 |
| $S_0 = 85$ | 0.7942 | 0.5345 | 0.9458 |
| $S_0 = 90$ | 1.2792 | 0.9697 | 1.3915 |
| $S_0 = 100$ | 2.5869 | 2.3768 | 2.4472 |
| $S_0 = 110$ | 4.1671 | 4.2224 | 3.4162 |
| $S_0 = 115$ | 4.8778 | 5.0815 | 3.7442 |

**Table 6.** The value of $B_{\mathrm{DKC},1}(0, S_0, X_0; T, K)$

| | Regime switching model | Black-Scholes model | |
|---|---|---|---|
| | | Regime 1 | Regime 2 |
| $S_0 = 85$ | 0.5751 | 0.4198 | 0.5034 |
| $S_0 = 90$ | 1.1863 | 0.9254 | 1.1576 |
| $S_0 = 100$ | 2.5630 | 2.3736 | 2.3831 |
| $S_0 = 110$ | 3.9526 | 4.2303 | 3.3974 |
| $S_0 = 115$ | 4.4816 | 5.0933 | 3.7316 |

## 5 Conclusion

In this paper, we derive numerically convenient Fourier integral formulas of pricing single and double barrier knock-out call options in Markovian regime switching Black-Scholes models with specific rebates. The integral representations of the rebates and the option prices, which are based on the results of the matrix Wiener-Hopf factorization for the first-passage time densities, turn out to be easy to implement. Numerical results demonstrate the computational accuracy and effects on pricing barrier options in regime switching models with rebates.

## Conflict of Interest

The authors declare no conflict of interest.

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