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Existence of Solutions for a Quasilinear Schrödinger Equation with Potential Vanishing

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Abstract We study the following quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \Delta(u^2)u = K(x)g(u), \qquad x \in \mathbb{R}^3,$$

where the nonlinearity $g(u)$ is asymptotically cubic at infinity, the potential $V(x)$ may vanish at infinity. Under appropriate assumptions on $K(x)$, we establish the existence of a nontrivial solution by using the mountain pass theorem.

Keywords quasilinear Schrödinger equation; vanishing potential; asymptotically cubic; mountain pass theorem

2000 MR Subject Classification 35J60; 35J62; 35B09

1 Introduction

In this paper, we are concerned with the existence results for the following quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \Delta(u^2)u = K(x)g(u), \qquad x \in \mathbb{R}^N, \tag{1.1}$$

where $N = 3$, the potential $V(x)$ vanishes at infinity, $K(x)$ is continuous and the nonlinearity $g(t)$ behaves like t^3 at infinity.

Problem (1.1) has been studied widely. As far as we know, it was firstly considered in [17], where the existence of a positive ground state solution was obtained via minimization methods. In [13], by using a constrained minimization argument, a positive ground state solution has been proved for problem (1.1) with $K(x)g(u) = \lambda|u|^{q-1}u$, $4 \le q+1 < 2\cdot 2^*$, where $2^* = 2N/(N-2)$ is the Sobolev critical exponent. In [12], the quasilinear equation was reduced to a semilinear one by utilizing a change in variables, and an Orlicz space framework was used to prove the existence of a positive solutions via the mountain pass theorem. The same method was also used in [3], but the usual Sobolev space $H^1(\mathbb{R}^N)$ framework was used as the working space. Taking into account the behavior of the potential $V(x)$, we find several works concerning on problem (1.1). Regarding asymptotically periodic, we cite the paper of Silva and Vieira[19], Xue and Tang[21, 22]. In [19], when the nonlinearity is critical, they obtain a non-trivial solution by using the mountain pass theorem. In [21, 22], the authors use a Nehari-type constraint to get a ground state solution under subcritical or critical growth. With a finite potential well, we mention the results in [2, 6, 10]. Cassani and Wang[2] studied blow-up phenomena and

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asymptotic profiles for problem (1.1). Dong and Mao[6] dealt with the more general quasilinear Schrödinger equation by using a perturbation method which was developed in [14]. They established a new convergence theorem and introduced a weighted space together with Moser's iteration to recover the compactness. Then, they proved the existence of a sequence of solutions with negative energy values which converges to 0. In [10], the authors considered the general Berestycki-Lions type assumptions on the nonlinearity by using Jeanjean′s monotonicity trick.

All the aforementioned articles concerned with the case that the nonlinearity is super-cubic or sub-cubic. We are interested in the case that $g(t)$ is asymptotically cubic at infinity. This type of nonlinearity was studied by Liu et. al.[7–9, 12, 15, 16]. The first one is reference [12], which states, among other results, the existence of positive solutions for the autonomous nonlinearity $K(x)g(t) = t^3$ under different kind of hypothesis on the potential $V(x)$. Fang and Szulkin[7] consider $K(x)g(t) = q(x)t^3$ and obtained the existence of infinite solutions under some symmetry conditions on the potential $V(x)$. The case of coercive potential were discussed in [9, 16], an Orlicz space framework was used to prove the existence of a nontrivial solution via the mountain pass theorem. We finally mention a paper of Maia et. al.[15] where they consider the case that the potential $V(x)$ changes sign by employing the mountain pass theorem to obtain the existence of a non-trivial solution. There are few papers which deal with vanishing potential except [1, 4], the nonlinearities in this two articles are quasi-critical and critical respectively. Up to our knowledge, there are no results concerning with the asymptotically cubic framework under the vanishing potential for problem (1.1). Inspired by [9, 11, 16], we discuss this kind of problem (1.1).

Throughout the paper we shall assume that $V(x)$, $K(x)$ and $g(s)$ satisfy the following conditions:

(V) $V(x) \in C(\mathbb{R}^3, \mathbb{R})$, and there exist $a > 0,\ A > 0,\ 0 < \alpha < 2$ such that

$$\frac{a}{1 + |x|^\alpha} \le V(x) \le A.$$

(K) $K(x)$ is a positive continuous function and

$$\lim_{|x| \to +\infty} \frac{V(x)}{K(x)} = +\infty.$$

(g_1) g is a positive continuous function $g(s) \equiv 0$ for all $s \le 0$, and

$$\lim_{s \to 0^+} \frac{g(s)}{s} = 0.$$

(g_2) There is $l \in (0, +\infty)$ such that

$$\lim_{s \to +\infty} \frac{g(s)}{s^3} = l.$$

(g_3) $\frac{1}{4} g(s)s - G(s) \ge 0$, where $G(s) = \int_0^s g(t)dt$.

In order to state our main theorem we need to introduce the following weighted Sobolev space

$$H = \Big\{ u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} \big(|\nabla u|^2 + V(x)u^2\big) dx < \infty \Big\}.$$

Then, H is a Hilbert space and $H^1(\mathbb{R}^3) \subset H$[11]. The scalar product and norm in H are given by

$$(u, v) = \int_{\mathbb{R}^3} \big(\nabla u \nabla v + V(x)uv\big) dx, \qquad \|u\|^2 = \int_{\mathbb{R}^3} \big(|\nabla u|^2 + V(x)u^2\big) dx.$$

We are able to state the main result of this paper as follows.

Theorem 1.1. *Suppose that $(V),(K)$ hold and the function $g(s)$ satisfies $(g_1)-(g_3)$. Let $l>\mu$ with*

$$\mu=\inf\Big\{\int_{\mathbb{R}^3}\big(|\nabla u|^2+V(x)u^2\big)dx : u\in H,\ \int_{\mathbb{R}^3}K(x)u^2dx=1\Big\}.$$

Then problem (1.1) possesses a nontrivial solution.

Remark 1.2. In [9, 16], the potential $V(x)$ is coercive, so the embedding $H^1(\mathbb{R}^3)\hookrightarrow L^p(\mathbb{R}^3)$ is compact for $2\le p<2^*$. This ensures that the bounded (C) sequence converges strongly to a nontrivial solution of problem (1.1). In this paper we cannot use the compact embedding theorem as in [9, 16]. Motivated by [11], we establish a compactness result, see Lemma 3.4 below, which ensures that the bounded (C) sequence converges to a nontrivial solution $u\in H$.

Remark 1.3. We reduce the restrictions on the nonlinearity. The conditions $(g_1)-(g_3)$ are much weaker than those in previous articles for the asymptotically linear problem.

Example 1.4. There are functions satisfying our conditions. Let $g(s)=\frac{ls^5}{1+s^2}$ for some $l>0$. By direct calculations, we have

$$G(s)=\frac{l}{2}\Big[\frac{s^4}{2}-s^2+\ln(1+s^2)\Big].$$

Define $H(s):=\frac{1}{4}g(s)s-G(s)$, then

$$H(0)=0,\qquad H'(s)=\frac{ls^5}{2(1+s^2)^2}\ge 0,\qquad \text{for}\ \ s\ge 0.$$

Hence $H(s)\ge 0$ for $s\ge 0$. Hereafter, we see that the function g satisfies the conditions $(g_1)-(g_3)$. Let

$$V(x)=\frac{1}{\ln\ln(3+|x|)},\qquad K(x)=\frac{1}{\ln(3+|x|^2)},$$

then conditions $(V),(K)$ hold.

2 Some Preliminary Results

We observe that formally problem (1.1) is the Euler-Lagrange equation associated with the energy functional

$$J(u)=\frac{1}{2}\int_{\mathbb{R}^3}\big[(1+2u^2)|\nabla u|^2\big]dx+\frac{1}{2}\int_{\mathbb{R}^3}V(x)u^2dx-\int_{\mathbb{R}^3}K(x)G(u)dx.$$

From the variational point of view, the first difficulty we have to deal with problem (1.1) is to find an appropriate function space where the above functional is well defined. In the spirit of the argument developed in [3]. We make a change of variables $v:=f^{-1}(u)$, where f is defined by

$$f'(t)=\frac{1}{(1+2f^2(t))^{1/2}},\qquad t\in[0,+\infty),$$
$$f(t)=-f(-t),\qquad t\in(-\infty,0].$$

After the change of variables from J, we obtain the following functional:

$$I(v)=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2dx+\frac{1}{2}\int_{\mathbb{R}^3}V(x)f^2(v)dx-\int_{\mathbb{R}^3}K(x)G(f(v))dx.$$

Then $I(v) = J(u) = J(f(v))$ and I is well defined on H, $I \in C^1(H, \mathbb{R})$ under the hypotheses (V), (K) and $(g_1) - (g_3)$. Moreover, we observe that if v is a critical point of the functional I, then the function $u = f(v)$ is a solution of problem (1.1)[3].

Below we summarize the properties of f, which have been proved in [3] and [19].

Lemma 2.1. *The function f satisfies the following properties:*

(1) f is uniquely defined, C^∞ and invertible;

(2) $|f'(t)| \le 1$ for all $t \in \mathbb{R}$;

(3) $|f(t)| \le |t|$ for all $t \in \mathbb{R}$;

(4) $\frac{f(t)}{t} \to 1$ as $t \to 0$;

(5) $\frac{f(t)}{\sqrt{t}} \to 2^{1/4}$ as $t \to \infty$;

(6) $\frac{f(t)}{2} \le tf'(t) \le f(t)$ for all $t > 0$;

(7) $|f(t)| \le 2^{1/4}|t|^{1/2}$ for all $t \in \mathbb{R}$;

(8) $f^2(t) - f(t)f'(t)t \ge 0$ for all $t \in \mathbb{R}$;

(9) there exists a positive constant C such that $|f(t)| \ge C|t|$ for $|t| \le 1$ and $|f(t)| \ge C|t|^{1/2}$ for $|t| \ge 1$.

Lemma 2.2. *Suppose that (g_1) and (g_2) hold, then for each $\epsilon > 0$, there is $C_\epsilon > 0$, $C^* > 0$ such that for all $s \in \mathbb{R}^+$,*

$$g(s) \le \epsilon|s| + C_\epsilon|s|^3, \tag{2.1}$$

$$G(s) \le \frac{\epsilon}{2}|s|^2 + C^*|s|^6. \tag{2.2}$$

Lemma 2.3 (Mountain pass theorem[20])**.** *Let E be a real Banach space with its dual space E^* and suppose that $I \in C^1(E, \mathbb{R})$ satisfies*

$$\max\{I(0), I(e)\} \le \mu < \eta \le \inf_{\|u\|=\rho} I(u),$$

for some $\mu < \eta$, $\rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $c \ge \eta$ be characterized by $c = \inf\limits_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t))$, where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ \gamma(1) = e\}$ is the set of continuous paths joining 0 and e, then there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to c \ge \eta, \qquad (1 + \|u_n\|)\|I'(u_n)\| \to 0.$$

3 Proof of Theorem 1.1

Lemma 3.1. *If the Conditions (V), (K), $(g_1) - (g_3)$ hold, then there exist $\rho > 0$, $\eta > 0$ such that $\inf_{\|v\|=\rho} I(v) > \eta$.*

Proof. Thanks to Lemma 2.1-(9), we can deduce that there is $C_1 > 0$, such that

$$f^2(t) \ge C_1(|t|^2 - |t|^6). \tag{3.1}$$

It follows from (K) that there exists $M > 0$ such that

$$K(x) \le MV(x). \tag{3.2}$$

By (3.1), (3.2), (2.2), Lemma 2.1-(3) and the Sobolev inequality, we have

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x) f^2(v) dx - \int_{\mathbb{R}^3} K(x) G(f(v)) dx$$

$$
\begin{aligned}
&\geq \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2dx+\frac{1}{2}\int_{\mathbb{R}^3}V(x)(C_1|v|^2-C_1|v|^6)dx\\
&\quad -M\int_{\mathbb{R}^3}V(x)\Big(\frac{\epsilon}{2}|f(v)|^2+C^*|f(v)|^6\Big)dx\\
&\geq \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2dx+\Big(\frac{C_1}{2}-\frac{M\epsilon}{2}\Big)\int_{\mathbb{R}^3}V(x)v^2dx-\Big(\frac{C_1}{2}+MC^*\Big)\int_{\mathbb{R}^3}V(x)v^6dx\\
&\geq \min\Big\{\frac{1}{2},\frac{C_1}{2}-\frac{M\epsilon}{2}\Big\}\|v\|^2-\Big(\frac{C_1}{2}+MC^*\Big)AS^{-3}\|v\|^6,
\end{aligned}
$$

where A is the constant in condition (V) and S is the optimal constant in the Sobolev embeddings. Let $0<\epsilon<\frac{C_1}{M}$, then we conclude that there is $\rho>0$ small enough, such that $I(v)>0$ whenever $\|v\|\leq\rho$, $v\neq 0$. And there exists $\eta>0$ such that for any $\|v\|=\rho$, one has $I(v)\geq\eta>0$. □

Lemma 3.2. *Suppose (V), (K), $(g_1)-(g_3)$ are satisfied. Then there exists $e\in H$ with $\|e\|>\rho$, such that $I(e)<0$, where ρ is given by Lemma 3.1.*

Proof. By the definition of μ and $l>\mu$, there is $v\in H$ such that $v\geq 0$, $\int_{\mathbb{R}^3}K(x)v^2dx=1$ and $\mu\leq\|v\|^2<l$. Then, by Lemma 2.1-(3)(5), (g_2) and the Fatou lemma we deduce that

$$
\begin{aligned}
\lim_{t\to\infty}\frac{I(tv)}{t^2}\leq&\frac{1}{2}\|v\|^2-\lim_{t\to\infty}\int_{\mathbb{R}^3}K(x)\frac{G(f(tv))}{f^4(tv)}\frac{f^4(tv)}{t^2v^2}v^2dx\\
\leq&\frac{1}{2}\|v\|^2-\frac{1}{2}l\int_{\mathbb{R}^3}K(x)v^2dx=\frac{1}{2}(\|v\|^2-l)<0,
\end{aligned}
$$

and the lemma is proved by taking $e=t_0v$ with $t_0>0$ large enough. □

Lemma 3.3. *Suppose (V), (K), $(g_1)-(g_3)$ are satisfied. Then there exists $\{v_n\}\subset H$ such that $I(v_n)\to c$, $(1+\|v_n\|)\|I'(v_n)\|\to 0$ and $\{v_n\}$ is bounded in H.*

Proof. It follows from Lemma 3.1, Lemma 3.2 and Lemma 2.3 that, there exists a sequence $\{v_n\}\subset H$ such that $I(v_n)\to c$, $(1+\|v_n\|)\|I'(v_n)\|\to 0$. We only need to prove that $\{v_n\}$ is bounded.

First of all, we observe that if a sequence $\{v_n\}\subset H$ satisfies

$$
\gamma(v_n):=\int_{\mathbb{R}^3}|\nabla v_n|^2dx+\int_{\mathbb{R}^3}V(x)f^2(v_n)dx\leq C_2,
$$

for some constant $C_2>0$, then the sequence $\{v_n\}$ is bounded in H. In fact, by Lemma 2.1-(9), (V) and the Sobolev inequality, we observe that

$$
\int_{\{x:|v_n(x)|\leq 1\}}V(x)v_n^2dx\leq\frac{1}{C^2}\int_{\{x:|v_n(x)|\leq 1\}}V(x)f^2(v_n)dx\leq\frac{\gamma(v_n)}{C^2},
$$

and

$$
\int_{\{x:|v_n(x)|>1\}}V(x)v_n^2dx\leq A\int_{\{x:|v_n(x)|>1\}}v_n^6dx\leq AS^{-3}(\gamma(v_n))^3.
$$

Thus,

$$
\int_{\mathbb{R}^3}V(x)v_n^2dx\leq\frac{\gamma(v_n)}{C^2}+AS^{-3}(\gamma(v_n))^3,
$$

that is,

$$\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \int_{\mathbb{R}^3} V(x) v_n^2 dx \le \Big(1 + \frac{1}{C^2}\Big)\gamma(v_n) + AS^{-3}(\gamma(v_n))^3.$$

Therefore, it remains to show that $\gamma(v_n)$ is bounded.

Let $\{v_n\} \subset H$ be an arbitrary Cerami sequence for I at level $c > 0$, that is $I(v_n) \to c$ and $(1 + \|v_n\|)\|I'(v_n)\| \to 0$, namely

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x) f^2(v_n) dx - \int_{\mathbb{R}^3} K(x) G(f(v_n)) dx = c + o_n(1), \tag{3.3}$$

and for any $\varphi \in H$,

$$\begin{aligned}\langle I'(v_n), \varphi\rangle = &\int_{\mathbb{R}^3} \nabla v_n \cdot \nabla \varphi dx + \int_{\mathbb{R}^3} V(x) f(v_n) f'(v_n) \varphi dx \\ &- \int_{\mathbb{R}^3} K(x) g(f(v_n)) f'(v_n) \varphi dx = o_n(1).\end{aligned}$$

Choosing

$$\varphi = \varphi_n = \sqrt{1 + 2f^2(v_n)} f(v_n) = \frac{f(v_n)}{f'(v_n)},$$

from Lemma 2.1-(6), we get $\|\varphi_n\|_2 \le 2\|v_n\|_2$ and

$$|\nabla \varphi_n| = \Big(1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)}\Big) |\nabla v_n| \le 2|\nabla v_n|.$$

Thus there exists a constant $C_3 > 0$ such that $\|\varphi_n\| \le C_3\|v_n\|$. Recalling that $\{v_n\} \subset H$ is a (C) sequence, we get

$$\begin{aligned}\langle I'(v_n), \varphi_n\rangle = &\int_{\mathbb{R}^3} \Big(1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)}\Big)|\nabla v_n|^2 dx + \int_{\mathbb{R}^3} V(x) f^2(v_n) dx \\ &- \int_{\mathbb{R}^3} K(x) g(f(v_n)) f(v_n) dx = o_n(1).\end{aligned} \tag{3.4}$$

By computing (3.3)$-\frac{1}{4}$(3.4), one gets

$$\begin{aligned}c + o_n(1) = &\frac{1}{4}\int_{\mathbb{R}^3} \frac{1}{1 + 2f^2(v_n)}|\nabla v_n|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} V(x) f^2(v_n) dx \\ &+ \int_{\mathbb{R}^3} K(x)\Big(\frac{1}{4} g(f(v_n)) f(v_n) - G(f(v_n))\Big) dx.\end{aligned}$$

Thanks to (K) and (g_3), we get

$$\frac{1}{4}\int_{\mathbb{R}^3} \frac{1}{1 + 2f^2(v_n)}|\nabla v_n|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} V(x) f^2(v_n) dx \le c + o_n(1). \tag{3.5}$$

Denote $w_n = f(v_n)$, then $|\nabla v_n|^2 = (1 + 2w_n^2)|\nabla w_n|^2$. We can rewrite (3.3), (3.5) as follows.

$$\frac{1}{2}\int_{\mathbb{R}^3} (1 + 2w_n^2)|\nabla w_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x) w_n^2 dx - \int_{\mathbb{R}^3} K(x) G(w_n) dx = c + o_n(1), \tag{3.6}$$

and

$$\frac{1}{4}\int_{\mathbb{R}^3} |\nabla w_n|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} V(x) w_n^2 dx \le c + o_n(1). \tag{3.7}$$

From (3.7), we can see that $\{w_n\}$ is bounded in H. Therefore $\|w_n\|_6$ is bounded by the Sobolev inequality. It follows from (2.2), (3.2) and (V) that

$$\begin{aligned}\int_{\mathbb{R}^3} K(x)G(w_n)dx \leq & M\int_{\mathbb{R}^3} V(x)\Big(\frac{\epsilon}{2}|w_n|^2 + C^*|w_n|^6\Big)dx \\ \leq & \frac{1}{2}M\epsilon\int_{\mathbb{R}^3} V(x)|w_n|^2dx + MC^*A\int_{\mathbb{R}^3}|w_n|^6dx.\end{aligned}$$

We can deduce that there is a constant $C_4 > 0$, such that

$$\int_{\mathbb{R}^3} K(x)G(w_n)dx \leq C_4.$$

By the above inequality and (3.6), one has

$$\frac{1}{2}\int_{\mathbb{R}^3}(1+2w_n^2)|\nabla w_n|^2dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)w_n^2dx \leq C_4 + c + o_n(1),$$

namely

$$\frac{1}{2}\int_{\mathbb{R}^3}|\nabla v_n|^2dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)f^2(v_n)dx \leq C_4 + c + o_n(1).$$

This completes the proof. □

Lemma 3.4. *Let (V), (K) $(g_1)-(g_3)$ hold. Then for any $\epsilon > 0$, there exist $R(\epsilon) > 0$ and $n(\epsilon) > 0$ such that*

$$\int_{\{x:|x|\geq R\}}|\nabla v_n|^2dx + \int_{\{x:|x|\geq R\}} V(x)v_n^2dx \leq \epsilon,$$

for all $R \geq R(\epsilon)$ and $n \geq n(\epsilon)$, where the sequence $\{v_n\}$ is a (C) sequence of the functional I.

Proof. Thanks to (K), for any $\eta > 0$, there exists $R_0 > 0$ such that for all $|x| \geq R_0$,

$$K(x) < \eta V(x). \tag{3.8}$$

Define

$$C_1(R_0,\alpha,a) := \sup\Big\{\frac{1+(2R)^\alpha}{aR^2} : R \geq R_0\Big\} = \frac{1+(2R_0)^\alpha}{aR_0^2},$$

and

$$C_2(R_0,\alpha,a) := \sup\Big\{\frac{1+(2R)^\alpha}{aR^\alpha} : R \geq R_0\Big\} = \frac{1+(2R_0)^\alpha}{aR_0^\alpha},$$

where α and a are given by (V). Then, by the above two inequalities and (V), for all $R \geq R_0$, we have

$$1/R^2 \leq C_1(R_0,\alpha,a)V(x), \qquad \text{for all} \quad |x| \leq 2R, \tag{3.9}$$

and

$$1/R^\alpha \leq C_2(R_0,\alpha,a)V(x), \qquad \text{for all} \quad |x| \leq 2R. \tag{3.10}$$

Let $\xi_R : \mathbb{R}^3 \to [0,1]$ be a smooth function such that

$$\xi_R(x) = \begin{cases} 0, & 0 \le |x| \le R, \\ 1, & |x| \ge 2R. \end{cases}$$

Moreover there is a constant $C_0 > 0$ independent of R such that

$$|\nabla \xi_R(x)| \le \frac{C_0}{R}, \qquad \text{for all} \quad x \in \mathbb{R}^3.$$

Then by the Young inequality and (3.9), for all $n \in \mathbb{N}$ and $R \ge R_0$, one has

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla(v_n\xi_R)|^2 dx =& \int_{\mathbb{R}^3} \big(\xi_R^2|\nabla v_n|^2 + v_n^2|\nabla\xi_R|^2 + 2\xi_R v_n \nabla\xi_R \nabla v_n\big)dx \\ \le& \int_{\mathbb{R}^3} \big(\xi_R^2|\nabla v_n|^2 + v_n^2|\nabla\xi_R|^2\big)dx + \int_{\mathbb{R}^3} \big((\xi_R\nabla v_n)^2 + (v_n\nabla\xi_R)^2\big)dx \\ \le& 2\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \frac{2C_0^2}{R^2}\int_{\{x:R\le|x|\le 2R\}} |v_n|^2 dx \\ \le& [2 + 2C_0^2 C_1(R_0,\alpha,a)]\|v_n\|^2. \end{aligned} \tag{3.11}$$

This implies that for all $n \in \mathbb{N}$ and $R \ge R_0$, we have

$$\|v_n\xi_R\| \le [3 + 2C_0^2 C_1(R_0,\alpha,a)]^{1/2}\|v_n\|. \tag{3.12}$$

By $0 < \alpha < 2$, for any $\epsilon > 0$, there exists $R(\epsilon) \ge R_0$ such that

$$R^{\alpha-2} \le \frac{4\epsilon^2}{C_0^2 C_2(R_0,\alpha,a)}, \qquad \text{for all} \ \ R \ge R(\epsilon). \tag{3.13}$$

Since $\{v_n\}$ is a (C) sequence, $\|I'(v_n)\|_{H^{-1}}\|v_n\| \to 0$ as $n \to \infty$. Hence for any $\epsilon > 0$, there is $n(\epsilon) > 0$ such that

$$\|I'(v_n)\|_{H^{-1}}\|v_n\| \le \frac{\epsilon}{[3 + 2C_0^2 C_1(R_0,\alpha,a)]^{1/2}}, \qquad \text{for all} \ \ n \ge n(\epsilon). \tag{3.14}$$

Combining (3.14) with (3.12), we have

$$|\langle I'(v_n), v_n\xi_R\rangle| \le \|I'(v_n)\|_{H^{-1}}\|v_n\xi_R\| \le \epsilon, \qquad \text{for all} \ \ n \ge n(\epsilon), \quad R \ge R_0. \tag{3.15}$$

For $R \ge R(\epsilon)$, using (3.10) and (3.13), one has

$$\frac{C_0^2 C_2(R_0,\alpha,a)}{R^2} \le \frac{4\epsilon^2}{R^\alpha} \le 4\epsilon^2 C_2(R_0,\alpha,a)V(x), \qquad \text{for all} \ \ |x| \le 2R,$$

that is,

$$\frac{C_0^2}{R^2} \le 4\epsilon^2 V(x), \qquad \text{for all} \ \ |x| \le 2R.$$

Therefore, for all $n \in \mathbb{N}$ and $R \ge R(\epsilon)$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} |\, v_n\nabla v_n\nabla\xi_R \,|\, dx \le& \epsilon\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \frac{1}{4\epsilon}\int_{\{x:|x|\le 2R\}} \frac{C_0^2}{R^2}|v_n|^2 dx \\ \le& \epsilon\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \epsilon\int_{\{x:|x|\le 2R\}} V(x)|v_n|^2 dx\end{aligned}$$

$$\leq \epsilon \|v_n\|^2. \tag{3.16}$$

It follows from (2.1), (3.8) and Lemma 2.1-(3)(7)(8) that, for all $n \in \mathbb{N}$ and $R \geq R(\epsilon)$,

$$\begin{aligned}\int_{\mathbb{R}^3} | K(x)g(f(v_n))f'(v_n)v_n\xi_R | \, dx \leq& \eta \int_{\mathbb{R}^3} V(x)(\epsilon|f(v_n)| + C_\epsilon|f(v_n)|^3)f'(v_n)v_n\xi_R dx \\ \leq& (\eta\epsilon + 2C_\epsilon\eta) \int_{\mathbb{R}^3} V(x)v_n^2\xi_R dx. \end{aligned} \tag{3.17}$$

Combining (3.16), (3.17) with Lemma 2.1-(6), for all $n \in \mathbb{N}$ and $R \geq R(\epsilon) \geq R_0$, we have

$$\begin{aligned}|\langle I'(v_n), v_n\xi_R\rangle| \geq& \langle I'(v_n), v_n\xi_R\rangle \\ =& \int_{\mathbb{R}^3} \xi_R|\nabla v_n|^2 dx + \int_{\mathbb{R}^3} V(x)f(v_n)f'(v_n)v_n\xi_R dx \\ &+ \int_{\mathbb{R}^3} v_n\nabla v_n \nabla\xi_R dx - \int_{\mathbb{R}^3} K(x)g(f(v_n))f'(v_n)v_n\xi_R dx \\ \geq& \int_{\mathbb{R}^3} \xi_R|\nabla v_n|^2 dx - \epsilon\|v_n\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(x)f^2(v_n)\xi_R dx \\ &- (\eta\epsilon + 2C_\epsilon\eta)\int_{\mathbb{R}^3} V(x)v_n^2\xi_R dx.\end{aligned}$$

Since $\{v_n\}$ is bounded by Lemma 3.3, we can get the boundedness of $\int_{\mathbb{R}^3} V(x)f^2(v_n)\xi_R dx$ and $\int_{\mathbb{R}^3} V(x)v_n^2\xi_R dx$. Let η small enough, one gets

$$\frac{1}{2}\int_{\mathbb{R}^3} V(x)f^2(v_n)\xi_R dx - (\eta\epsilon + 2C_\epsilon\eta)\int_{\mathbb{R}^3} V(x)v_n^2\xi_R dx > 0.$$

Hence combining the above inequality with (3.15), there are $C_5 > 0,\ C_6 > 0$ such that

$$\int_{\mathbb{R}^3} |\nabla v_n|^2\xi_R dx \leq C_5\epsilon, \tag{3.18}$$

and

$$\frac{1}{2}\int_{\mathbb{R}^3} V(x)f^2(v_n)\xi_R dx - (\eta\epsilon + 2C_\epsilon\eta)\int_{\mathbb{R}^3} V(x)v_n^2\xi_R dx \leq C_6\epsilon.$$

Furthermore

$$\int_{\mathbb{R}^3} V(x)f^2(v_n)\xi_R dx \leq C_7\epsilon, \tag{3.19}$$

for some constant C_7. By the Sobolev inequality and (3.18), we can deduce

$$\int_{\mathbb{R}^3} v_n^6\xi_R dx \leq C_8\epsilon. \tag{3.20}$$

It follows from (3.1), (V), (3.19) and (3.20) that

$$\begin{aligned}C_1\int_{\mathbb{R}^3} V(x)v_n^2\xi_R dx \leq& C_1\int_{\mathbb{R}^3} V(x)v_n^6\xi_R dx + \int_{\mathbb{R}^3} V(x)f^2(v_n)\xi_R dx \\ \leq& C_1 A\int_{\mathbb{R}^3} v_n^6\xi_R dx + C_7\epsilon \leq (C_1AC_8 + C_7)\epsilon.\end{aligned} \tag{3.21}$$

Thanks to (3.18) and (3.21), we can get the conclusion. □

Proof of Theorem 1.1. By Lemma 3.3, the (C) sequence $\{v_n\} \subset H$ is bounded. We may assume that, up to a subsequence, $v_n \rightharpoonup v$ in H for some $v \in H$. In order to prove our theorem, it is now sufficient to show that $\|v_n\| \to \|v\|$ as $n \to \infty$. Since $\{v_n\}$ is a (C) sequence, we have

$$\begin{aligned}\langle I'(v_n), v_n\rangle =& \int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \int_{\mathbb{R}^3} V(x)v_n^2 dx - \int_{\mathbb{R}^3} V(x)v_n^2 dx \\ &+ \int_{\mathbb{R}^3} V(x)f(v_n)f'(v_n)v_n dx - \int_{\mathbb{R}^3} K(x)g(f(v_n))f'(v_n)v_n dx = o_n(1),\end{aligned}$$

and

$$\begin{aligned}\langle I'(v_n), v\rangle =& \int_{\mathbb{R}^3} \nabla v_n \nabla v dx + \int_{\mathbb{R}^3} V(x)v_n v dx - \int_{\mathbb{R}^3} V(x)v_n v dx \\ &+ \int_{\mathbb{R}^3} V(x)f(v_n)f'(v_n)v dx - \int_{\mathbb{R}^3} K(x)g(f(v_n))f'(v_n)v dx = o_n(1).\end{aligned}$$

So to show $\|v_n\| \to \|v\|$ is equivalent to proving the following three equalities.

$$\int_{\mathbb{R}^3} V(x)v_n(v_n - v)dx = o_n(1), \tag{3.22}$$

$$\int_{\mathbb{R}^3} V(x)f(v_n)f'(v_n)(v_n - v)dx = o_n(1), \tag{3.23}$$

$$\int_{\mathbb{R}^3} K(x)g(f(v_n))f'(v_n)(v_n - v)dx = o_n(1). \tag{3.24}$$

(i) The proof of (3.22). For any $\epsilon > 0$, by the Hölder inequality, Lemma 3.3 and Lemma 3.4, for n large enough, one has

$$\begin{aligned}&\int_{\{x:|x|\geq R(\epsilon)\}} V(x)v_n(v_n - v)dx \\ \leq& \Big(\int_{\{x:|x|\geq R(\epsilon)\}} V(x)v_n^2 dx\Big)^{\frac{1}{2}} \Big(\int_{\{x:|x|\geq R(\epsilon)\}} V(x)(v_n - v)^2 dx\Big)^{\frac{1}{2}} \leq C_9\epsilon.\end{aligned}$$

This and the compactness of the embedding $H \hookrightarrow L^2_{\text{loc}}(\mathbb{R}^3)$ imply (3.22).

(ii) The proof of (3.23). It follows from Lemma 2.1-(2)(3) that

$$\int_{\mathbb{R}^3} V(x)f(v_n)f'(v_n)(v_n - v)dx \leq \int_{\mathbb{R}^3} V(x)v_n(v_n - v)dx.$$

Hence (3.23) can be deduced by (3.22) easily.

(iii) The proof of (3.24). Thanks to (3.2), (2.1) and Lemma 2.1-(2)(3)(7), one has

$$\begin{aligned}&\int_{\{x:|x|\geq R(\epsilon)\}} K(x)g(f(v_n))f'(v_n)(v_n - v)dx \\ \leq& \int_{\{x:|x|\geq R(\epsilon)\}} MV(x)[\epsilon|f(v_n)| + C_\epsilon|f(v_n)|^3](v_n - v)dx \\ \leq& M\epsilon \int_{\{x:|x|\geq R(\epsilon)\}} V(x)v_n(v_n - v)dx + 2^{3/4}C_\epsilon M \int_{\{x:|x|\geq R(\epsilon)\}} V(x)v_n^{3/2}(v_n - v)dx \\ \leq& M\epsilon\Big(\int_{\{x:|x|\geq R(\epsilon)\}} V(x)v_n^2 dx\Big)^{1/2}\Big(\int_{\{x:|x|\geq R(\epsilon)\}} V(x)(v_n - v)^2 dx\Big)^{1/2} \\ &+ 2^{3/4}C_\epsilon M\Big(\int_{\{x:|x|\geq R(\epsilon)\}} V(x)v_n^3 dx\Big)^{1/2}\Big(\int_{\{x:|x|\geq R(\epsilon)\}} V(x)(v_n - v)^2 dx\Big)^{1/2}.\end{aligned}$$

Since $\{v_n\} \subset H$ is bounded, we can get the boundedness of $\int_{\mathbb{R}^3} V(x)v_n^3 dx$ from the Sobolev inequality and the interpolation inequality. Thereafter,

$$\int_{\{x:|x|\geq R(\epsilon)\}} K(x)g(f(v_n))f'(v_n)(v_n - v)dx \leq \epsilon.$$

This and the compactness of the embedding $H \hookrightarrow L^2_{\text{loc}}(\mathbb{R}^3)$ imply (3.24). □

Conflict of Interest

The authors declare no conflict of interest.

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