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On the Turán Density of Uniform Hypergraphs

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Abstract Let p, q be two positive integers. The 3-graph $F(p,q)$ is obtained from the complete 3-graph K_p^3 by adding q new vertices and $p\binom{q}{2}$ new edges of the form vxy for which $v \in V(K_p^3)$ and $\{x,y\}$ are new vertices. It frequently appears in many literatures on the Turán number or Turán density of hypergraphs. In this paper, we first construct a new class of r-graphs which can be regarded as a generalization of the 3-graph $F(p,q)$, and prove that these r-graphs have the same Turán density under some situations. Moreover, we investigate the Turán density of the $F(p,q)$ for small p, q and obtain some new bounds on their Turán densities.

Keywords Hypergraph; Turán density; Bound
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1 Introduction

Let $[n]$ denote the set $\{1, 2, \cdots, n\}$. An r-uniform hypergraph (or r-graph) H of order n is a family of r-element subsets of $[n]$. We denote the vertex set and the edge set of H by $V(H)$ and $E(H)$, respectively. Given an r-graph F, we say that an r-graph H is F-free if H does not contain F as a subgraph. The Turán number $\mathrm{ex}_r(n,F)$ is the maximum number of edges of an F-free r-graph on n vertices. Meanwhile the Turán density of F is defined as

$$\pi(F) = \lim_{n\to\infty} \frac{ex_r(n,F)}{\binom{n}{r}}.$$

A well-known fact is that the Turán density exists for any r-graph F, since that if we let $\tau(n) = \mathrm{ex}_r(n,F)/\binom{n}{r}$, then $\{\tau(n)\}$ is a bounded and monotonically non-increasing sequence (see [8] Section 2 for detail).

Determining Turán number or Turán density is perhaps the most fundamental open problem in the extremal hypergraph theory. In the case of 2-graphs, the Erdős-Stone-Simonovits Theorem [3, 5] answers that the Turán density of a 2-graph F is $\pi(F) = 1 - \frac{1}{\chi(F)-1}$, where $\chi(F)$ denotes the chromatic number of F. However, for $r \geq 3$, only a few cases have been settled. In fact, the Turán density of the complete r-graph K_p^r is still unknown for any pair of $p > r \geq 3$.

In many previous literatures, a key tool used frequently in determining Turán density is so-called the 'blow up' of an r-graph. Given an r-graph F, the *t-blow-up* $F(t)$ is a transformation of F that replacing each vertex of F by t copies of itself and each edge by corresponding complet r-partite r-graph of these copies. The celebrated supersaturation result of Erdős and Simonovists [4] implies that $\pi(F(t)) = \pi(F)$.

An intuitive question is that, given an r-graph, apart from the 'blow-up', is there any other way to construct new classes of r-graphs which keep invariant Turán density? In this paper, our main purpose is to consider this problem and give such a new constructing method for r-graphs.

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The main method used in this paper is the link multigraph method which is developed by de Caen and Füredi in [2]. By this method, they give a surprisingly short proof of a conjecture proposed by Sós[13] on the Turán density of the Fano plane, that is $\pi(\text{Fano}) = \frac{3}{4}$. After that, there are some results on Turán density building upon this method successfully. For example, Füredi, Pikhurko and Simonovits[7] determined that $\pi(F(3,2)) = \frac{4}{9}$, where $F(3,2) = (\{a,b,c,d\}, \{abc, ade, bde, cde\})$. Mubayi and Rödl[10] determined that $\pi(F(3,3)) = \frac{3}{4}$, where $F(3,3) = (\{a,b,c,x,\ y,z\}, \{abc, xya, xyb, xyc, xza, xzb, xzc,\ yza, yzb, yzc\})$.

The rest of this paper is organized as follows. In Section 2, we give some definitions and notations, and then prove several lemmas needed for proving main results. Our main results and their proofs are presented in Section 3 and 4. Specially, some new bounds on the Turán density of $F(p,q)$ for small p,q will be presented in Section 4.

2 Preliminaries

Throughout this section we let H be an r-graph and G be an $(r-1)$-graph, where $r \geq 3$. An r-uniform *multi-hypergraph* is the r-graph which allows for multi-edges. For convenience, $|E(H)|$ is sometimes referred to as $|H|$. Let S be a vertex set. We use $H - S$ to denote the r-graph with vertex set $V(H)\backslash S$ and edge set $\{e : e \in E(H) \text{ and } e \subset V(H)\backslash S\}$, i.e, $H - S$ is the induced subgraph of H with vertex set $V(H)\backslash S$.

Definition 2.1[9]**.** *For a vertex $x \in V(H)$, the link of x in H is an $(r-1)$-graph with vertex set $V(H)\backslash\{x\}$ and edge set $\{e\backslash\{x\} : x \in e \in E(H)\}$, denoted by $L_H(x)$. The degree of x in H is $d_H(x) = |L_H(x)|$. We omit the subscript if there is no confusion.*

Definition 2.2. *For a vertex set $S \subseteq V(H)$, the link of S in H is an $(r-1)$-uniform multi-hypergraph with vertex set $V(H)\backslash S$ and edge set $\{e\backslash\{x\} : e \in E(H) \text{ and } \{x\} = e \cap S\}$, denoted by $L_H(S)$. Let l be an edge of $L_H(S)$, we use $M(l)$ to denote the multiple number of l in $L_H(S)$.*

For a set P and an integer k, we write $\binom{P}{k}$ for the family of k-element subsets of P. The 3-graph $F(p,q)$ is defined as follows.

Definition 2.3[10]**.** *Let p,q be two positive integers. Then $F(p,q)$ is the 3-graph with vertex set $P \cup Q$, where $P = [p]$ and $Q = [p+q] - [p]$, and edge set $\binom{P}{3} \cup \{xyz | x \in P, y,z \in Q\}$.*

Now we give a transformation for r-graphs, which can be regarded as a generalization of 3-graph $F(p,q)$.

Definition 2.4. *Given an $(r-1)$-graph G and a set P. Let $B(P,G)$ be the r-graph with vertex set $P \cup V(G)$ and edge set $\{e \cup \{x\} : e \in E(G), x \in P\}$.*

Definition 2.5. *Let F be an r-graph and G be an $(r-1)$-graph with $V(F) \cap V(G) = \emptyset$. For a vertex set $P \subseteq V(F)$, the r-graph $F(P,G)$ is constructed as follows:*

1. The vertex set of $F(P,G)$ is $V(F) \cup V(G)$.

2. The edge set of $F(P,G)$ is $E(F) \cup E(B(P,G))$, where the $B(P,G)$ is constructed as in Definition 2.4.

Note that if $F = K_p^3, P = V(F)$ and $G = K_q^2$, then it's easy to see that $F(P,G) = F(p,q)$. So the r-graph $F(P,G)$ can be regarded as an generalization of the 3-graph $F(p,q)$.

For preparations before presenting our main results, we need the following lemmas.

Proposition 2.1. *Let F and H be r-graphs. If $F \subseteq H$, then $\pi(F) \leq \pi(H)$.*

Lemma 2.2. *Let r, n be positive integers. If $r \geq 2$ and n is sufficiently large, then*

$$n^{r-1} - (n-1)^{r-2} > (n-1)^{r-1}.$$

Proof. By the Newton's Binomial Theorem, we obtain that

$$\begin{aligned} n^{r-1} - (n-1)^{r-2} &= (n-1+1)^{r-1} - (n-1)^{r-2} \\ &> (n-1)^{r-1} + (n-1)^{r-2} - (n-1)^{r-2} \\ &\geq (n-1)^{r-1}. \end{aligned}$$

□

Lemma 2.3. *For any constant $c > 0$, $0 < \varepsilon < a < a + \varepsilon < 1$ and positive integer r, there exist n_0 and $c_0 = \binom{n_0}{r} - (a+\varepsilon)\binom{n_0}{r} - cn_0^{r-1}$, such that $f(n) = (a+\varepsilon)\binom{n}{r} + cn^{r-1} + c_0$ satisfies:*

1. *$f(n_0) = \binom{n_0}{r}$.*
2. *For $n > n_0$, $f(n) < \binom{n}{r}$.*

Proof. The first part of the lemma is obvious. Next we prove the second part.

For any positive number x, we let

$$\binom{x}{r} = \frac{x(x-1)\cdots(x-r+1)}{r!}$$

and

$$g(x) = \binom{x}{r} - (a+\varepsilon)\binom{x}{r} - cx^{r-1}.$$

Then the derivative of $g(x)$ with respect to x is

$$g'(x) = \frac{(1-a-\varepsilon)}{(r-1)!}x^{r-1} + O(x^{r-2}).$$

It is easy to see that when x is large enough, we have $g'(x) > 0$. That is, there exists an x_0 such that $g'(x) > 0$ when $x > x_0$. It means $g(x)$ is an increasing function on interval $(x_0, +\infty)$. Set $n_0 > x_0$ and $c_0 = \binom{n_0}{r} - (a+\varepsilon)\binom{n_0}{r} - cn_0^{r-1}$. Then if $n > n_0$, we have $g(n) > g(n_0)$. Thus,

$$\binom{n}{r} - f(n) = g(n) - c_0 > g(n_0) - c_0 = 0.$$

It implies $f(n) < \binom{n}{r}$ when $n > n_0$. □

Lemma 2.4. *Given positive integers $p \leq k < n$. Let $A_1, A_2, \cdots, A_p$ be the subsets of $[n]$. If $|A_i| \geq (1 - \frac{1}{k})n$ for any $1 \leq i \leq p$, then $|\bigcap_{1\leq i\leq p} A_i| \geq (1 - \frac{p}{k})n$.*

Proof. By induction on p. For $p = 2$, by the inclusion-exclusion principle we obtain that

$$\begin{aligned} |A_1 \cap A_2| &\geq |A_1| + |A_2| - n \\ &= \Big(1 - \frac{1}{k}\Big)n + \Big(1 - \frac{1}{k}\Big)n - n = \Big(1 - \frac{2}{k}\Big)n. \end{aligned}$$

Suppose the lemma holds for $p - 1$. Then

$$\begin{aligned} \Big|\bigcap_{1\leq i\leq p} A_i\Big| = \Big|\Big(\bigcap_{1\leq i\leq p-1} A_i\Big) \cap A_p\Big| &\geq \Big|\bigcap_{1\leq i\leq p-1} A_i\Big| + |A_p| - n \\ &\geq \Big(1 - \frac{p-1}{k}\Big)n + \Big(1 - \frac{1}{k}\Big)n - n = \Big(1 - \frac{p}{k}\Big)n, \end{aligned}$$

where the first inequality follows from the inclusion-exclusion principle and the second inequality follows from the induction hypothesis. This completes the proof. □

3 Properties on Turán Density $F(P,G)$

Now we start to present our main theorem.

Theorem 3.1. *Given positive integers $r \geq 3, p \geq 1$. Let G be an $(r-1)$-graph and F be an r-graph. Let $P \subseteq V(F)$, and $|P| = p$.*

1. *If $\pi(F) \geq \frac{p-1+\pi(G)}{p}$, then $\pi(F(P,G)) = \pi(F)$;*
2. *If $\pi(F) < \frac{p-1+\pi(G)}{p}$, then $\pi(F(P,G)) \leq \frac{p-1+\pi(G)}{p}$.*

Proof. According to Proposition 2.1, we have $\pi(F(P,G)) \geq \pi(F)$. Now we first prove that if $\pi(F) \geq \frac{p-1+\pi(G)}{p}$, then $\pi(F(P,G)) \leq \pi(F)$.

Suppose that $\pi(F) \geq \frac{p-1+\pi(G)}{p}$. For any $0 < \varepsilon < 1 - \pi(F)$, there exists $n_1 = n(\varepsilon)$ such that any positive integer $n \geq n_1$ satisfies

$$ex_r(n, F) < (\pi(F) + \varepsilon)\binom{n}{r}.$$

In additional, for any $p\varepsilon > 0$, there exists $n_2 = n(\varepsilon)$ such that any positive integer $n \geq n_2$ satisfies

$$ex_{r-1}(n, G) < (\pi(G) + p\varepsilon)\binom{n}{r-1}.$$

Let $|V(F)| = s$. By Lemma 2.3, we can find $n_0 > max\{n_1, n_2\} + s$ and $c_0 = \binom{n_0}{r} - (\pi(F) + \varepsilon)\binom{n_0}{r} - (s-1)n_0^{r-1}$ such that

$$(\pi(F) + \varepsilon)\binom{n}{r} + (s-1)n^{r-1} + c_0 \leq \binom{n}{r},$$

where $n \geq n_0$. So we can assume that H is an r-graph with n vertices and $(\pi(F)+\varepsilon)\binom{n}{r} + (s-1)n^{r-1} + c_0$ edges.

Next we use induction on n to prove that H contains a copy of $F(P,G)$.

If $n = n_0$, then $|E(H)| = \binom{n_0}{r}$. It implies that H is a complete r-graph of order n_0. So H contains a copy of $F(P,G)$.

Suppose the result holds for all r-graphs with $n-1$ vertices, where $n-1 \geq n_0$. Next we prove that the result holds for r-graph H with n vertices.

Since $n > n_0 > n_1$, we have $|E(H)| > ex_r(n, F)$. So $F \subseteq H$. Take a copy of F in H and let $P = \{v_1, v_2, \cdots, v_p\} \subseteq V(F)$. For $1 \leq i \leq p$, let $L(v_i)$ denote the link of v_i in H and $L'(v_i) = L(v_i) - V(F)$.

If for any v_i, we have

$$|L'(v_i)| \geq \Big(1 - \frac{1-\pi(G)-p\varepsilon}{p}\Big)\binom{n-s}{r-1}.$$

By Lemma 2.4, we deduce that

$$\Big|\bigcap_{1\leq i\leq p} L'(v_i)\Big| \geq (\pi(G) + p\varepsilon)\binom{n-s}{r-1}.$$

Since $\bigcap_{1\leq i\leq p} L'(v_i)$ is an $(r-1)$-graph with $n-s$ vertices and $n-s > n_0 - s > n_2$, we have

$$G \subseteq \bigcap_{1\leq i\leq p} L'(v_i).$$

Thus, the subgraph of H induced by the vertex set $P \cup V(G)$ contains a copy of $F(P,G)$.

If there exists a $v_i \in P$ such that

$$|L'(v_i)| < (1 - \frac{1-\pi(G)-p\varepsilon}{p})\binom{n-s}{r-1},$$

Then we have

$$\begin{aligned} d_H(v_i) &\le \Big(\frac{p-1+\pi(G)}{p} + \varepsilon\Big)\binom{n-s}{r-1} + (s-1)\binom{n}{r-2} \\ &\le \Big(\frac{p-1+\pi(G)}{p} + \varepsilon\Big)\binom{n-1}{r-1} + (s-1)n^{r-2}. \end{aligned}$$

Let $H' = H - v_i$. Then we have

$$\begin{aligned} |E(H')| &= |E(H)| - d_H(v_i) \\ &\ge (\pi(F)+\varepsilon)\binom{n}{r} + (s-1)n^{r-1} + c_0 - \Big(\frac{p-1+\pi(G)}{p} + \varepsilon\Big)\binom{n-1}{r-1} - (s-1)n^{r-2} \\ &\ge (\pi(F)+\varepsilon)\binom{n-1}{r} + (s-1)(n-1)^{r-1} + c_0. \end{aligned}$$

The last inequality follows from Lemma 2.2 and the assumption $\pi(F) \ge \frac{p-1+\pi(G)}{p}$. By induction hypothesis, we deduced that $F(P,G) \subseteq H' \subseteq H$. Therefore, for any $n \ge n_0$, we have

$$ex_r(n, F(P,G)) < (\pi(F)+\varepsilon)\binom{n}{r} + (s-1)n^{r-1} + c_0.$$

It means for any $0 < \varepsilon < 1 - \pi(F)$, we have $\pi(F(P,G)) \le \pi(F) + \varepsilon$. Thus we can obtain conclusion 1 by combining $\pi(F(P,G)) \ge \pi(F)$.

As for conclusion 2, one can proceed to prove it in a similar way. That is, we can use induction to get an r-graph H of order n with $(\frac{p-1+\pi(G)}{p} + \varepsilon)\binom{n}{r} + (s-1)n^{r-1} + c_0$ edges which contains a copy of $F(P,G)$. It immediately follows that

$$\pi(F(P,G)) \le \frac{p-1+\pi(G)}{p}.$$

The proof is completed. □

Theorem 3.1 implies that as long as the condition $\pi(F) \ge \frac{p-1+\pi(G)}{p}$ holds, one can always use the construction method in Definition 2.5 repeatedly to get a new r-graph which keeps invariant Turán density. Furthermore, if $F \subseteq F' \subseteq F(P,G)$, then $\pi(F') = \pi(F)$.

We here remark that the conditions in Theorem 3.1 is sufficient but not necessary. Let us illustrate it with an example to be presented in the following Theorem 3.3. We denote by $Fano$ the Fano plane with vertex set $\{x, y, z, x', y', z', a\}$ and edge set $\{xyz', xy'z, x'yz, axx', ayy', azz', x'y'z'\}$. The $Fano'$ is a 3-graph obtained from $Fano$ by adding three new vertices 1,2,3 and eleven new edges $12x, 12y, 12z, 12a, 13x, 13y, 13z, 13a, 23x, 23y, 23z$.

Next we prove the Turán densities of $Fano'$ and $Fano$ are equal. To prove it, we first give a lemma. A multigraph is a graph allowing for multi-edges.

Lemma 3.2. *Let t be a positive integer. Let G be a multigraph on n vertices such that every 3 vertices span at most $3t+1$ edges. Then G has at most $t\binom{n}{2} + (t+1)n$ edges.*

Proof. We use induction on n. The cases for $n = 2, 3$ are obvious. If G has no edge of multiple number at least $t+1$, then $|E(G)| \le t\binom{n}{2}$. Thus, we may assume the multiple numbers of xy is at least $t+1$. It leads to that for every vertex $v \in V(G)\backslash\{x, y\}$, there are at most $2t$ edges connecting to $\{x, y\}$. Let $G' = G - \{x, y\}$. Then

$$\begin{aligned}|E(G)| &\le |E(G')| + 2t(n-2) + 3t + 1\\ &\le t\binom{n-2}{2} + (t+1)(n-2) + 2t(n-2) + 3t + 1\\ &\le t\binom{n}{2} + (t+1)n,\end{aligned}$$

where the second inequality is according to the induction hypothesis. □

We remark that Lemma 3.2 is a special case in [6], here we only give a simple proof for this special case.

Theorem 3.3. $\pi(Fano') = \frac{3}{4}$.

Proof. For any constant $0 < \varepsilon < \frac{1}{4}, c_0 > 0$, let H be a 3-graph of order n with $(\frac{3}{4} + \varepsilon)\binom{n}{3} + 7n^2 + c_0$ edges.

The Turán density of $Fano$ is $\frac{3}{4}$ which is given by de Cean and Füredi[2]. Thus one can make sure that there exists n_1, such that $Fano \subseteq H$ when $n > n_1$.

By Lemma 2.3, we can find $n_0 > n_1$ and $c_0 = \left(\frac{1}{4} - \varepsilon\right)\binom{n_0}{3} - 7n_0^2$ such that $\left(\frac{3}{4} + \varepsilon\right)\binom{n}{3} + 7n^2 + c_0 \le \binom{n}{3}$, where $n \ge n_0$. So the 3-graph H which has n vertices and $\left(\frac{3}{4} + \varepsilon\right)\binom{n}{3} + 7n^2 + c_0$ edges is reasonable.

We use induction on n to prove $Fano' \subseteq H$. If $n = n_0$, then H is a complete 3-graph. It implies $Fano' \subseteq H$. Suppose the result holds for all 3-graphs H with the number of vertices lying in the interval $(n_0,\ n)$. Now what we need to prove is that the result holds for 3-graph H of order n.

Take a copy of $Fano$ in H whose vertices are labelled as mentioned above. Let $L(\{x, y, z, a\})$ be the link of $\{x, y, z, a\}$ in H. Let $L'(\{x, y, z, a\}) = L(\{x, y, z, a\}) - V(Fano)$.

If there exist three vertices in $L'(\{x, y, z, a\})$ span eleven edges, then the subgraph induced by these three vertices and $V(Fano)$ contains a copy of $Fano'$, it implies $Fano' \subseteq H$.

Hence, every three vertices in $L'(\{x, y, z, a\})$ span at most ten edges. By Lemma 3.2, we deduce that $L'(\{x, y, z, a\})$ has at most $3\binom{n-7}{2} + 4(n-7)$ edges. Thus there is a vertex, say $v \in \{x, y, z, a\}$, such that

$$d_H(v) \le \frac{3}{4}\binom{n-7}{2} + (n-7) + 6(n-7) + 15 < \frac{3}{4}\binom{n-1}{2} + 7n.$$

Let $H' = H - v$, we have

$$\begin{aligned}|E(H')| &= |E(H)| - d_H(v)\\ &> \left(\frac{3}{4} + \varepsilon\right)\binom{n}{3} + 7n^2 + c_0 - \frac{3}{4}\binom{n-1}{2} - 7n\\ &> \left(\frac{3}{4} + \varepsilon\right)\binom{n-1}{3} + 7(n-1)^2 + c_0.\end{aligned}$$

Thus, by the induction hypothesis, we have $Fano' \subseteq H' \subseteq H$. It means for any $\varepsilon > 0$, we have

$$\pi(Fano') \le \lim_{n\to\infty} \frac{(\frac{3}{4} + \varepsilon)\binom{n}{3} + 7n^2 + c_0}{\binom{n}{3}} = \frac{3}{4} + \varepsilon.$$

On the other hand, by Proposition 2.1, we have $\pi(Fano') \ge \pi(Fano) = \frac{3}{4}$. It follows that $\pi(Fano') = \frac{3}{4}$. This completes the proof. □

4 More on Turán Density of $F(p,q)$ for Small p,q

The Erdős-Stone-Simonovits Theorem [3, 5] tells us that $\pi(K_q^2) = 1 - \frac{1}{q-1}$ and Sidorenko [12] prove that $\pi(K_p^3) \geq 1 - \frac{4}{(p-1)^2}$. As if $F = K_p^3, P = V(F)$ and $G = K_q^2$, then $F(P,G) = F(p,q)$. So apply Theorem 3.1, we have

Corollary 4.1. *Given integers $p \geq 3$ and $q \geq 2$. If $q \leq \lfloor \frac{p+2}{4} \rfloor$, then $\pi(F(p,q)) = \pi(K_p^3)$.*

Proof. By Theorem 3.1, we have $\pi(F(p,q)) = \pi(K_p^3)$ when

$$\pi(K_p^3) \geq \frac{p-1+\pi(K_q^2)}{p}.$$

Since $\pi(K_q^2) = 1 - \frac{1}{q-1}$ and $\pi(K_p^3) \geq 1 - \frac{4}{(p-1)^2}$, we still have $\pi(F(p,q)) = \pi(K_p^3)$ if

$$1 - \frac{4}{(p-1)^2} \geq \frac{p-1+(1-\frac{1}{q-1})}{p}.$$

It's equivalent to

$$q \leq \frac{p}{4} + \frac{1}{2} + \frac{1}{4p}.$$

Since q is an integer and $p \geq 3$, we deduce that $q \leq \lfloor \frac{p+2}{4} \rfloor$.

□

Consequently, by Corollary 4.1, we have $\pi(F(p,3)) = \pi(K_p^3)$ for $p \geq 10$. So a natural question is that what's the relationship between $\pi(F(p,3))$ and $\pi(K_p^3)$ for $3 \leq p \leq 9$? It is proved by Mubayi and Rödl [10] that $\pi(F(3,3)) = \frac{3}{4}$. So we have $0 = \pi(K_3^3) < \pi(F(3,3)) = \frac{3}{4}$. For $p = 4$, we can present the following bounds for $\pi(F(4,3))$.

Theorem 4.2. $\frac{3}{4} \leq \pi(F(4,3)) \leq \frac{4}{5}$.

Proof. By Proposition 2.1, we have $\pi(F(4,3)) \geq \pi(F(3,3)) = \frac{3}{4}$, which settles the case of the lower bound.

As for the case of the upper bound, we prove it by showing that a 3-graph of order n with $\frac{4}{5}\binom{n}{3} + 5n^2 + c_0$ edges contains a copy of $F(4,3)$, where c_0 is constant. Suppose that H is the 3-graph of order n with $\frac{4}{5}\binom{n}{3} + 5n^2 + c_0$ edges.

It has been proven by Baber [1] that $\pi(K_5^3) \leq 0.76954$. Thus, there exists n_1 such that $K_5^3 \subseteq H$ for $n > n_1$.

By Lemma 2.3, we can find $n_0 > n_1$ and we set $c_0 = \frac{1}{5}\binom{n_0}{3} - 5n_0^2$, such that $\frac{4}{5}\binom{n}{3} + 5n^2 + c_0 \leq \binom{n}{3}$, where $n \geq n_0$. So a 3-graph H which has n vertices and $\frac{4}{5}\binom{n}{3} + 5n^2 + c_0$ edges is reasonable.

We use induction on n to prove $F(4,3) \subseteq H$. If $n = n_0$, then H is a complete 3-graph. It implies $F(4,3) \subseteq H$. Suppose the result holds for all 3-graphs H with the number of vertices lying in the interval $(n_0,\ n)$. What we need to prove is that the result holds for 3-graph H of order n.

Take a copy of K_5^3 in H. Let $L(V(K_5^3))$ be the link of $V(K_5^3)$ in H.

If there exist three vertices in $L(V(K_5^3))$ span fourteen edges, then one can check that the subgraph induced by these three vertices and $V(K_5^3)$ contains a copy of $F(4,3)$, it implies $F(4,3) \subseteq H$.

If every three vertices in $L(V(K_5^3))$ span at most thirteen edges. By Lemma 3.2, we deduce that $L(V(K_5^3))$ has at most $4\binom{n-5}{2} + 5(n-5)$ edges. Thus there is a vertex, say $v \in V(K_5^3)$, such that

$$d_H(v) \leq \frac{4}{5}\binom{n-5}{2} + (n-5) + 4(n-5) + 6 < \frac{4}{5}\binom{n-1}{2} + 5n.$$

Let $H' = H - v$, we have

$$\begin{aligned} |E(H')| &= |E(H)| - d_H(v) \\ &> \frac{4}{5}\binom{n}{3} + 5n^2 + c_0 - \frac{4}{5}\binom{n-1}{2} - 5n \\ &> \frac{4}{5}\binom{n-1}{3} + 5(n-1)^2 + c_0. \end{aligned}$$

Thus, by the induction hypothesis, we have $F(4,3) \subseteq H' \subseteq H$. So we have

$$ex_3(n, F(4,3)) < \frac{4}{5}\binom{n}{3} + 5n^2 + c_0.$$

It means

$$\pi(F(4,3)) \leq \lim_{n\to\infty} \frac{\frac{4}{5}\binom{n}{3} + 5n^2 + c_0}{\binom{n}{3}} = \frac{4}{5}.$$

This completes the proof of the theorem. □

It has been proven by Razborov [11] that $\pi(K_4^3) \leq 0.5616$. So when $p = 4$, we have $\pi(K_4^3) \leq 0.561 < \frac{3}{4} \leq \pi(F(4,3))$. Here we propose the following problem.

Problem 4.3. *Does* $\pi(F(4,3)) = \frac{3}{4}$*?*

In the case for $p = 5$, we also give the following bounds on $\pi(F(5,3))$.

Theorem 4.4. $0.8176 \leq \pi(F(5,3)) \leq 0.9$.

Proof. For the lower bound, we construct an $F(5,3)$-free 3-graph $H(n,a,b)$, where n is the number of vertices of H and $0 < a < 1,\ 0 < b < 1$.

The vertex set of $H(n,a,b)$ is partitioned into two vertex sets A and B, where $|A| = an$, and meanwhile A is divided into two sets A_1 and A_2 with $|A_1| = abn$ and $|A_2| = a(1-b)n$, respectively. The edges of $H(n,a,b)$ are

- all uvw, where $\{u,v,w\} \cap A \neq \emptyset$ and $\{u,v,w\} \cap B \neq \emptyset$;
- and all xyz, where $|\{x,y,z\} \cap A_1| = 1$ and $|\{x,y,z\} \cap A_2| = 2$.

We assert that $H(n,a,b)$ is $F(5,3)$-free. Otherwise, suppose $H(n,a,b)$ contains an $F(5,3)$. Then there is a $K_5^3 \subseteq H(n,a,b)$. It yields $|V(K_5^3) \cap A_1| = 1$, $|V(K_5^3) \cap A_2| = 2$ and $|V(K_5^3) \cap B| = 2$. But now we can not find the other 3 vertices in $H(n,a,b)$ to make up an $F(5,3)$. So $H(n,a,b)$ is $F(5,3)$-free.

By counting the edge number of $H(n,a,b)$, we have

$$|E(H(n,a,b))| = (1-a)n\binom{an}{2} + an\binom{(1-a)n}{2} + |abn|\binom{(1-b)an}{2}.$$

Set $a = \frac{9-3\sqrt{5}}{4}$ and $b = \frac{1}{3}$, we obtain

$$|E(H(n,a,b))| \geq (0.8176 + o(1))\binom{n}{3}.$$

Thus, we deduce that $\pi(F(5,3)) \geq 0.8176$.

As for the upper bound, by the conclusion 2 of Theorem 3.1, we obtain that

$$\pi(F(5,3)) \leq \frac{5-1+\pi(K_3^2)}{5} = 0.9.$$

This completes the proof. □

So for $p = 5$, we have $\pi(K_5^3) \leq 0.76954 < 0.8176 \leq \pi(F(5,3))$, where $\pi(K_5^3) \leq 0.76954$ is given by Baber[1].

For $6 \leq p \leq 9$, there is no result about the relationship between $\pi(F(p,3))$ and $\pi(K_p^3)$ so far. Here we also propose the following problem.

Problem 4.5. *If $6 \leq p \leq 9$, then $\pi(F(p,3)) = \pi(K_p^3)$?*

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Conflict of Interest

The authors declare no conflict of interest.

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