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On the Existence of Ground State Solutions to a Quasilinear Schrödinger Equation involving p-Laplacian

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Abstract We consider the following quasilinear Schrödinger equation involving p-Laplacian

$$-\Delta_p u + V(x)|u|^{p-2}u - \Delta_p(|u|^{2\eta})|u|^{2\eta-2}u = \lambda\frac{|u|^{q-2}u}{|x|^{\mu}} + \frac{|u|^{2\eta p^*(\nu)-2}u}{|x|^{\nu}} \quad \text{in } \mathbb{R}^N,$$

where $N > p > 1$, $\eta \geq \frac{p}{2(p-1)}$, $p < q < 2\eta p^*(\mu)$, $p^*(s) = \frac{p(N-s)}{N-p}$, and λ, μ, ν are parameters with $\lambda > 0$, $\mu, \nu \in [0, p)$. Via the Mountain Pass Theorem and the Concentration Compactness Principle, we establish the existence of nontrivial ground state solutions for the above problem.

Keywords quasilinear Schrödinger equation; critical Hardy-Sobolev exponent; ground state solutions; singularities

2000 MR Subject Classification 35J62; 35J60; 35J20

1 Introduction

We are concerned with the problem of existence of the ground state solutions satisfying the quasilinear Schrödinger equation with p-Laplacian,

$$-\Delta_p u + V(x)|u|^{p-2}u - \Delta_p(|u|^{2\eta})|u|^{2\eta-2}u = \lambda\frac{|u|^{q-2}u}{|x|^{\mu}} + \frac{|u|^{2\eta p^*(\nu)-2}u}{|x|^{\nu}} \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N > p > 1$, $\eta \geq \frac{p}{2(p-1)}$, $p < q < 2\eta p^*(\mu)$, $p^*(s) = \frac{p(N-s)}{N-p}$, and λ, μ, ν are parameters with $\lambda > 0$, $\mu, \nu \in [0, p)$. The p-Laplacian operator is written as $\Delta_p u := \nabla \cdot \big(|\nabla u|^{p-2}\nabla u\big)$, with $|\nabla u|^{p-2} = \big[\big(\frac{\partial u}{\partial x_1}\big)^2 + \cdots + \big(\frac{\partial u}{\partial x_n}\big)^2\big]^{\frac{p-2}{2}}$. Furthermore, we always need the following assumptions on the potential $V(x)$:

(V) $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf V(x) = V_0 > 0$, and for each $M > 0$, $\operatorname{meas}\{x \in \mathbb{R}^N : V(x) \leq M\} < +\infty$, where V_0 is a constant and meas denotes the Lebesgue measure in $\mathbb{R}^N$.

We also recall that the nontrivial solutions with the least energy to (1.1) are called the ground state solutions of (1.1).

Our motivation of investigating (1.1) comes from the quasilinear Schrödinger equations involving Laplacian:

$$i\partial_t\phi = -\Delta\phi + W(x)\phi - \tilde{f}(|\phi|^2)\phi - \kappa\Delta h(|\phi|^2)h'(|\phi|^2)\phi, \tag{1.2}$$

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where $\phi:\ \mathbb{R}\times\mathbb{R}^N\to\mathbb{C}$, $W:\ \mathbb{R}^N\to\mathbb{R}$ is a given potential, κ is a real constant and $f,h:\mathbb{R}^+\to\mathbb{R}$ are suitable functions. Such equations arise in various branches of mathematical physics. The case $h(s)=s$ was used for the time evolution of the condensate wave function of super-fluid film equation in plasma physics [14, 16]. When $h(s)=(1+s)^{1/2}$, (1.2) models the self-channeling of a high-power ultra short laser in matter, see [4, 8, 11, 22]. Meanwhile, (1.2) was also used in the theory of Heisenberg ferromagnets and magnons [13] and in condensed matter theory [20]. For further physical backgrounds and applications, we refer readers to [3, 6, 17, 21] and references therein.

Let us consider the case $h(s)=s$ and $\kappa>0$ in (1.2). Once we substitute $\phi(t,x)=\exp(-i\beta t)u(x)$ into (1.2) with $\beta\in\mathbb{R}$, we can obtain an equation of elliptic type which has the following formal structure

$$-\Delta u+V(x)u-\kappa(\Delta|u|^2)u=f(u)\ \text{ in }\mathbb{R}^N, \tag{1.3}$$

where $V(x)=W(x)-\beta$ is the potential function and $f(u)=\tilde{f}(|u|^2)u$ is the nonlinearity. According to this substitution of $\phi(t,x)$, u is a solution of (1.3) if and only if ϕ is a standing wave solution to (1.2).

Recently, there are many fruitful mathematical studies focusing on the existence of solutions for (1.3) which is accompanied with different kinds of nonlinearities. In [21], Poppenberg et al. studied (1.3) with $f(u)=\lambda|u|^{q-2}u$ and $\kappa=1$, i.e.,

$$-\Delta u+V(x)u-(\Delta|u|^2)u=\lambda|u|^{q-2}u. \tag{1.4}$$

In particular, $V(x)$ is a bounded potential and $q>2$, $\lambda>0$. They established the existence of positive ground state solutions in one dimensional case via the constrained variational method. By a change of variables in [18], Liu et al. transferred (1.4) to a semilinear equation and proved the existence of positive solution with different settings on the potential $V(x)$ in the Orlicz space when $4<q<22^*$ and $\lambda>0$, where $2^*=\frac{2N}{N-2}$ is the critical Sobolev exponent. do Ó et al. in [12] considered the critical exponent case:

$$-\Delta u+V(x)u-(\Delta|u|^2)u=|u|^{22^*-2}u+|u|^{q-2}u, \tag{1.5}$$

where $4<q<22^*$, $N\geq 3$. Applying a change of variables and the Mountain Pass Theorem, they obtained the existence results for positive solutions with different classes of nonlinearities. Liu et al. [15] extended the method in [12] and studied the general Schrödinger equations with critical growth:

$$-\sum_{i,j=1}^{N}D_j(a_{ij}(u)D_iu)+\frac{1}{2}\sum_{i,j=1}^{N}D_sa_{ij}(u)D_iuD_ju+V(x)u=|u|^{22^*-2}u+|u|^{q-2}u, \tag{1.6}$$

where $4<q<22^*$, $N\geq 3$. It is observed that (1.6) can be reduced into (1.5) with $a_{ij}(u)=(1+2u^2)\delta_{ij}$. The existence of positive solutions with bounded potential was established through the Nehari method. In a recent article [28], through some classical variation techniques, the authors improved the results of [12] with unbounded potential $V(x)$, and they relaxed the restriction on q. Moreover, in [25] we obtained the existence results with the nonlinearity $f(u)$ including some singularities, which extended the results of [28].

We now describe our approach of proving the main result. Recall that the nontrivial solutions of (1.1) correspond to the nontrivial critical points of the following energy functional

$$\begin{aligned}I(u)=\frac{1}{p}\int_{\mathbb{R}^N}\big(1+(2\eta)^{p-1}|u|^{p(2\eta-1)}\big)|\nabla u|^pdx+\frac{1}{p}\int_{\mathbb{R}^N}V(x)|u|^pdx\\ -\frac{\lambda}{q}\int_{\mathbb{R}^N}\frac{|u|^q}{|x|^\mu}dx-\frac{1}{2\eta p^*(\nu)}\int_{\mathbb{R}^N}\frac{|u|^{2\eta p^*(\nu)}}{|x|^\nu}dx.\end{aligned} \tag{1.7}$$

From a variational viewpoint, the major difficulties stem from two aspects: one is that the energy functional (1.7) is not well defined in its domain $X = \{u \in W^{1,p}(\mathbb{R}^N) : u^{2\eta} \in W^{1,p}(\mathbb{R}^N), |V|^{\frac{1}{p}} u \in L^p(\mathbb{R}^N)\}$; the other is the absence of the compactness of the embedding $X \hookrightarrow L^{2\eta p^*(\nu)}(\mathbb{R}^N, |x|^{-\nu})$ for each fixed ν, with the critical-like exponent $2\eta p^*(\nu)$. Therefore, our approach takes a different route. Namely, inspired by the work [9, 18] and [24], we first use a change of variable $v = f^{-1}(u)$ to reformulate the quasilinear problem (1.1) to a semilinear problem. We find that the corresponding functional (defined in (2.1)) is well-defined in a suitable subspace E (which we will define later) of the Sobolev space $W^{1,p}$ and satisfy the geometric structure of the Mountain Pass Theorem (see [1]). And it is equivalent to find the critical point of the functional (2.1) instead of searching the critical points of energy functional (1.7). With application of a version of the Mountain Pass Theorem (see Theorem 3.1, also see [23, 27]) without compactness condition, we obtain the existence of a Cerami sequence associated with the minimax level c. Eventually, we get the existency of nontrivial critical point of (2.1) by taking advantage of the Cerami sequence and some technical results from Lions (see [10, 27]). In other words, it gives the existence of nontrivial solution to (1.1). Furthermore, we can also show that the value of the functional (1.7) evaluated at this critical point is less than or equal to the mountain pass minimax level which is attained. Now we may state our main result.

Theorem 1.1. *Let $N > p > 1$, $\eta \geq \frac{p}{2(p-1)}$. Suppose that the assumption (V) is satisfied. For every fixed μ, $\nu \in [0, p)$, we have the following statements.*

(I) If $2\eta p^*(\mu) - \frac{p}{p-1} < q < 2\eta p^*(\mu)$, the problem (1.1) possesses a ground state solution for any $\lambda > 0$.

(II) If $p < q \leq 2\eta p^*(\mu) - \frac{p}{p-1}$, there exists a positive constant λ^*, the problem (1.1) possesses a ground state solution for $\lambda > \lambda^*$.

The outline of the paper is as follows. We give some preliminaries and a reformulation of the problem (1.1) in Section 2. Section 3 is devoted to the existence result via the verification of the Mountain Pass geometric structure. In Section 4 we prove Theorem 1.1.

Notation. In this paper, we will use the following notations frequently:

- C denotes the universal positive constant unless specified.
- $\mathcal{C}^\infty$ denotes the space of the functions which are infinitely differentiable on $\mathbb{R}^N$.
- $\mathcal{C}_0^\infty(\mathbb{R}^N)$ denotes the space of the functions which are infinitely differentiable and compactly supported in $\mathbb{R}^N$.
- $L^s(\mathbb{R}^N, |x|^{-\sigma})$, $1 < s < \infty$, $0 \leq \sigma < p$, denotes the Lebesgue space with the norms

$$|u|_s^s = \int_{\mathbb{R}^N} \frac{|u|^s}{|x|^\sigma} dx.$$

- $\mathcal{D}^{1,p}(\mathbb{R}^N)$ is the closure of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_{\mathcal{D}}^p = \int_{\mathbb{R}^N} |\nabla u|^p dx$, where $\mathcal{D}$ denotes the closure of $\mathcal{C}_0^\infty$.
- $W^{1,p}(\mathbb{R}^N)$ denotes the usual Sobolev spaces modeled in $L^p(\mathbb{R}^N)$ with the inner product

$$\langle u, v\rangle_W = \int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv\right) dx,$$

  and the norm $\|u\|_W^p = \langle u, u\rangle_W$.

2 Reformulation of the Problem and Preliminaries

In this section, as we mentioned before, we will use a change of variable to transfer the problem (1.1) to a semilinear problem at first. This guarantees that we can find an appropriate function space as our working station. Then we give some property on this function space.

Firstly, we define a new function space E as below

$$E=\Big\{u\in W^{1,p}(\mathbb{R}^N):\int_{\mathbb{R}^N}V(x)|u|^p dx<\infty\Big\}$$

endowed with the inner product

$$\langle u,v\rangle=\int_{\mathbb{R}^N}\big(|\nabla u|^{p-2}\nabla u\nabla v+V(x)|u|^{p-2}uv\big)dx,$$

and the associated norm $\|u\|^p=\langle u,u\rangle$. By the assumption (V), it is easy to see that both $W^{1,p}(\mathbb{R}^N)$ and E are Banach spaces, and we have the continuous imbedding $E\hookrightarrow W^{1,p}(\mathbb{R}^N)$. Moreover, analogous to [7] (or [29]) and [25], we can obtain the continuity and compactness for the embedding from E to $L^s(\mathbb{R}^N,|x|^{-\sigma})$ with ease in the following lemma.

Lemma 2.1. *Let $0\le\sigma<p$. The embedding $E\hookrightarrow L^s(\mathbb{R}^N,|x|^{-\sigma})$ is continuous for $p\le s\le p^*(\sigma)$ and compact for $p\le s<p^*(\sigma)$ provided $V(x)$ satisfies the assumption (V).*

Motivated by [9] and [18], we define a $\mathcal{C}^\infty$ function f as below:

$$f(-t)=-f(t)\ \text{ on } (-\infty,0],\qquad f'(t)=\frac{1}{\big(1+(2\eta)^{p-1}|f(t)|^{p(2\eta-1)}\big)^{1/p}}\ \text{ on } [0,+\infty).$$

After a change of variable $v=f^{-1}(u)$, we can transfer the functional $I(u)$ into the following

$$\begin{aligned}J(v)=&\frac{1}{p}\int_{\mathbb{R}^N}\big[|\nabla v|^p+V(x)|f(v)|^p\big]dx-\frac{\lambda}{q}\int_{\mathbb{R}^N}\frac{|f(v)|^q}{|x|^\mu}dx\\&-\frac{1}{2\eta p^*(\nu)}\int_{\mathbb{R}^N}\frac{|f(v)|^{2\eta p^*(\nu)}}{|x|^\nu}dx,\end{aligned}\tag{2.1}$$

which is well defined on E. Via a standard argument, it is readily to see that $J\in\mathcal{C}^1(E,\mathbb{R})$ and

$$\begin{aligned}\langle J'(v),\varphi\rangle\ =&\int_{\mathbb{R}^N}\Big[|\nabla v|^{p-2}\nabla v\nabla\varphi+V(x)|f(v)|^{p-2}f(v)f'(v)\varphi\\&-\lambda\frac{|f(v)|^{q-2}}{|x|^\mu}f(v)f'(v)\varphi-\frac{|f(v)|^{2\eta p^*(\nu)-2}}{|x|^\nu}f(v)f'(v)\varphi\Big]dx,\end{aligned}\tag{2.2}$$

for all v, $\varphi\in E$. According to this reformulation, we observe that if v is a critical point of the functional J, the function $u=f(v)$ is then a solution to problem (1.1).

Finally, we conclude this section recording some properties of f which will be required in the subsequent of the paper.

Proposition 2.2. *The function $f(t)$ enjoys the following properties:*

(f_1) *f is a uniquely defined, invertible $\mathcal{C}^\infty$-function;*
(f_2) *$0<f'(t)\le 1$ for all $t\in\mathbb{R}$;*
(f_3) *$|f(t)|\le|t|$ for all $t\in\mathbb{R}$;*
(f_4) *$|f(t)|\le(2\eta)^{\frac{1}{2\eta p}}|t|^{\frac{1}{2\eta}}$ for all $t\in\mathbb{R}$;*
(f_5) *There exists a positive constant C such that*

$$|f(t)|\ge\begin{cases}C|t|, & |t|\le 1,\\ C|t|^{\frac{1}{2\eta}}, & |t|\ge 1;\end{cases}$$

(f_6) $\frac{1}{2\eta}f(t) \le f'(t)t \le f(t)$ *for all* $t \ge 0$;

(f_7) $|f(t)|^{2\eta-1}|f'(t)| \le (2\eta)^{\frac{1-p}{p}}$ *for all* $t \in \mathbb{R}$.

The proof of properties (f_1)–(f_6) is similar to that in [24] and [26]. We can show that property (f_7) is also true through the property (f_6) and a direct calculation. We leave the proof for the interested readers.

3 Mountain Pass Geometric Structure

In this section, we will first state a version of the Mountain Pass Theorem (see Theorem 3.1) which gives the framework of our proof to the main result. It is clear that the proof involves a combination of various ingredients. We start with the verification of the geometric structure of the Mountain Pass Theorem, and present some properties to the Cerami sequences of the associated functional, that is, the boundedness of the Cerami sequences.

3.1 A version of the Mountain Pass Theorem

Let D be a real Banach space and $\mathcal{I} : D \to \mathbb{R}$ a functional of class $\mathcal{C}^1$. For a given $b \in \mathbb{R}$, we define $\mathcal{I}^b = \{u \in D : \mathcal{I}(u) \le b\}$. As we mentioned in the introduction, the functional $I(u)$ in (1.7) does not satisfy a compactness condition of Palais-Smale type. Therefore, we may apply a version of Mountain Pass Theorem (see [23]). Recall that $\{v_n\} \subset D$ is a Cerami sequence of $\mathcal{I}$, denoted by $(C)_c$-sequence, if $\mathcal{I}(v_n) \to c$ and $(1 + \|v_n\|)\mathcal{I}'(v_n) \to 0$ as $n \to \infty$, for any $c \in \mathbb{R}$. Now we recall the modified Mountain Pass Theorem in [23] for convenience.

Theorem 3.1. *Let D be a real Banach space and $\mathcal{I} \in \mathcal{C}^1(D, \mathbb{R})$. Let S be a closed subset of D which disconnects (archwise) D in distinct connected components D_1 and D_2. Moreover, assume that $\mathcal{I}(0) = 0$ and*

(M1) $0 \in D_1$ and there exists $\alpha > 0$ such that $\mathcal{I}\big|_S \ge \alpha > 0$,

(M2) there is $e \in D_2$ such that $\mathcal{I}(e) \le 0$.

Then $\mathcal{I}$ possesses a $(C)_c$-sequence with $c \ge \alpha > 0$ provided

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{I}(\gamma(t)),$$

where $\Gamma = \left\{\gamma \in \mathcal{C}([0, 1], D)\big|\gamma(0) = 0, \gamma(1) \in \mathcal{I}^0 \bigcap D_2\right\}$.

3.2 Verification of Mountain Pass Geometry

Define

$$S(\rho) = \Big\{v \in E : \int_{\mathbb{R}^N} \big[|\nabla v|^p + V(x)|f(v)|^p\big] dx = \rho^p\Big\}.$$

The lemma below will show that the functional $J(v)$ in (2.1) exhibits the Mountain Pass geometric structure.

Lemma 3.2. *Let $\mu, \nu \in [0, p)$ be fixed. Suppose that (V) and $(f_1) - (f_7)$ are satisfied. Then the functional J in (2.1) satisfies $J(0) = 0$, and conditions $(M1)$ and $(M2)$ of Theorem 3.1.*

Proof. Firstly, we note that $J(0) = 0$. For any $\rho \in \mathbb{R}$ and the definition of $S(\rho)$ given above, it is clear that $S(\rho)$ is a closed subset which disconnects the space E. From (f_7), we have

$$\int_{\mathbb{R}^N} |\nabla(f^{2\eta}(v))|^p dx =(2\eta)^p \int_{\mathbb{R}^N} |f(v)|^{p(2\eta-1)}|f'(v)|^p|\nabla v|^p dx$$

$$\leq 2\eta \int_{\mathbb{R}^N} |\nabla v|^p dx$$
$$\leq 2\eta \int_{\mathbb{R}^N} \big[|\nabla v|^p + V(x)|f(v)|^p\big] dx. \tag{3.1}$$

For any $v \in S(\rho)$, by the Sobolev-Hardy inequality and (3.1), it then follows that

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|f(v)|^{2\eta p^*(\mu)}}{|x|^\mu} dx &\leq C\Big[\int_{\mathbb{R}^N} \big|\nabla(f^{2\eta}(v))\big|^p dx\Big]^{\frac{p^*(\mu)}{p}} \\ &\leq C\Big[\int_{\mathbb{R}^N} \big(|\nabla v|^p + V(x)|f(v)|^p\big) dx\Big]^{\frac{p^*(\mu)}{p}} \\ &\leq C\rho^{p^*(\mu)} .\end{aligned} \tag{3.2}$$

Similarly, we also have

$$\int_{\mathbb{R}^N} \frac{|f(v)|^{2\eta p^*(\nu)}}{|x|^\nu} dx \leq C\rho^{p^*(\nu)}. \tag{3.3}$$

Furthermore, for any fixed $\mu \in [0, p)$ and $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that

$$\frac{|t|^q}{|x|^\mu} \leq \varepsilon \frac{|t|^p}{|x|^\mu} + C(\varepsilon)\frac{|t|^{2\eta p^*(\mu)}}{|x|^\mu},$$

with $p < q < 2\eta p^*(\mu)$. Together with the inequality above, it follows from (f_3), (3.2) and the Sobolev-Hardy inequality that

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|f(v)|^q}{|x|^\mu} dx &\leq \varepsilon \int_{\mathbb{R}^N} \frac{|f(v)|^p}{|x|^\mu} dx + C(\varepsilon)\int_{\mathbb{R}^N} \frac{|f(v)|^{2\eta p^*(\mu)}}{|x|^\mu} dx \\ &\leq C\varepsilon \int_{\mathbb{R}^N} |\nabla v|^p dx + C\cdot C(\varepsilon)\rho^{p^*(\mu)} \\ &\leq C\varepsilon\rho^p + C\cdot C(\varepsilon)\rho^{p^*(\mu)}.\end{aligned} \tag{3.4}$$

Thus, it follows from (3.2)–(3.4) that for sufficiently small enough $\varepsilon > 0$, we have

$$\begin{aligned}J(v) &= \frac{1}{p}\int_{\mathbb{R}^N} \big[|\nabla v|^p + V(x)|f(v)|^p\big] dx - \frac{\lambda}{q}\int_{\mathbb{R}^N} \frac{|f(v)|^q}{|x|^\mu} dx - \frac{1}{2\eta p^*(\nu)}\int_{\mathbb{R}^N} \frac{|f(v)|^{2\eta p^*(\nu)}}{|x|^\nu} dx \\ &\geq \Big[\frac{1}{p} - \frac{\lambda}{q}C\varepsilon\Big]\rho^p - \frac{\lambda}{q}C\cdot C(\varepsilon)\rho^{p^*(\mu)} - \frac{1}{2\eta p^*(\nu)}C\rho^{p^*(\nu)} \\ &\geq \frac{1}{2p}\rho^p - C\cdot C(\varepsilon)\rho^{p^*(\mu)} - C\rho^{p^*(\nu)}.\end{aligned}$$

Taking a suitable positive ρ_0 with $\frac{1}{2p}\rho_0^p - C\cdot C(\varepsilon)\rho_0^{p^*(\mu)} - C\cdot\rho_0^{p^*(\nu)} > 0$, we denote this expression by α and it is clear that $\alpha > 0$ by the choice of ρ_0 and ε. We then conclude that $J(v) \geq \alpha > 0$ for all $v \in S(\rho_0)$. This shows that the condition $(M1)$ is satisfied.

For the verification of condition $(M2)$, it suffices to show that $J(t\psi)$ tends to $-\infty$ as t goes to $+\infty$ for a given $\psi \in E \cap L^{2\eta p^*(\nu)}(\mathbb{R}^N)$ with $0 < \psi \leq 1$. Indeed, from (f_6) of Proposition 2.2, it implies that $\frac{f(t)}{t}$ is decreasing for $t > 0$. This in turn yields that

$$\frac{f(t\psi(x))}{t\psi(x)} \geq \frac{f(t)}{t} \qquad \text{for} \quad t \geq t\psi(x) > 0,$$

i.e.

$$f(t\psi(x)) \geq f(t)\psi(x). \tag{3.5}$$

Therefore, from (f_3) and (f_5) of Proposition 2.2 and (3.5), we obtain for $t \geq 1$,

$$\begin{aligned}
J(t\psi) &\leq \frac{1}{p}\int_{\mathbb{R}^N}\left[|\nabla(t\psi)|^p + V(x)|f(t\psi)|^p\right]dx - \frac{1}{2\eta p^*(\nu)}\int_{\mathbb{R}^N}\frac{|f(t\psi)|^{2\eta p^*(\nu)}}{|x|^\nu}dx \\
&\leq \frac{t^p}{p}\int_{\mathbb{R}^N}\left[|\nabla\psi|^p + V(x)|\psi|^p\right]dx - \frac{1}{2\eta p^*(\nu)}\int_{\mathbb{R}^N}\frac{|f(t)\psi|^{2\eta p^*(\nu)}}{|x|^\nu}dx \\
&\leq \frac{t^p}{p}\int_{\mathbb{R}^N}\left[|\nabla\psi|^p + V(x)|\psi|^p\right]dx - Ct^{p^*(\nu)}\int_{\mathbb{R}^N}\frac{|\psi|^{2\eta p^*(\nu)}}{|x|^\nu}dx \\
&\to -\infty, \quad \text{as } t \to +\infty,
\end{aligned}$$

since $p^*(\nu) > p$ for $\nu \in [0, p)$. We thus verify condition $(M2)$, and finish our proof. □

3.3 Cerami Sequences

In this part we begin by showing the boundedness of the $(C)_c$-sequence associated to the functional J in (2.1).

Lemma 3.3. *Every $(C)_c$-sequence in E corresponding to J is bounded in $W^{1,p}(\mathbb{R}^N)$.*

Proof. Let $\{v_n\} \subset E$ be any $(C)_c$-sequence of J at level c, that is,

$$J(v_n) \to c \ \text{ and } \ (1 + \|v_n\|)J'(v_n) \to 0 \qquad \text{as } \ n \to \infty.$$

Choose $\varphi_n = \frac{f(v_n)}{f'(v_n)}$. By the definition of f' and the fact $v_n \in E$, it is easy to see that $\varphi_n \in E$. It then follows from (f_3) and (f_4) that $\|\varphi_n\| \leq 2\eta\|v_n\|$ and $\langle J'(v_n), \varphi_n\rangle \to 0$ as $n \to \infty$.

Now we define two functions $\psi(t,x), \Psi(t,x) \in \mathbb{R}$ with

$$\psi(t,x) = \lambda\frac{|t|^{q-2}t}{|x|^\mu} + \frac{|t|^{2\eta p^*(\nu)-2}t}{|x|^\nu}, \qquad \Psi(t,x) = \int_0^t \psi(s,x)ds.$$

Considering $\sigma = \max\{\mu,\nu\}$, we find a constant $\tau \in (2\eta p, 2\eta p^*(\nu))$ such that

$$\lim_{t\to 0}\frac{|x|^\sigma[t\psi(t,x) - \tau\Psi(t,x)]}{t^p} = 0,$$

and

$$\lim_{|t|\to+\infty}\frac{|x|^\sigma[t\psi(t,x) - \tau\Psi(t,x)]}{t^\tau} = +\infty \ \text{ uniformly for } x \in \mathbb{R}^N.$$

Therefore, for any $x \in \mathbb{R}^N$, there exists $r > 0$ such that

$$t\psi(t,x) - \tau\Psi(t,x) \geq 0, \quad \text{for any } |t| > r. \tag{3.6}$$

Furthermore, by the fact $\eta \geq \frac{p}{2(p-1)} > \frac{1}{2}$, we have $\tau > 2\eta p > p$. It then follows that, for any $\varepsilon > 0$, there exists a positive constant $C(\varepsilon)$ such that

$$|t\psi(t,x) - \tau\Psi(t,x)| \leq \varepsilon\frac{|t|^p}{|x|^\sigma} + C(\varepsilon)\frac{|t|^{2\eta p^*(\nu)}}{|x|^\sigma}, \quad \text{for any } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^N. \tag{3.7}$$

Therefore, we may obtain from (3.6) that

$$
\begin{aligned}
c+o(1) &= J(v_n)-\frac{1}{\tau}\langle J'(v_n),\varphi_n\rangle \\
&= \frac{1}{p}\int_{\mathbb{R}^N}|\nabla v_n|^p dx-\frac{1}{\tau}\int_{\mathbb{R}^N}\Big[1+\frac{(2\eta)^{p-1}(2\eta-1)|f(v_n)|^{p(2\eta-1)}}{1+(2\eta)^{p-1}|f(v_n)|^{p(2\eta-1)}}\Big]|\nabla v_n|^p dx \\
&\quad+\Big(\frac{1}{p}-\frac{1}{\tau}\Big)\int_{\mathbb{R}^N}V(x)|f(v_n)|^p dx+\int_{\mathbb{R}^N}\Big[\frac{1}{\tau}\psi(f(v_n),x)f(v_n)-\Psi(f(v_n),x)\Big]dx \\
&\geq \Big(\frac{1}{p}-\frac{2\eta}{\tau}\Big)\int_{\mathbb{R}^N}|\nabla v_n|^p dx+\Big(\frac{1}{p}-\frac{1}{\tau}\Big)\int_{\mathbb{R}^N}V(x)|f(v_n)|^p dx \\
&\quad+\int_{B_f}\Big[\frac{1}{\tau}\psi(f(v_n),x)f(v_n)-\Psi(f(v_n),x)\Big]dx,
\end{aligned}
\tag{3.8}
$$

where $B_f=\big\{x\in\mathbb{R}^N:|f(v_n)|\leq r\big\}$. In order to conclude the proof, we need to estimate some terms of the above inequality. We can find that, from (3.7), there is a constant $M>V_0$ such that

$$
\Big|\frac{1}{\tau}t\psi(t,x)-\Psi(t,x)\Big|\leq\Big(\frac{1}{2p}-\frac{1}{2\tau}\Big)M\frac{|t|^p}{|x|^\sigma},\qquad \text{for any}\ \ |t|\leq r,\ \ x\in\mathbb{R}^N, \tag{3.9}
$$

where V_0 is given in (V).

Denote $B_V:=\{x\in\mathbb{R}^N:V(x)\leq M\}$. Applying the estimate (3.9) together with the condition (V), we get

$$
\begin{aligned}
&\Big(\frac{1}{2p}-\frac{1}{2\tau}\Big)\int_{\mathbb{R}^N}V(x)|f(v_n)|^p dx+\int_{B_f}\Big[\frac{1}{\tau}\psi(f(v_n),x)f(v_n)-\Psi(f(v_n),x)\Big]dx \\
\geq&\Big(\frac{1}{2p}-\frac{1}{2\tau}\Big)\int_{B_f\cap\{x\in\mathbb{R}^N:|x|>1\}}V(x)|f(v_n)|^p dx-\Big(\frac{1}{2p}-\frac{1}{2\tau}\Big)M\int_{B_f}\frac{|f(v_n)|^p}{|x|^\sigma}dx \\
=&\Big(\frac{1}{2p}-\frac{1}{2\tau}\Big)\int_{B_f\cap\{x\in\mathbb{R}^N:|x|>1\}}V(x)|f(v_n)|^p dx \\
&-\Big(\frac{1}{2p}-\frac{1}{2\tau}\Big)M\int_{B_f\cap\{x\in\mathbb{R}^N:|x|>1\}}\frac{|f(v_n)|^p}{|x|^\sigma}dx \\
&-\Big(\frac{1}{2p}-\frac{1}{2\tau}\Big)M\int_{B_f\cap\{x\in\mathbb{R}^N:|x|\leq1\}}\frac{|f(v_n)|^p}{|x|^\sigma}dx \\
\geq&\Big(\frac{1}{2p}-\frac{1}{2\tau}\Big)\int_{B_f\cap\{x\in\mathbb{R}^N:|x|>1\}}\big(V(x)-M\big)|f(v_n)|^p dx \\
&-\Big(\frac{1}{2p}-\frac{1}{2\tau}\Big)M\int_{B_f\cap\{x\in\mathbb{R}^N:|x|\leq1\}}\frac{|f(v_n)|^p}{|x|^\sigma}dx \\
\geq&\Big(\frac{1}{2p}-\frac{1}{2\tau}\Big)\int_{B_V\cap B_f\cap\{x\in\mathbb{R}^N:|x|>1\}}(V_0-M)r^p dx \\
&-\Big(\frac{1}{2p}-\frac{1}{2\tau}\Big)M\int_{B_f\cap\{x\in\mathbb{R}^N:|x|\leq1\}}\frac{r^p}{|x|^\sigma}dx \\
\geq&\Big(\frac{1}{2p}-\frac{1}{2\tau}\Big)(V_0-M)r^p\text{meas}\,\big(B_V\cap B_f\cap\{x\in\mathbb{R}^N:\ |x|>1\}\big) \\
&-\Big(\frac{1}{2p}-\frac{1}{2\tau}\Big)Mr^p\int_{\{x\in\mathbb{R}^N:|x|\leq1\}}\frac{1}{|x|^\sigma}dx \\
\geq&\Big(\frac{1}{2p}-\frac{1}{2\tau}\Big)(V_0-M)r^p\text{meas}\,(B_V)-\Big(\frac{1}{2p}-\frac{1}{2\tau}\Big)Mr^p\int_0^1\rho^{N-\sigma-1}d\rho
\end{aligned}
$$

$$\geq\Big(\frac{1}{2p}-\frac{1}{2\tau}\Big)(V_0-M)r^p\text{meas}\,(B_V)-\Big(\frac{1}{2p}-\frac{1}{2\tau}\Big)Mr^p\ ,$$

which implies that

$$\begin{aligned}&\Big(\frac{1}{p}-\frac{2\eta}{\tau}\Big)\int_{\mathbb{R}^N}|\nabla v_n|^p dx+\Big(\frac{1}{2p}-\frac{1}{2\tau}\Big)\int_{\mathbb{R}^N}V(x)|f(v_n)|^p dx\\ \leq&\Big(\frac{1}{2p}-\frac{1}{2\tau}\Big)r^p\big[(M-V_0)\text{meas}(B_V)+M\big]+c+o(1).\end{aligned}\tag{3.10}$$

Since $meas(B_V)$ is finite according to the assumption (V), it follows that we have

$$\int_{\mathbb{R}^N}\big[|\nabla v_n|^p+V(x)|f(v_n)|^p\big]dx\leq C.$$

Consequently, by (f_5) and Sobolev inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^N}|v_n|^p dx=&\int_{\{|v_n|\leq 1\}}|v_n|^p dx+\int_{\{|v_n|>1\}}|v_n|^p dx\leq C\int_{\mathbb{R}^N}V(x)|f(v_n)|^p dx+\int_{\mathbb{R}^N}|v_n|^{p^*}dx\\ \leq& C\int_{\mathbb{R}^N}V(x)|f(v_n)|^p dx+C\Big[\int_{\mathbb{R}^N}|\nabla v_n|^p dx\Big]^{\frac{p^*}{p}}<+\infty,\end{aligned}$$

where $p^*=\frac{Np}{N-p}$ is the critical Sobolev exponent. Hence $\{v_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$, which implies the desired result. □

The other important property of the $(C)_c$-sequence is nonvanishing. It is a crucial tool to show that the critical point of (2.1) is nonzero.

Proposition 3.4. *Let $N>p>1$, $\eta\geq\frac{p}{2(p-1)}$ and $\mu,\ \nu\in[0,p)$ be fixed. Suppose $\{v_n\}\subset E$ be a $(C)_c$-sequence of J with*

$$c<\frac{p-\nu}{2\eta p(N-\nu)}S_\nu^{\frac{N-\nu}{p-\nu}}.\tag{3.11}$$

Then there exist positive constants R and ξ, and a sequence $\{y_n\}\subset\mathbb{R}^N$, such that

$$\limsup_{n\to\infty}\int_{B_R(y_n)}|v_n|^p dx\geq\xi.$$

Proof. Suppose that the conclusion is not true. It then follows from Lemma 1.21 in [27] that $v_n\to 0$ in $L^s(\mathbb{R}^N)$ for all $p<s<p^*$. By Hölder's inequality, Hardy's inequality and Lemma 3.3, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{|v_n|^q}{|x|^\mu}dx\leq&\Big(\int_{\mathbb{R}^N}\frac{|v_n|^p}{|x|^p}dx\Big)^{\frac{\mu}{p}}\Big(\int_{\mathbb{R}^N}|v_n|^{\frac{p(q-\mu)}{p-\mu}}dx\Big)^{\frac{p-\mu}{p}}\\ \leq& C\Big(\int_{\mathbb{R}^N}|v_n|^{\frac{p(q-\mu)}{p-\mu}}dx\Big)^{\frac{p-\mu}{p}}.\end{aligned}$$

By the fact $p<q<p^*(\mu)$ and $0\leq\mu<p$, it yields that $p<\frac{p(q-\mu)}{p-\mu}<p^*$. Then we can obtain $v_n\to 0$ in $L^{\frac{p(q-\mu)}{p-\mu}}(\mathbb{R}^N)$, which implies that

$$v_n\to 0\text{ in }L^q(\mathbb{R}^N,|x|^{-\mu}),\qquad\text{for}\ \ p<q<p^*(\mu).$$

By (f_3), (f_4) and the interpolation inequality, this in turn implies that

$$f(v_n) \to 0 \text{ in } L^q(\mathbb{R}^N, |x|^{-\mu}), \qquad \text{for } \ p < q < 2\eta p^*(\mu). \tag{3.12}$$

Thanks to Lemma 3.3, we may assume that, passing to a subsequence of $\{v_n\} \subset E$ (still denoted by$\{v_n\}$),

$$\int_{\mathbb{R}^N} \Big(1 + \frac{(2\eta)^{p-1}(2\eta-1)|f(v_n)|^{p(2\eta-1)}}{1+(2\eta)^{p-1}|f(v_n)|^{p(2\eta-1)}}\Big)|\nabla v_n|^p dx + \int_{\mathbb{R}^N} V(x)|f(v_n)|^p dx \to b,$$

and

$$\int_{\mathbb{R}^N} \frac{|f(v_n)|^{2\eta p^*(\nu)}}{|x|^\nu} dx \to d.$$

On the other hand, we know that

$$\begin{aligned} S_\nu\Big(\int_{\mathbb{R}^N} \frac{|f(v_n)|^{2\eta p^*(\nu)}}{|x|^\nu} dx\Big)^{\frac{p}{p^*(\nu)}} &\le \int_{\mathbb{R}^N} |\nabla(f^{2\eta}(v_n))|^p dx \\ = \int_{\mathbb{R}^N} \frac{(2\eta)^p|f(v_n)|^{p(2\eta-1)}}{1+(2\eta)^{p-1}|f(v_n)|^{p(2\eta-1)}}|\nabla v_n|^p dx \\ \le \int_{\mathbb{R}^N} \Big(1 + \frac{(2\eta)^{p-1}(2\eta-1)|f(v_n)|^{p(2\eta-1)}}{1+(2\eta)^{p-1}|f(v_n)|^{p(2\eta-1)}}\Big)|\nabla v_n|^p dx \\ + \int_{\mathbb{R}^N} V(x)|f(v_n)|^p dx, \end{aligned} \tag{3.13}$$

that is,

$$\begin{aligned} S_\nu\Big(\int_{\mathbb{R}^N} \frac{|f(v_n)|^{2\eta p^*(\nu)}}{|x|^\nu} dx\Big)^{\frac{p}{p^*(\nu)}} \le \int_{\mathbb{R}^N} \Big(1 + \frac{(2\eta)^{p-1}(2\eta-1)|f(v_n)|^{p(2\eta-1)}}{1+(2\eta)^{p-1}|f(v_n)|^{p(2\eta-1)}}\Big)|\nabla v_n|^p dx \\ + \int_{\mathbb{R}^N} V(x)|f(v_n)|^p dx. \end{aligned} \tag{3.14}$$

Taking limit $n \to \infty$ to the both sides of the inquality (3.14), we obtain $S_\nu d^{\frac{p}{p^*(\nu)}} \le b$. Moreover, in view of (3.12), it follows that $0 = \lim_{n\to\infty} \langle J'(v_n), w_n\rangle = b - d$, where $w_n = \frac{f(v_n)}{f'(v_n)}$. Therefore, $b = d \ge S_\nu^{\frac{N-\nu}{p-\nu}}$.

On the other hand, by (3.12) again, we get that

$$\begin{aligned} c &= \lim_{n\to\infty} J(v_n) \\ &= \lim_{n\to\infty} \Big[\frac{1}{p}\int_{\mathbb{R}^N} \big(|\nabla v_n|^p + V(x)|f(v_n)|^p\big)dx - \frac{1}{2\eta p^*(\nu)}\int_{\mathbb{R}^N} \frac{|f(v_n)|^{2\eta p^*(\nu)}}{|x|^\nu} dx\Big] \\ &\ge \lim_{n\to\infty} \Big[\frac{1}{2\eta p}\int_{\mathbb{R}^N} \Big(1 + \frac{(2\eta)^{p-1}(2\eta-1)|f(v_n)|^{p(2\eta-1)}}{1+(2\eta)^{p-1}|f(v_n)|^{p(2\eta-1)}}\Big)|\nabla v_n|^p dx + \frac{1}{2\eta p}\int_{\mathbb{R}^N} V(x)|f(v_n)|^p dx \\ &\qquad - \frac{1}{2\eta p^*(\nu)}\int_{\mathbb{R}^N} \frac{|f(v_n)|^{2\eta p^*(\nu)}}{|x|^\nu} dx\Big] \\ &= \Big(\frac{1}{2\eta p} - \frac{1}{2\eta p^*(\nu)}\Big)d \ge \frac{p-\nu}{2\eta p(N-\nu)} S_\nu^{\frac{N-\nu}{p-\nu}}, \end{aligned}$$

that is,

$$c \ge \frac{p-\nu}{2\eta p(N-\nu)} S_\nu^{\frac{N-\nu}{p-\nu}},$$

contradicting to (3.11). Hence we finish the proof. □

3.4 Estimate for the Minimax Level

In this part we verify the condition (3.11) to guarantee the employment of the modified Mountain Pass Theorem (see Theorem 3.1). To show this, we use some appropriate test functions. With some auxiliary properties of these chosen functions, we then prove the main result of this part.

For given $\epsilon > 0$, we consider the function $w_\epsilon : \mathbb{R}^N \to \mathbb{R}$ defined by

$$w_\epsilon(x) = \frac{C_{\nu,\epsilon}}{(\epsilon + |x|^{\frac{p-\nu}{p-1}})^{\frac{N-p}{p-\nu}}},$$

where

$$C_{\nu,\epsilon} = [(N-\nu)(N-p)^{p-1}(p-1)^{p-1}\epsilon]^{\frac{N-p}{p(p-\nu)}}.$$

It is well known that the minimization problem

$$S_\nu = \inf\Big\{ \int_{\mathbb{R}^N} |\nabla v|^p dx : v \in \mathcal{D}^{1,p}(\mathbb{R}^N), \int_{\mathbb{R}^N} \frac{|v|^{p^*(\nu)}}{|x|^\nu} dx = 1 \Big\} \tag{3.15}$$

can be achieved by w_ϵ defined above. And the infimum S_ν actually is the best constant of the Sobloev embedding $\mathcal{D}^{1,p} \hookrightarrow L^{p^*(\nu)}(\mathbb{R}^N, |x|^{-\nu})$ for $1 < p < +\infty$, $0 \le \nu < p$. Moreover, define a new setting on $w_\epsilon(x)$, that is, let $\bar{w}_\epsilon = w_\epsilon^{\frac{1}{2\eta}}$. We observe that $\bar{w}_\epsilon^{2\eta}$ satisfies the following equation

$$-\Delta_p w = \frac{w^{p^*(\nu)-1}}{|x|^\nu},$$

it is also a minimizer of the minimization problem (3.15). Now let $0 < R < 1$. We consider a smooth cut-off function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N, [0,1])$ such that $\varphi(x) = 1$ for $|x| \le R$, $0 < \varphi(x) < 1$ for $R < |x| < 2R$, and $\varphi(x) = 0$ for $|x| \ge 2R$, and define

$$u_\epsilon = \varphi \bar{w}_\epsilon. \tag{3.16}$$

By a similar argument as the one in [2, 5] with $\eta \ge \frac{p}{2(p-1)}$, we have the following lemma. Since the process is standard, we omit the proof here.

Lemma 3.5. *Let $N > p > 1$, $\eta \ge \frac{p}{2(p-1)}$, and u_ϵ as defined in (3.16). Then, we have:*

$$\int_{\mathbb{R}^N} |\nabla(u_\epsilon^{2\eta})|^p dx = S_\nu^{\frac{N-\nu}{p-\nu}} + O\big(\epsilon^{\frac{N-p}{p-\nu}}\big), \qquad \int_{\mathbb{R}^N} \frac{|u_\epsilon|^{2\eta p^*(\nu)}}{|x|^\nu} dx = S_\nu^{\frac{N-\nu}{p-\nu}} + O\big(\epsilon^{\frac{N-\nu}{p-\nu}}\big), \tag{3.17}$$

$$\int_{\mathbb{R}^N} |\nabla u_\epsilon|^p dx \le O\big(\epsilon^{\frac{N-p}{2\eta(p-\nu)}} |\ln \epsilon|\big), \qquad \int_{\mathbb{R}^N} |u_\epsilon|^p dx \le O\big(\epsilon^{\frac{N-p}{2\eta(p-\nu)}} |\ln \epsilon|\big), \tag{3.18}$$

and

$$\int_{\mathbb{R}^N} \frac{|u_\epsilon|^q}{|x|^\mu} dx = O\big(\epsilon^{\frac{[2\eta(N-\mu)p-(N-p)q](p-1)}{2\eta p(p-\nu)}}\big), \qquad \text{for} \quad \frac{2\eta(N-\mu)(p-1)}{N-p} < q < \frac{2\eta p(N-\mu)}{N-p}. \tag{3.19}$$

Furthermore, we define the Mountain Pass level c of J by

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t)), \tag{3.20}$$

where $\Gamma = \{\gamma \in C([0,1], E) \mid \gamma(0) = 0, \gamma(1) \neq 0, J(\gamma(1)) < 0\}$. It is easy to see that $c > 0$ by Lemma 3.2. As we have done in [25], it follows that

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J(\gamma(t)) \leq \sup_{t\geq 0} J(f^{-1}(tu_\epsilon)) = \sup_{t\geq 0} I(tu_\epsilon).$$

Hence we have the following Lemma which provides a delicate estimate on the Mountain Pass level c of the functional J.

Lemma 3.6. *Let $N > p > 1$, $\eta \geq \frac{p}{2(p-1)}$ and μ, $\nu \in [0, p)$ be fixed. Then,*

(i) if $2\eta p^*(\mu) - \frac{p}{p-1} < q < 2\eta p^*(\mu)$, (3.11) is satisfied for any $\lambda > 0$;

(ii) if $p < q \leq 2\eta p^*(\mu) - \frac{p}{p-1}$, there exists a positive constant λ^* such that (3.11) holds for $\lambda > \lambda^*$.

Proof. (i) Firstly, there exists $t_\epsilon > 0$ such that $I(t_\epsilon u_\epsilon) = \max_{t\geq 0} I(tu_\epsilon)$ by $I(0) = 0$ and $\lim_{t\to\infty} I(tu_\epsilon) = -\infty$. We then have the following claim.

Claim. There exist positive constants t_1 and t_2 such that $t_\epsilon \in [t_1, t_2]$ for $\epsilon \in (0, \epsilon_0)$.

Indeed, by (3.17)–(3.19), there is a sufficiently small $\epsilon_2 > 0$ such that

$$\begin{aligned} I(tu_\epsilon) &\leq \frac{t^p}{p}\int_{\mathbb{R}^N} \left[|\nabla u_\epsilon|^p + V(x)|u_\epsilon|^p\right] dx + \frac{t^{2\eta p}}{2\eta p}\int_{\mathbb{R}^N} |\nabla(u_\epsilon^{2\eta})|^p dx - \frac{t^{2\eta p^*(\nu)}}{2\eta p^*(\nu)}\int_{\mathbb{R}^N} \frac{|u_\epsilon|^{2\eta p^*(\nu)}}{|x|^\nu} dx \\ &\leq \frac{t^p}{p} + \frac{t^{2\eta p}}{p} S_\nu^{\frac{N-\nu}{p-\nu}} - \frac{t^{2\eta p^*(\nu)}}{2\eta p \cdot p^*(\nu)} S_\nu^{\frac{N-\nu}{p-\nu}}, \end{aligned} \tag{3.21}$$

for any $\epsilon \in (0, \epsilon_2)$. Consequently, we have

$$\frac{t_\epsilon^{2\eta p^*(\nu)}}{2\eta p^*(\nu)} S_\nu^{\frac{N-\nu}{p-\nu}} \leq t_\epsilon^p + t_\epsilon^{2\eta p} S_\nu^{\frac{N-\nu}{p-\nu}},$$

which implies that there exists a constant $t_2 > 0$ such that $t_\epsilon \leq t_2$ for any $\epsilon \in (0, \epsilon_2)$.

We now establish the lower bound for t_ϵ. By the fact that $\eta \geq \frac{p}{2(p-1)}$ and $\mu \in [0, p)$, we have $\frac{2\eta(N-\mu)(p-1)}{N-p} \leq 2\eta p^*(\mu) - \frac{p}{p-1} < q < 2\eta p^*(\mu)$. Then it follows from (3.17)–(3.19) that there exists $\epsilon_1 \in (0, \epsilon_2)$ such that

$$\begin{aligned} I(tu_\epsilon) &\geq \frac{t^{2\eta p}}{2\eta p}\int_{\mathbb{R}^N} |\nabla(u_\epsilon^{2\eta})|^p dx - \lambda\frac{t^q}{q}\int_{\mathbb{R}^N} \frac{|u_\epsilon|^q}{|x|^\mu} dx - \frac{t^{2\eta p^*(\nu)}}{2\eta p^*(\nu)}\int_{\mathbb{R}^N} \frac{|u_\epsilon|^{2\eta p^*(\nu)}}{|x|^\nu} dx \\ &\geq \frac{1}{4\eta p} S_\nu^{\frac{N-\nu}{p-\nu}} t^{2\eta p} - \lambda C \epsilon^{\frac{[2\eta(N-\mu)p-(N-p)q](p-1)}{2\eta p(p-\nu)}} t^q - \frac{1}{\eta p^*(\nu)} S_\nu^{\frac{N-\nu}{p-\nu}} t^{2\eta p^*(\nu)}, \end{aligned}$$

for all $\epsilon \in (0, \epsilon_1)$.

Set

$$\chi = \max_{0\leq t\leq 1}\left[\frac{1}{4\eta p} t^{2\eta p} - \frac{1}{\eta p^*(\nu)} t^{2\eta p^*(\nu)}\right] S_\nu^{\frac{N-\nu}{p-\nu}}.$$

With this setting, it is easy to check that $\chi > 0$. Moreover, thanks to the positivity of the fraction $\frac{[2\eta(N-\mu)p-(N-p)q](p-1)}{2\eta p(p-\nu)}$, we can find a small ϵ_0 with $\epsilon_0 < \epsilon_1$ such that $\lambda C \epsilon^{\frac{[2\eta(N-\mu)p-(N-p)q](p-1)}{2\eta p(p-\nu)}} \leq \frac{\chi}{2}$ for all $\epsilon \in (0, \epsilon_0)$. Hence, we obtain

$$I(t_\epsilon u_\epsilon) \geq \max_{0\leq t\leq 1}\left\{\frac{1}{4\eta p} S_\nu^{\frac{N-\nu}{p-\nu}} t^{2\eta p} - \lambda C \epsilon^{\frac{[2\eta(N-\mu)p-(N-p)q](p-1)}{2\eta p(p-\nu)}} t^q - \frac{1}{\eta p^*(\nu)} S_\nu^{\frac{N-\nu}{p-\nu}} t^{2\eta p^*(\nu)}\right\} \geq \frac{\chi}{2}.$$

Combining the above inequality with (3.21), we deduce that

$$\frac{\chi}{2} \le \frac{t^p}{p} + \frac{t^{2\eta p}}{p} S_\nu^{\frac{N-\nu}{p-\nu}} - \frac{t^{2\eta p^*(\nu)}}{2\eta p \cdot p^*(\nu)} S_\nu^{\frac{N-\nu}{p-\nu}},$$

which gives a positive lower bound t_1 for t_ϵ when $\epsilon \in (0, \epsilon_0)$. Hence we prove our claim.

We now proceed with proving part (i). From (3.17)–(3.19), we get that

$$\begin{aligned} I(t_\epsilon u_\epsilon) &\le \frac{t_\epsilon^{2\eta p}}{2\eta p} \int_{\mathbb{R}^N} |\nabla(u_\epsilon^{2\eta})|^p dx - \frac{t_\epsilon^{2\eta p^*(\nu)}}{2\eta p^*(\nu)} \int_{\mathbb{R}^N} \frac{|u_\epsilon|^{2\eta p^*(\nu)}}{|x|^\nu} dx \\ &\quad - \frac{\lambda}{q} t_1^q \int_{\mathbb{R}^N} \frac{|u_\epsilon|^q}{|x|^\mu} dx + \frac{t_2^p}{p} \int_{\mathbb{R}^N} \big[|\nabla u_\epsilon|^p + V(x)|u_\epsilon|^p\big] dx \\ &\le \Big(\frac{t_\epsilon^{2\eta p}}{2\eta p} - \frac{t_\epsilon^{2\eta p^*(\nu)}}{2\eta p^*(\nu)}\Big) S_\nu^{\frac{N-\nu}{p-\nu}} + O\big(\epsilon^{\frac{N-p}{2\eta(p-\nu)}} |\ln \epsilon|\big) - C\epsilon^{\frac{[2\eta(N-\mu)p-(N-p)q](p-1)}{2\eta p(p-\nu)}} \\ &\le \frac{p-\nu}{2\eta p(N-\nu)} S_\nu^{\frac{N-\nu}{p-\nu}} + O\big(\epsilon^{\frac{N-p}{2\eta(p-\nu)}} |\ln \epsilon|\big) - C\epsilon^{\frac{[2\eta(N-\mu)p-(N-p)q](p-1)}{2\eta p(p-\nu)}} \\ &< \frac{p-\nu}{2\eta p(N-\nu)} S_\nu^{\frac{N-\nu}{p-\nu}}, \end{aligned}$$

for $\epsilon \in (0, \epsilon_0)$ sufficiently small and $\frac{[2\eta(N-\mu)p-(N-p)q](p-1)}{2\eta p(p-\nu)} < \frac{N-p}{2\eta(p-\nu)}$ if $2\eta p^*(\mu) - \frac{p}{p-1} < q$. Therefore we are able to find a small enough $\bar\epsilon > 0$ such that

$$\sup_{t\ge 0} J(f^{-1}(tu_{\bar\epsilon})) = \sup_{t\ge 0} I(tu_{\bar\epsilon}) = I(t_{\bar\epsilon} u_{\bar\epsilon}) < \frac{p-\nu}{2\eta p(N-\nu)} S_\nu^{\frac{N-\nu}{p-\nu}}.$$

Furthermore, we conclude by (3.21) that $J(f^{-1}(tu_{\bar\epsilon})) = I(tu_{\bar\epsilon}) \to -\infty$ as $t \to +\infty$, which indicates that there exists a $\bar t > 0$ such that $J(f^{-1}(\bar t u_{\bar\epsilon})) < 0$. Taking $\bar\gamma(t) = f^{-1}(t\bar t u_{\bar\epsilon})$, we have $\bar\gamma \in \Gamma$ and $c \le \max_{t\in[0,1]} J(\bar\gamma(t)) < \frac{p-\nu}{2\eta p(N-\nu)} S_\nu^{\frac{N-\nu}{p-\nu}}$ for any $\lambda > 0$ as required.

(ii) We first rewrite I as I_λ. Define $u_0 \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ with $u_0 \ne 0$ and $t_\lambda > 0$ such that $I_\lambda(t_\lambda u_0) = \sup_{t\ge 0} I_\lambda(tu_0)$. We then claim that $t_\lambda \to 0$ as $\lambda \to +\infty$. We will prove this claim by contradiction. Suppose that there exists a constant $t_0 > 0$ and a sequence $\{\lambda_n\}$ such that $t_{\lambda_n} \ge t_0$ as $\lambda_n \to +\infty$ for all n. Without loss of generality, we may assume that $\lambda_n \ge 1$ for all n. Let $t_n = t_{\lambda_n}$ and $I_1 = I_\lambda|_{\lambda=1}$. Then $0 \le I_{\lambda_n}(t_n u_0) \le I_1(t_n u_0)$ for any n, which implies that t_n is bounded from above. On the other hand, we also have

$$\begin{aligned} I_{\lambda_n}(t_n u_0) &\le \frac{t_n^{2\eta p}}{2\eta p} \int_{\mathbb{R}^N} |\nabla(u_0^{2\eta})|^p dx + \frac{t_n^p}{p} \int_{\mathbb{R}^N} \big[|\nabla u_0|^p + V(x)|u_0|^p\big] dx - \lambda_n \frac{t_n^q}{q} \int_{\mathbb{R}^N} \frac{|u_0|^q}{|x|^\mu} dx \\ &\le C - \lambda_n \frac{t_0^q}{q} \int_{\mathbb{R}^N} \frac{|u_0|^q}{|x|^\mu} dx \\ &\to -\infty, \end{aligned} \tag{3.22}$$

as $n \to \infty$, and this contradicts the nonnegativity of $I_{\lambda_n}(t_n u_0)$. Hence our claim holds.

Since $t_\lambda \to 0$ as $\lambda \to +\infty$, and

$$I_\lambda(t_\lambda u_0) \le \frac{t_\lambda^{2\eta p}}{2\eta p} \int_{\mathbb{R}^N} |\nabla(u_0^{2\eta})|^p dx + \frac{t_\lambda^p}{p} \int_{\mathbb{R}^N} \big[|\nabla u_0|^p + V(x)|u_0|^p\big] dx,$$

we can obtain that $I_\lambda(t_\lambda u_0)$ converges to zero as λ tends to infinity. As a consequence, there exists $\lambda^* > 0$ such that $\sup\limits_{t\geq 0} I_\lambda(tu_0) < \frac{p-\nu}{2\eta p(N-\nu)} S_\nu^{\frac{N-\nu}{p-\nu}}$ for any $\lambda > \lambda^*$, which implies that $c < \frac{p-\nu}{2\eta p(N-\nu)} S_\nu^{\frac{N-\nu}{p-\nu}}$ for all $\lambda > \lambda^*$. Therefore, the proof of part (ii) of Lemma 3.6 is complete. □

4 Proof of the Main Result

In this section, we will apply Theorem 3.1 to prove the existence of the critical point of J. After establishing the existency, we then show that the critical point is nonzero.

Proof of Theorem 1.1. Let c be the Mountain Pass level given in (3.20). We infer by Lemma 3.2 and Theorem 3.1 that J has a $(C)_c$ sequence $\{v_n\} \subset E$. By Lemma 3.3, we may assume that $v_n \rightharpoonup v$ in $W^{1,p}(\mathbb{R}^N)$ and $f(v_n) \rightharpoonup f(v)$ in E. We *claim* that $J'(v) = 0$, that is, we only need to show that $\langle J'(v), \varphi\rangle = 0$ for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$.

Indeed, we note that

$$
\begin{aligned}
&\langle J'(v_n), \varphi\rangle - \langle J'(v), \varphi\rangle \\
=& \int_{\mathbb{R}^N} \big[|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v\big]\nabla\varphi \\
&+ \int_{\mathbb{R}^N} V(x)\big[|f(v_n)|^{p-2} f(v_n) f'(v_n) - |f(v)|^{p-2} f(v) f'(v)\big]\varphi dx \\
&- \lambda \int_{\mathbb{R}^N} \Big[\frac{|f(v_n)|^{q-2}}{|x|^\mu} f(v_n) f'(v_n) - \frac{|f(v)|^{q-2}}{|x|^\mu} f(v) f'(v)\Big]\varphi dx \\
&- \int_{\mathbb{R}^N} \Big[\frac{|f(v_n)|^{2\eta p^*(\nu)-2}}{|x|^\nu} f(v_n) f'(v_n) - \frac{|f(v)|^{2\eta p^*(\nu)-2}}{|x|^\nu} f(v) f'(v)\Big]\varphi dx.
\end{aligned}
$$

From the assumption (V) and Lemma 2.1, for any $0 \leq \sigma < p$, the embedding $E \hookrightarrow L^r(\mathbb{R}^N, |x|^{-\sigma})$ is continuous for $p \leq r \leq p^*(\sigma)$, and it is compact for $p \leq r < p^*(\sigma)$ as well. Consequently,

$$
\begin{aligned}
&f(v_n) \to f(v) \text{ in } L^s(\mathbb{R}^N, |x|^{-\mu}), \qquad \text{for } p \leq s < 2\eta p^*(\mu), \\
&f(v_n) \rightharpoonup f(v) \text{ weakly } \text{ in } L^{2\eta p^*(\nu)}(\mathbb{R}^N, |x|^{-\nu}),
\end{aligned}
$$

for $\mu, \nu \in [0, p)$. Hence, we have $\langle J'(v_n), \varphi\rangle \to \langle J'(v), \varphi\rangle = 0$ for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$. Since $J'(v_n) \to 0$, we conclude that $J'(v) = 0$, that is, v is a weak solution of the Euler-Lagrange equation of J.

Our final task is to show that the critical point is nonzero, i.e. $v \neq 0$. We conclude from Lemma 3.6 that for any $\mu, \nu \in [0, p)$, (3.11) holds when either of the following statement holds: (I) $2\eta p^*(\mu) - \frac{p}{p-1} < q < 2\eta p^*(\mu)$ and each $\lambda > 0$; (II) $p < q \leq 2\eta p^*(\mu) - \frac{p}{p-1}$ and each $\lambda > \lambda^*$ for some positive constant λ^*. Moreover, we infer by Proposition 3.4 that there exists a constant $\xi > 0$ such that

$$
\int_{\mathbb{R}^N} |v|^p dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^p dx \geq \xi > 0,
$$

which tells us that v is a nontrivial solution of the Euler-Lagrange equation of J. Hence $u = f(v)$ is a nontrivial solution of problem (1.1).

Finally, set $e = \inf\{J(v) : v \in E, v \neq 0, J'(v) = 0\}$, one readily sees that e is attained by the lower semi-continuity. The proof of Theorem 1.1 is completed.

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