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# Controllability of Nonlinear Discrete Systems with Degeneracy

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**Abstract** This paper concerns the controllability of autonomous and nonautonomous nonlinear discrete systems, in which linear parts might admit certain degeneracy. By introducing Fredholm operators and coincidence degree theory, sufficient conditions for nonlinear discrete systems to be controllable are presented. In addition, applications are given to illustrate main results.

**Keywords** nonlinear discrete systems; controllability; degeneracy; Fredholm operators; coincidence degree theory

**2000 MR Subject Classification** 39A05; 39A70; 93C65

## 1 Introduction

In this paper, we consider the following nonlinear autonomous discrete system

$$x(k+1) = Ax(k) + Bu(k) + f(x(k)), \qquad k \in \mathbf{Z}_+, \tag{1.1}$$

where $\mathbf{Z}_+$ is the set of all positive integers, $x(k) \in \mathbb{R}^n$ is the state variable, $u(k) \in \mathbb{R}^p$ is the control variable, $A$ is an $n \times n$ constant matrix; $B$ is an $n \times p$ constant matrix; $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping in $x$. The nonautonomous form is as follows

$$x(k+1) = A(k)x(k) + B(k)u(k) + f(k, x(k)), \qquad k \in \mathbf{Z}_+, \tag{1.2}$$

where $x(k) \in \mathbb{R}^n$ is the state variable, $u(k) \in \mathbb{R}^p$ is the control variable, $A(k)$ is an $n \times n$ matrix, $B(k)$ is an $n \times p$ matrix, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping.

The idea of controllability was put forward by Kalman in the 1960s[6] where he established a powerful fundamental principle. Similar problems for nonlinear systems became rapidly developed. For some classical liturate, see Chow[1], Hermann[4], Haynes & Hermes[3], Lobry[8], Sussman & Jurdevic[10], Krener[7], Hermann[5].

The controllability of nonlinear discrete systems result from some practical problems, for example, engineering, biology and economic fields, in which the state variable and the control variable in time are discrete. Therefore, discrete control problems are important in applications. In recent years, there have been some results in studying the controllability for discrete nonlinear systems. For instance, Tie Lin studied controllability, small-controllability and near-controllability of nonlinear systems, see [12–14]. Zhao and Sun[15] used a geometric method

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based on the differential geometric analysis and Lie group theory to study discrete impulsive nonlinear systems. In the usual study, the linear parts of nonlinear systems are required to be non-degenerate, more precisely, in (1.1),

$$\operatorname{rank}(B, AB, \cdots, A^{N-1}B) = n.$$

However, the rank of the control matrix is singular, i.e; its

$$\operatorname{rank}(B, AB, \cdots, A^{N-1}B) < n,$$

how controllability takes place. So far this topic has been paid rare attention. Tan and Li[11] studied the null controllability for nonlinear discrete control systems with degeneracy by using Cayley-Hamilton theorem and Brouwer's fixed point theorem.

The purpose of this paper is to establish sufficient conditions for the controllability of nonlinear discrete systems. Compared with the existing studies, we study the controllability of nonlinear discrete systems, in which linear parts might admit certain degeneracy. We give sufficient conditions guaranteeing the controllability of nonlinear discrete systems via Fredholm operators and coincidence degree theory. This is a new method for studying the controllability of discrete systems. In our approach, the linear part of systems under consideration might admit some degeneracy. We give some examples to verify our results. Those examples show that although sometimes linear parts admit degeneracy, systems are still controllable under certain prior estimate and transversality condition for nonlinear terms.

## 2 Fredholm Operator and Coincidence Degree Theorem

We begin with a brief summary of the Leray-Schauder degree of mapping theory. Let $\Phi : X \to X$ be a completely continuous mapping on a Banach space $X$. Let $\Omega$ be a bounded open set in $X$ with boundary $\partial\Omega$. Then there exists an integer value function: Leray-Schauder degree $\deg_{LS}(I - \Phi, \Omega, p)$, for any $p \notin (I - \Phi)(\partial\Omega)$. It possesses the following regularity: if $\deg_{LS}(I - \Phi, \Omega, p) \neq 0$, then the equation $x - \Phi(x) = p$ admits a solution in $\Omega$. If $X$ is a finite-dimensional spaces, Leray-Schauder degree defines Brouwer degree. In this paper, $\deg_B(\cdot, \cdot, \cdot)$ denotes Brouwer degree. For a more complete treatment of the Leray-Schauder degree theory[9].

Next we introduce the main methods used in this paper: Fredholm operators and coincidence degree theory[2].

**Definition 2.1.** *Let $X$ and $Z$ be real normed vector spaces. A linear mapping $L : \operatorname{dom} L \subseteq X \to Z$ is called Fredholm if the following conditions hold:*
(i) $\operatorname{Ker} L := L^{-1}\{0\}$ *has finite dimension;*
(ii) $\operatorname{Im} L = L(\operatorname{dom} L)$ *is closed and has finite codimension.*

**Definition 2.2.** *If $L$ is a Fredholm operator, then index of $L$ is the integer* $\operatorname{Ind} L = \dim \operatorname{Ker} L - \operatorname{codim} \operatorname{Im} L$, *where* $\operatorname{codim} \operatorname{Im} L = \dim(Z/\operatorname{Im} L)$.

Here $\operatorname{dom} L$ is the domain of definition for mapping $L$, $\operatorname{Ker} L := L^{-1}\{0\}$ is a kenel space of mapping $L$, $\operatorname{Im} L$ denotes the image of mapping $L$, $\dim \operatorname{Ker} L$ is the dimension of $\operatorname{Ker} L$, $\operatorname{codim} \operatorname{Im} L = \dim(Z/\operatorname{Im} L)$ is the codimension of $\operatorname{Im} L$.

Obviously, $L$ is called a Fredholm operator of index zero if $\operatorname{Im} L$ is closed and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$.

For a Fredholm operator $L : \operatorname{dom} L \subseteq X \to Z$, there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$, thereby $X = \operatorname{Ker} L \oplus \operatorname{Ker} P$ and $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$.

Let $L_P := L|_{\operatorname{dom} L \cap \operatorname{Ker} P}$. Then $L_P : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is a bijection. The inverse mapping is defined by $K_P := \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ and let $K_{P,Q} : Z \to \operatorname{dom} L \cap \operatorname{Ker} P$, $K_{P,Q} = K_P(I - Q)$.

If $L$ is a Fredholm operator of index zero, then $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L$. Hence there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} P$, let us define $H_{J,P,Q} : Z \to \operatorname{dom} L$ by $H_{J,P,Q} = JQ + K_{P,Q}$, then $H_{J,P,Q}$ is an algebraic isomorphism.

**Definition 2.3.** *Let $L : \operatorname{dom} L \subseteq X \to Z$ be a Fredholm operator of index zero, $E \subseteq X$. A continuous mapping $\mathscr{N} : E \to Z$ is called $L$-compact if $Q\mathscr{N} : E \to Z$ and $K_{P,Q}\mathscr{N} : E \to X$ are all compact. $\mathscr{N}$ is called $L$-completely continuous in $E$ if $\mathscr{N}$ is $L$-compact in each bounded subset.*

Let $L : \operatorname{dom} L \subseteq X \to Z$ be a Fredholm operator of index zero, $\Omega$ an open bounded subset of $X$, and $\mathscr{N} : \overline{\Omega} \to Z$ an $L$-completely continuous mapping. On $\operatorname{dom} L \cap \overline{\Omega}, H_{J,P,Q}(L - \mathscr{N}) = I - P - JQ\mathscr{N} - K_{P,Q}\mathscr{N} = I - T$, where $T := P + JQ\mathscr{N} + K_{P,Q}\mathscr{N}$. Since $P$ is a finite dimensional linear operator and $\mathscr{N}$ an $L$-completely continuous mapping, $T : \overline{\Omega} \to X$ is a completely continuous mapping. Suppose that $Lx \neq \mathscr{N}x, \forall x \in \operatorname{dom} L \cap \partial\Omega$ holds. Since $H_{J,P,Q} : Z \to \operatorname{dom} L$ is an algebraic isomorphism, we know that $Tx \neq x, \forall x \in \partial\Omega$. Hence, Leray-Schauder degree $\deg_{LS}(I - T, \Omega, 0)$ is well-defined.

**Definition 2.4.** $D[(L, \mathscr{N}), \Omega] = \deg_{LS}(I - T, \Omega, 0)$ *is called coincidence degree of operators $L$ and $\mathscr{N}$ in $\Omega$.*

The following properties are basic.

**Homotopy invariance:** If $H : \overline{\Omega} \times [0, 1] \to Z$ is $L$-compact in $\overline{\Omega} \times [0, 1]$, and $Lx \neq H(x, \lambda), \forall (x, \lambda) \in \operatorname{dom} L \cap \partial\Omega \times [0, 1]$, then $D[(L, H(\cdot, \lambda)), \Omega]$ is a constant for all $\lambda \in [0, 1]$.

**Normal property:** If $D[(L, \mathscr{N}), \Omega] \neq 0$, then there exists at least one solution $x \in \operatorname{dom} L \cap \Omega$ such that $Lx = \mathscr{N}x$.

## 3 Main Results

By nonlinear autonomous discrete control system (1.1), we have

$$\begin{aligned} x(1) &= Ax(0) + Bu(0) + f(x(0)), \\ x(2) &= Ax(1) + Bu(1) + f(x(1)) \\ &= A^2x(0) + ABu(0) + Bu(1) + Af(x(0)) + f(x(1)), \\ &\vdots \\ x(k) &= Ax(k-1) + Bu(k-1) + f(x(k-1)) = \cdots \\ &= A^k x(0) + \sum_{i=0}^{k-1} A^{k-1-i}Bu(i) + \sum_{i=0}^{k-1} A^{k-1-i}f(x(i)), \end{aligned}$$

i.e.,

$$x(N) = A^N x(0) + \sum_{i=0}^{N-1} A^{N-1-i}Bu(i) + \sum_{i=0}^{N-1} A^{N-1-i}f(x(i)). \tag{3.1}$$

Let $S, T \subset \mathbb{R}^n$ be two given sets.

**Definition 3.1.** *System (1.1) is said to be controllable with respect to $(S, T)$ if for any $\xi \in S, \eta \in T$ there exist a positive integer $N$ and a control sequence $u(0)$, $u(1), \cdots, u(N-1)$ such that the solution sequence $x(0)$, $x(1)$, $\cdots$, $x(N)$ of (1.1) starting at $x(0) = \xi$ satisfies $x(N) = \eta$. If $S = T = \mathbb{R}^n$, then (1.1) is said to be completely controllable.*

Put (3.1) into the vector form

$$(B, AB, \cdots, A^{N-1}B) \begin{pmatrix} u(N-1) \\ u(N-2) \\ \vdots \\ u(0) \end{pmatrix} = x(N) - A^N x(0) - \sum_{i=0}^{N-1} A^{N-1-i} f(x(i)), \tag{3.2}$$

where $N$ is a positive integer. Then

$$M := (B, AB, \cdots, A^{N-1}B)$$

is an $n \times m$ matrix with $m = Np$,

$$v := (u(N-1), u(N-2), \cdots, u(0))^{\mathrm{T}}$$

is an $m$ dimensional vector,

$$F(v, x(0), x(N)) := x(N) - A^N x(0) - \sum_{i=0}^{N-1} A^{N-1-i} f(x(i))$$

is continuous in $(v, x)$. Thus (3.2) can be written as

$$Mv = F(v, x(0), x(N)),$$

equivalently

$$\tilde{M}v := M^{\mathrm{T}} M v = M^{\mathrm{T}} F(v, x(0), x(N)) := \tilde{F}(v, x(0), x(N)). \tag{3.3}$$

Obviously, if for any given $x(0)$ and $x(N)$ there exists a vector $v$ satisfying (3.3), system (1.1) is controllable. We turn the controllability problem of system (1.1) into a solvability problem of (3.3). So we use Fredholm operators and coincidence degree to get the existence of solutions of (3.3).

**Theorem 3.2.** *Suppose that $f$ is continuous and there exists a bounded open non-empty subset $\Omega$ of $\mathbb{R}^m$ such that the following conditions hold:*
(i) *for any $\lambda \in (0,1)$, and $x(0) = \xi \in S$, $x(N) = \eta \in T$, equation*

$$\tilde{M}v = \lambda \tilde{F}(v, x(0), x(N)) \tag{3.4}$$

*has no solution $v$ on $\partial\Omega$;*
(ii) *for all $v \in \operatorname{Ker} \tilde{M} \cap \partial\Omega$, $Q\tilde{F}(v, x(0), x(N)) \neq 0$;*
(iii) $\deg_B(Q\tilde{F}|_{\operatorname{Ker} \tilde{M}}, \operatorname{Ker} \tilde{M} \cap \Omega, 0) \neq 0$.
*Then (3.3) has at least one solution, and nonlinear autonomous discrete control system (1.1) is controllable with respect to $(S, T)$, where mapping $Q : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous projector satisfying* $\operatorname{Ker} Q = \operatorname{Im} \tilde{M}$.

**Remark 3.3.** When $M := (B, AB, \cdots, A^{N-1}B)$ is singular, if the nonlinear term $f$ satisfies prior estimate condition (ii) and transversality condition (iii) of Theorem 3.1, then discrete control system (1.1) is controllability.

*Proof of Theorem 3.1.* Let $X = \mathbb{R}^m$ be endowed with the norms $\|v\| = \big(\sum_{i=1}^{m} v_i^2\big)^{1/2}$ Let $L : X \to X$, $Lv = \tilde{M}v$. Then $\dim \operatorname{Ker} L = n - \operatorname{rank}(\tilde{M})$ and $\dim \operatorname{Im} L = \operatorname{rank}(\tilde{M})$. Obviously,

$\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$. In addition, because $L$ is linear, $\operatorname{Im} L$ is closed. Thus $L$ is a Fredholm operator of index zero. Let

$$\mathscr{N} : \overline{\Omega} \subseteq X \to X, \qquad \mathscr{N} v = \tilde{F}(v, x(0), x(N)).$$

Define $H(v, \lambda)$ by

$$H(v, \lambda) = \lambda \mathscr{N} v + (1 - \lambda) Q \mathscr{N} v, \qquad \lambda \in [0,1], \quad v \in \Omega.$$

Since the operators under consideration are all finitely dimensional, they are $L$-completely.

Consider the auxiliary equation

$$Lv = H(v, \lambda). \tag{3.5}$$

**Case I.** $\lambda = 1$. (3.5) is equivalent to $Lv = \mathscr{N} v$. We can assume that there exists no solution on $\partial\Omega$, otherwise Theorem 3.1 holds.

**Case II.** $\lambda = 0$. (3.5) is equivalent to $Lv = Q\mathscr{N} v$. If there exists $v \in \partial\Omega$ such that (3.5) holds, then $Lv = 0$ and $Q\mathscr{N} v = 0$, which contradicts condition $(ii)$ of Theorem 3.1. Hence, for $\lambda = 0$ (3.5) has no solution on $\partial\Omega$.

**Case III.** $\lambda \in (0,1)$. If there exist $v_0 \in \partial\Omega$ and $\lambda_0 \in (0,1)$ such that (3.5) holds, then

$$Lv_0 = H(v_0, \lambda_0) = \lambda_0 \mathscr{N} v_0 + (1 - \lambda_0) Q \mathscr{N} v_0 = \lambda_0 (I - Q) \mathscr{N} v_0 + Q \mathscr{N} v_0.$$

It is equivalent to

$$Lv_0 = \lambda_0 (I - Q) \mathscr{N} v_0, \qquad Q\mathscr{N} v_0 = 0,$$

i.e. $Lv_0 = \lambda_0 \mathscr{N} v_0$, which contradicts condition $(i)$ of Theorem 3.1. Hence, for all $(v, \lambda) \in \partial\Omega \times [0,1]$, (3.5) has no solution, i.e. for all $(v, \lambda) \in \partial\Omega \times [0,1], Lv \neq H(v, \lambda)$. Consequently, coincidence degree $D[(L, H(\cdot, \lambda)), \Omega]$ is well-defined. And it follows from the homotopy invariance of coincidence degree that

$$D[(L, H(\cdot, \lambda)), \Omega] = D[(L, H(\cdot, 1)), \Omega] = D[(L, H(\cdot, 0)), \Omega].$$

Thus we just need to prove $D[(L, H(\cdot, 0)), \Omega] \neq 0$. Notice that $H(v, 0) = Q\mathscr{N} v$, and

$$H_{J,P,Q}(L - H(\cdot, 0)) = (JQ + K_{P,Q})(L - Q\mathscr{N}) = I - P - JQ\mathscr{N},$$

where $J$, $P$, $K_{P,Q}$ and $H_{J,P,Q}$ are defined in the section of Fredholm Operator and Coincidence Degree Theory. Consequently,

$$\begin{aligned} D[(L, H(\cdot, 0)), \Omega] &= \deg_{LS}(I - P - JQ\mathscr{N}, \Omega, 0) \\ &= \deg_B((I - P - JQ\mathscr{N})|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega, 0) \\ &= \deg_B((-JQ\mathscr{N})|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega, 0) \\ &= (-1)^{\dim \operatorname{Ker} L} \deg_B(JQ\mathscr{N})|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega, 0) \\ &= (-1)^{\dim \operatorname{Ker} L} \deg_B(Q\mathscr{N})|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega, 0) \\ &\neq 0. \end{aligned}$$

It follows that $D[(L, H(\cdot, 1)), \Omega] = D[(L, H(\cdot, 0)), \Omega] \neq 0$. By the normal property of coincidence degree, the equation $Lv = \mathscr{N} v$ has at least one solution in $\operatorname{dom} L \cap \Omega$. So (3.3) has at least one solution in $\overline{\Omega}$. Consequently, nonlinear autonomous discrete control system (1.1) is controllable with respect $(S, T)$. □

Applying Theorem 3.1 to the non-degenerate case: $\operatorname{rank}(B, AB, \cdots, A^{N-1}B) = n$, we have:

**Theorem 3.4.** *Suppose that $f$ is continuous and bounded in $x$, and $\tilde{M}$ nonsingular. Then nonlinear autonomous discrete control system (1.1) is completely controllable.*

*Proof.* First, we prove that the set

$$\Omega_1 = \{v \in \mathbb{R}^m : Lv = \lambda \mathscr{N} v, \lambda \in [0,1]\}$$

is bounded. Matrix $\tilde{M}$ is nonsingular, so the inverse matrix $\tilde{M}^{-1}$ is given. For any $v \in \Omega_1$, we have

$$\tilde{M}v = \lambda \tilde{F}(v, x(0), x(N)) = -\lambda\Big(-x(N) + A^N x(0) + \sum_{i=0}^{N-1} A^{N-1-i} f(x(i))\Big),$$
$$v = -\lambda \tilde{M}^{-1}\Big(-x(N) + A^N x(0) + \sum_{i=0}^{N-1} A^{N-1-i} f(x(i))\Big).$$

Thus, we have

$$\begin{aligned}\|v\| &\le |\lambda| \|\tilde{M}^{-1}\| \Big(\|-x(N)\| + \|A^N\|\|x(0)\| + \sum_{i=0}^{N-1} \|A^{N-1-i}\|\|f(x(i))\|\Big)\\ &\le \|\tilde{M}^{-1}\| \Big(\|x(N)\| + \|A^N\|\|x(0)\| + \sum_{i=0}^{N-1} \|A^{N-1-i}\|\|f(x(i))\|\Big).\end{aligned}$$

$f$ is bounded, since there exists a real number $f_0$ such that $\|f\| \le f_0$. For given $x(0)$ and $x(N)$,

$$\|v\| \le \|\tilde{M}^{-1}\| \Big(\|x(N)\| + \|A^N\|\|x(0)\| + \sum_{i=0}^{N-1} \|A^{N-1-i}\| f_0\Big).$$

Hence, $\Omega_1$ is bounded. We define $\gamma \in R$ and

$$\gamma = \|\tilde{M}^{-1}\| \Big(\|x(N)\| + \|A^N\|\|x(0)\| + \sum_{i=0}^{N-1} \|A^{N-1-i}\| f_0\Big).$$

From the proof of Theorem 3.1, $L$ is a Fredholm operator of index zero and $\mathscr{N}$ is $L$-completely continuous. Let $\Omega_0 = \{v \in \mathbb{R}^m : \|v\| \le \gamma + 1\}$. Then for any $v \in \operatorname{dom} L \cap \partial\Omega_0$ and any $\lambda \in [0,1]$, $Lv \ne \lambda \mathscr{N}$. Thus $D[(L, \mathscr{N}), \Omega_1] = D[(L, 0), \Omega_1] = \pm 1$. By the normal property of coincidence degree, the equation $Lv = \mathscr{N} v$ has at least one solution. Consequently, nonlinear autonomous discrete control system (1.1) is completely controllable. □

For nonautonomous form, we use similar approach to nonlinear autonomous discrete systems.

By nonlinear nonautonomous discrete system (1.2), we have

$$\begin{aligned}x(h+1) =& A(h)x(h) + B(h)u(h) + f(h, x(h)),\\ x(h+2) =& A(h+1)x(h+1) + B(h+1)u(h+1) + f(h+1, x(h+1))\\ =& \Phi(h+2, h)x(h) + \sum_{i=h}^{h+1} \Phi(h+2, i+1)B(i)u(i)\\ &+ \sum_{i=h}^{h+1} \Phi(h+2, i+1) f(i, x(i)),\end{aligned}$$

$$\begin{aligned}
&\vdots\\
x(h+k) =& A(h+k-1)x(h+k-1) + B(h+k-1)u(h+k-1)\\
&+ f(h+k-1, x(h+k-1))\\
=& \cdots\\
=& \Phi(h+k,h)x(h) + \sum_{i=h}^{h+k-1} \Phi(h+k,i+1)B(i)u(i)\\
&+ \sum_{i=h}^{h+k-1} \Phi(h+k,i+1)f(i,x(i)),
\end{aligned}$$

where the state-transition matrix $\Phi(k,h)$ $(k \geq h)$ is given by

$$\begin{cases} \Phi(k,h) = A(k-1)A(k-2)\cdots A(h), \\ \Phi(k,k) = I. \end{cases}$$

**Definition 3.5.** *System (1.2) is said to be controllable with respect to $(S,T)$ if for any $\xi \in S, \eta \in T$, there exist a positive integer $N$ and a control sequence $u(h)$, $u(h+1), \cdots, u(h+N-1)$ such that the solution sequence $x(h)$, $x(h+1)$, $\cdots$, $x(h+N)$ of (1.2) starting at $x(h) = \xi$ satisfies $x(h+N) = \eta$. If $S = T = \mathbb{R}^{\mathbf{n}}$. Then (1.2) is said to be completely controllable.*

Let

$$M_1 := (B(h+N-1)\ \Phi(h+N,h+N-1)B(h+N-2)\cdots\Phi(h+N,h+1)B(h)),$$

$$v := \begin{pmatrix} u(h+N-1) \\ u(h+N-2) \\ \vdots \\ u(h) \end{pmatrix},$$

$$F_1(v,x(h),x(h+N)) := x(h+N) - \Phi(h+N,h)x(h) - \sum_{i=h}^{h+N-1} \Phi(h+N,i+1)f(i,x(i)).$$

Then the following equation

$$\begin{aligned}
x(h+N) =& \Phi(h+N,h)x(h) + \sum_{i=h}^{h+N-1} \Phi(h+N,i+1)B(i)u(i)\\
&+ \sum_{i=h}^{h+N-1} \Phi(h+N,i+1)f(i,x(i)),
\end{aligned}$$

can be written as

$$M_1 v = F_1(v,x(h),x(h+N)),$$

equivalently,

$$\tilde{M_1} v := M_1^{\mathrm{T}} M_1 v = M_1^{\mathrm{T}} F_1(v,x(h),x(h+N)) := \tilde{F_1}(v,x(h),x(h+N)),$$

where $N$ is a positive integer. Then $M_1$ is an $n \times m$ matrix with $m = Np$, and $v \in \mathbb{R}^m$.

We turn the controllability problem of system (1.2) into a solvability one and get the following theorems with the same proof of Theorem 3.1.

**Theorem 3.6.** *Suppose $f$ is continuous and there exists a bounded open subset $\Omega$ of $\mathbb{R}^n$ such that the following conditions hold:*
(i) *for any $\lambda \in (0,1)$, and $x(0) = \xi \in S, x(N) = \eta \in T$, equation $\tilde{M}_1 v = \lambda \tilde{F}_1(v, x(h), x(h+N))$ has no solution on $\partial\Omega$;*
(ii) *for all $v \in \operatorname{Ker} \tilde{M}_1 \cap \partial\Omega$, and $x(0) = \xi \in S, x(N) = \eta \in T$, $Q\tilde{F}_1(v, x(h), x(h+N)) \neq 0$;*
(iii) $\deg_B(Q\tilde{F}_1|_{\operatorname{Ker} \tilde{M}_1}, \operatorname{Ker} \tilde{M}_1 \cap \Omega, 0) \neq 0$.
*Then nonlinear nonautonomous discrete control system (1.2) is controllable with respect to $(S,T)$ in step $h$, where mapping $Q : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous projector satisfying* $\operatorname{Ker} Q = \operatorname{Im} \tilde{M}_1$.

**Theorem 3.7.** *Suppose that $\tilde{M}_1$ is nonsingular, and $f$ bounded and continuous. Then nonlinear nonautonomous discrete control system (1.2) is completely controllable in step $h$.*

**Remark 3.8.** We have the following criteria: if one of the following conditions holds, then $\tilde{M}_1$ is nonsingular,
(i) $\operatorname{rank}(B(h+N-1), A(h+N-1)B(h+N-2), \cdots, A(h+N-1)\cdots A(h+1)B(h)) = n$;
(ii) $M_1^{\mathrm{T}} M_1$ is a positive-definite matrix.

## 4 Applications

For a specific nonlinear discrete control system, to apply Theorem 3.1, one can follow the following steps.

1. Based on the specific control equation, for any given $x(0) \in S$ and $x(N) \in T$ we can get

$$F(v, x(0), x(N)) := x(N) - A^N x(0) - \sum_{i=0}^{N-1} A^{N-1-i} f(x(i))$$

and

$$Mv := (B, AB, \cdots, A^{N-1}B) \begin{pmatrix} u(N-1) \\ u(N-2) \\ \vdots \\ u(0) \end{pmatrix},$$

equivalently,

$$\tilde{M}v = M^{\mathrm{T}} M v;$$

2. By parameter equation $\tilde{M}v = \lambda \tilde{F}(v, x(0), x(N))$, we construct the set $\Omega$ of $v$, such that for any $\lambda \in (0,1)$ the parameter equation has no solution on $\partial\Omega$;

3. Based on $M := (B \ AB \ \cdots \ A^{N-1}B)$, we get the continuous projection mapping $Q$ satisfying $\operatorname{Ker} Q = \operatorname{Im} \tilde{M}$, with $\tilde{M} = M^{\mathrm{T}} M$;

4. Work out $\operatorname{Ker} \tilde{M} \cap \partial\Omega$, and for $v \in \operatorname{Ker} \tilde{M} \cap \partial\Omega$ verify $Q\tilde{F}(v, x(0), x(N)) \neq 0$.

5. Calculate $\deg_B(Q\tilde{F}|_{\operatorname{Ker} \tilde{M}}, \operatorname{Ker} \tilde{M} \cap \Omega, 0)$ and verify $\deg_B(Q\tilde{F}|_{\operatorname{Ker} \tilde{M}}, \operatorname{Ker} \tilde{M} \cap \Omega, 0) \neq 0$.

Notice that when $M$ and $M_1$ are square matrices, we need to assume $\dim \operatorname{Ker} M + \operatorname{rank} M = n$ and $\dim \operatorname{Ker} M_1 + \operatorname{rank} M_1 = n$, which ensure the operators $M$ and $M_1$ are of zero-index.

The following we illustrate some applications by examples. Since $M$ is a square matrix, we consider the equation $Mv = F(v, x(0), x(N))$.

**Example 1.** Consider the controllability of the following nonlinear autonomous discrete control system:

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k) + \begin{pmatrix} x_2^3(k) \\ x_1(k)x_2(k) \end{pmatrix}. \tag{4.1}$$

Discrete control system (4.1) shows that

$$n = 2, \quad p = 1, \qquad A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We deduce that $v = (u(1), u(0))^{\mathrm{T}}$,

$$M = (B\ AB) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and

$$F(v, x(0), x(2)) = \begin{pmatrix} x_1(2) - x_1(0) - x_2^3(0) - [u(0) + x_1(0)x_2(0)]^3 \\ x_2(2) - [x_1(0) + x_2^3(0)][u(0) + x_1(0)x_2(0)] \end{pmatrix}.$$

So the linear part of system (4.1) is degenerate. Obviously, $\operatorname{Ker} M = \operatorname{Im} M = \{(0, y_2)^{\mathrm{T}} \in \mathbb{R}^2 : \forall y_2 \in \mathbb{R}\}$. Let

$$Q : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (y_1, y_2)^{\mathrm{T}} \mapsto (y_1, 0)^{\mathrm{T}}.$$

Then mapping $Q$ is a continuous projector satisfying $\operatorname{Ker} Q = \operatorname{Im} M$.

For any $x(0) = (x_1(0), x_2(0))^{\mathrm{T}}, x(2) = (x_1(2), x_2(2))^{\mathrm{T}} \in \mathbb{R}^2$, by (3.4)

$$\begin{cases} u(1) = \lambda[x_1(2) - x_1(0) - x_2^3(0)]^{4/3}, \\ u(0) = \sqrt[3]{x_1(2) - x_1(0) - x_2^3(0)} - x_1(0)x_2(0). \end{cases}$$

Let

$$\begin{aligned} \Omega := \{v \in \mathbb{R}^2 : \|v\| < ([x_1(2) - x_1(0) - x_2^3(0)]^{8/3} \\ + [-\sqrt[3]{x_1(2) - x_1(0) - x_2^3(0)} - x_1(0)x_2(0)]^2)^{1/2} + 1\}. \end{aligned}$$

For any $\lambda \in (0, 1)$, (3.4) has no solution on $\partial\Omega$, because (3.4) has only one solution $v = (u(1), u(0))^{\mathrm{T}}$ and $v = (u(1), u(0))^{\mathrm{T}} \notin \partial\Omega$. Note that

$$\begin{aligned} &\operatorname{Ker} M \cap \partial\Omega = \big\{ \pm (0, ([x_1(2) - x_1(0) - x_2^3(0)]^{8/3} \\ &\qquad + [-\sqrt[3]{x_1(2) - x_1(0) - x_2^3(0)} - x_1(0)x_2(0)]^2)^{1/2} + 1)\big\}, \\ &QF\big(x(0), \pm(0, ([x_1(2) - x_1(0) - x_2^3(0)]^{8/3} \\ &+ \big[ - \sqrt[3]{x_1(2) - x_1(0) - x_2^3(0)} - x_1(0)x_2(0)\big]^2)^{1/2} + 1)\big) \neq 0, \end{aligned}$$

and

$$QF(v, x(0), x(2)) = \begin{pmatrix} x_1(2) - x_1(0) - x_2^3(0) - [u(0) + x_1(0)x_2(0)]^3 \\ 0 \end{pmatrix}.$$

The determinant of Jacobi matrix $J_{QF}(v)$ is

$$J_{QF}(v) = \begin{vmatrix} 0 & 3[u(0) + x_1(0)x_2(0)]^2 \\ 0 & 0 \end{vmatrix} = 0,$$

thus $N_{QF} = \{v \in \operatorname{Ker} M \cap \Omega : J_{QF}(v) = 0\} = \operatorname{Ker} M \cap \Omega$ and $0 \in QF(N_{QF})$. The auxiliary function is

$$(QF)_\varepsilon(x(0), v) = \begin{pmatrix} x_1(2) - x_1(0) - x_2^3(0) - [u(0) + x_1(0)x_2(0)]^3 \\ \varepsilon u(1) \end{pmatrix},$$

where $\varepsilon > 0$. Then

$$J_{(QF)_\varepsilon}(v) = \begin{vmatrix} 0 & 3[u(0) + x_1(0)x_2(0)]^2 \\ \varepsilon & 0 \end{vmatrix} = -3\varepsilon[u(0) + x_1(0)x_2(0)]^2.$$

There is only a solution $v^* = (0, -\sqrt[3]{x_1(0) + x_2^3(0)} - x_1(0)x_2(0))$ of equation $(QF)_\varepsilon(v) = 0$ in $\operatorname{Ker} M \cap \Omega$, and $J_{(QF)_\varepsilon}(v^*) < 0$. So

$$\deg_B((QF)_\varepsilon|_{\operatorname{Ker} M}, \operatorname{Ker} M \cap \Omega, 0) = -1 \neq 0,$$

and

$$\deg_B(QF|_{\operatorname{Ker} M}, \operatorname{Ker} M \cap \Omega, 0) = \deg_B((QF)_\varepsilon|_{\operatorname{Ker} M}, \operatorname{Ker} M \cap \Omega, 0) = -1 \neq 0,$$

where $\varepsilon$ is sufficiently small. By Theorem 3.1, nonlinear autonomous discrete control system (4.1) is completely controllable.

**Example 2.** Consider the controllability of the following nonlinear autonomous discrete control system:

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1(k) \\ u_2(k) \end{pmatrix} + \begin{pmatrix} -x_2^3(k) \\ 0 \\ 0 \\ -x_3^3(k) \end{pmatrix}. \tag{4.2}$$

Discrete control system (4.2) shows that $n = 4$, $p = 2$,

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let $v = (u_1(1), u_2(1), u_1(0), u_2(0))^{\mathrm{T}}$,

$$M = (B, AB) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

and

$$F(v, x(0), x(2)) = \begin{pmatrix} x_1(2) - x_1(0) + x_3^3(0) + [x_3(0) + u_1(0)]^3 \\ x_2(2) - x_2(0) \\ x_3(2) - x_3(0) \\ x_4(2) - x_4(0) + x_2^3(0) + [x_2(0) + u_2(0)]^3 \end{pmatrix}.$$

So the linear part of system (4.2) is degenerate. Obviously,

$$\operatorname{Ker} M = \{(y_1, y_2, y_3, y_4)^{\mathrm{T}} \in \mathbb{R}^4 : y_1 + y_4 = 0, y_2 + y_3 = 0\}$$

and

$$\operatorname{Im}(M) = \{(y_1, y_2, y_3, y_4)^{\mathrm{T}} \in \mathbb{R}^4 : y_1 = y_2, y_3 = y_4\}.$$

Let

$$Q : \mathbb{R}^4 \to \mathbb{R}^4, \quad (y_1, y_2, y_3, y_4)^{\mathrm{T}} \mapsto (y_1 - y_2, 0, 0, y_4 - y_3)^{\mathrm{T}}.$$

Then mapping $Q$ is a continuous projector satisfying $\operatorname{Ker} Q = \operatorname{Im} M$.

For any $x(0) = (x_1(0), x_2(0), x_3(0), x_4(0))^{\mathrm{T}} \in \mathbb{R}^4$, by (3.4)

$$\begin{cases} u_1(1) = \lambda x_2(2) + (1-\lambda)x_2(0) - \sqrt[3]{x_4(0) - x_4(2) + x_3(2) - x_3(0) - x_2^3(0)}, \\ u_2(1) = \lambda x_3(2) + (1-\lambda)x_3(0) - \sqrt[3]{x_1(0) - x_1(2) + x_2(2) - x_2(0) - x_3^3(0)}, \\ u_1(0) = -x_3(0) + \sqrt[3]{x_1(0) - x_1(2) + x_2(2) - x_2(0) - x_3^3(0)}, \\ u_2(0) = -x_2(0) + \sqrt[3]{x_4(0) - x_4(2) + x_3(2) - x_3(0) - x_2^3(0)}. \end{cases}$$

Let

$$\begin{aligned} \Omega :=& \{v \in \mathbb{R}^4 : \|v\| < (4(x_3^2(0) + x_2^2(0) + [x_1(0) - x_1(2) + x_2(2) - x_2(0) - x_3^3(0)]^{2/3} \\ &+ x_3^2(2) + x_2^2(2) + [x_4(0) - x_4(2) + x_3(2) - x_3(0) - x_2^3(0)]^{2/3}))^{1/2}\}. \end{aligned}$$

For any $\lambda \in (0,1)$ (3.4) has no solution on $\partial\Omega$, because (3.4) has only one solution $v = (u_1(1), u_2(1), u_1(0), u_2(0))^{\mathrm{T}}$ and $v = (u_1(1), u_2(1), u_1(0), u_2(0))^{\mathrm{T}} \notin \partial\Omega$. Note that

$$\begin{aligned} \operatorname{Ker} M \cap \partial\Omega = \{&(y_1, y_2, y_3, y_4)^{\mathrm{T}} \in \mathbb{R}^4 : \; y_1 + y_4 = 0, \; y_2 + y_3 = 0, \\ &y_1^2 + y_2^2 + y_3^2 + y_4^2 = 4(x_3^2(0) + x_2^2(0) \\ &+ [x_1(0) - x_1(2) + x_2(2) - x_2(0) - x_3^3(0)]^{2/3} \\ &+ x_3^2(2) + x_2^2(2) + [x_4(0) - x_4(2) + x_3(2) - x_3(0) - x_2^3(0)]^{2/3})\}, \end{aligned}$$

$$QF(v, x(0), x(2)) = \begin{pmatrix} x_2(0) - x_2(2) + x_1(2) - x_1(0) + x_3^3(0) + [x_3(0) + u_1(0)]^3 \\ 0 \\ 0 \\ x_3(0) - x_3(2) + x_4(2) - x_4(0) + x_2^3(0) + [x_2(0) + u_2(0)]^3 \end{pmatrix}.$$

Let $QF(v, x(0), x(2)) = 0$. Then

$$\begin{aligned} v =& (u_1(1), u_2(1), -x_3(0) + \sqrt[3]{x_1(0) - x_1(2) + x_2(2) - x_2(0) - x_3^3(0)}, \\ &- x_2(0) + \sqrt[3]{x_4(0) - x_4(2) + x_3(2) - x_3(0) - x_2^3(0)})^{\mathrm{T}}, \end{aligned}$$

where $u_1(1)$ and $u_2(1)$ are arbitrary. The equation $QF(v, x(0), x(2)) = 0$ has a unique solution

$$\begin{aligned} v^* = \big(&x_2(0) - \sqrt[3]{x_4(0) - x_4(2) + x_3(2) - x_3(0) - x_2^3(0)}, \\ &x_3(0) - \sqrt[3]{x_1(0) - x_1(2) + x_2(2) - x_2(0) - x_3^3(0)}, \\ &- x_3(0) + \sqrt[3]{x_1(0) - x_1(2) + x_2(2) - x_2(0) - x_3^3(0)}, \end{aligned}$$

$$- x_2(0) + \sqrt[3]{x_4(0) - x_4(2) + x_3(2) - x_3(0) - x_2^3(0)}\Big)^{\mathrm{T}},$$

and $v^* \notin \operatorname{Ker} M \cap \partial\Omega$, so for any $v \in \operatorname{Ker} M \cap \partial\Omega$, $QF(v, x(0), x(2)) \neq 0$. The determinant of Jacobi matrix $J_{QF}(v)$ is

$$J_{QF}(v) = \begin{vmatrix} 0 & 0 & 3[x_3(0) + u_1(0)]^2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3[x_2(0) + u_2(0)]^2 \end{vmatrix} = 0,$$

thus $N_{QF} = \{v \in \operatorname{Ker} M \cap \Omega : J_{QF}(v) = 0\} = \operatorname{Ker} M \cap \Omega$ and $0 \in QF(N_{QF})$. The auxiliary function is

$$(QF)_\varepsilon(x(0), v) = \begin{pmatrix} x_2(0) + x_1(2) - x_2(2) - x_1(0) + x_3^3(0) + [x_3(0) + u_1(0)]^3 \\ \varepsilon_1[u_1(1) - x_2(0) + \sqrt[3]{x_4(0) - x_4(2) + x_3(2) - x_3(0) - x_2^3(0)}] \\ \varepsilon_2[u_2(1) - x_3(0) + \sqrt[3]{x_1(0) - x_1(2) + x_2(2) - x_2(0) - x_3^3(0)}] \\ x_3(0) + x_4(2) - x_3(2) - x_4(0) + x_2^3(0) + [x_2(0) + u_2(0)]^3 \end{pmatrix},$$

where $\varepsilon_1 > 0$, $\varepsilon_2 > 0$. Then

$$\begin{aligned} J_{(QF)_\varepsilon}(v) &= \begin{vmatrix} 0 & 0 & 3[x_3(0) + u_1(0)]^2 & 0 \\ \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_2 & 0 & 0 \\ 0 & 0 & 0 & 3[x_2(0) + u_2(0)]^2 \end{vmatrix} \\ &= 9\varepsilon_1\varepsilon_2 2[x_3(0) + u_1(0)]^2[x_2(0) + u_2(0)]^2. \end{aligned}$$

$v^*$ is the solution of equation $(QF)_\varepsilon(v) = 0$ and $QF(v) = 0$ in $\operatorname{Ker} M \cap \Omega$, and $J_{(QF)_\varepsilon}(v^*) > 0$. So

$$\deg_B((QF)_\varepsilon|_{\operatorname{Ker} M}, \operatorname{Ker} M \cap \Omega, 0) = 1 \neq 0,$$

and

$$\deg_B(QF|_{\operatorname{Ker} M}, \operatorname{Ker} M \cap \Omega, 0) = \deg_B((QF)_\varepsilon|_{\operatorname{Ker} M}, \operatorname{Ker} M \cap \Omega, 0) = 1 \neq 0,$$

where $\varepsilon_1$ and $\varepsilon_2$ are sufficiently small. By Theorem 3.1, nonlinear autonomous discrete system (4.2) is completely controllable.

**Example 3.** Consider the controllability of the following nonlinear nonautonomous discrete system:

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} u(k) + \begin{pmatrix} g_1(x_1(k), x_2(k)) \\ g_2(x_1(k), x_2(k)) \end{pmatrix}, \tag{4.3}$$

where $b_1^2 + b_2^2 \neq 0$, $g_1$ and $g_2$ are continuous, $\mid g_1(x_1(k), x_2(k)) \mid \leq l_1$, $\mid g_2(x_1(k), x_2(k)) \mid \leq l_2$, $l_1, l_2$ are constants.

From (4.3), $n = 2, p = 1$,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

and

$$M = (B, AB) = \begin{pmatrix} b_1 & b_2 \\ b_2 & -b_1 \end{pmatrix}.$$

The matrix $M$ is full rank by $b_1^2 + b_2^2 \neq 0$. We can obtain that nonlinear nonautonomous discrete system (4.3) is controllable by applying Theorem 3.2.

## 5 Conclusions

In this paper, the controllability of nonlinear discrete control systems is studied. Sufficient conditions for the systems to be controllable are presented via Fredholm operators and coincidence degree theory. Applications are given to illustrate results of this paper. In future work, we should more in-depth study the controllability of nonlinear discrete control systems and find necessary conditions.

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