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Generalized Lagrangian Duality in Set-valued Vector Optimization via Abstract Subdifferential

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Abstract In this paper, we investigate dual problems for nonconvex set-valued vector optimization via abstract subdifferential. We first introduce a generalized augmented Lagrangian function induced by a coupling vector-valued function for set-valued vector optimization problem and construct related set-valued dual map and dual optimization problem on the basic of weak efficiency, which used by the concepts of supremum and infimum of a set. We then establish the weak and strong duality results under this augmented Lagrangian and present sufficient conditions for exact penalization via an abstract subdifferential of the object map. Finally, we define the sub-optimal path related to the dual problem and show that every cluster point of this sub-optimal path is a primal optimal solution of the object optimization problem. In addition, we consider a generalized vector variational inequality as an application of abstract subdifferential.

Keywords Nonconvex set-valued vector optimization; abstract subdifferential; generalized augmented Lagrangian duality; exact penalization; sub-optimal path.
2000 MR Subject Classification 90C30; 90C46

1 Introduction

It is well known that augmented Lagrangian methods are useful for solving nonconvex optimization problems. Rockafellar and Wets[12] considered a primal problem of minimizing an extended real-valued function and proposed and analyzed a dual approach via augmented Lagrangians which is convex. They also presented strong duality and a criterion for exact penalty representation (see [12], Theorems 11.59 and 11.61). Wang et al. (see [16], Sect. 3.1) studied an augmented Lagrangian type function via an auxiliary coupling function, and proposed a valley-at-zero type property in the derivative of the coupling function with respect to the penalty parameter. Burachik and Iusem[2] considered a primal problem of minimizing an extended real-valued function in a Hausdorff topological space. With abstract convexity tools, they proposed duality scheme induced by a generalized augmented Lagrangian function and analyzed a valley-at-zero type property on the coupling (augmenting) function, which generalizes the valley-at-zero type property proposed in the related literature (e.g., Burachik and Rubinov[3] and references therein). Recently, Huang and Yang[13] extended augmented Lagrangian approach introduced by Rockafellar to vector optimization. Huy and Kim[4] developed augmented Lagrangian duality theory in set-valued vector optimization. By using the concepts of the supremum and infimum of a set and conjugate duality of a set-valued map on the basis of weak efficiency, they proposed an augmented Lagrangian function for a vector optimization problem with set-valued data and

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established weak and strong Lagrangian duality results under assumptions of R^m_+-lower semi-continuity and R^m_+-lower boundness. Except mentioned above, there are also many researches concerning with nonconvex set-valued optimization problems (see [10, 11]).

In the present paper, we consider dual problems for nonconvex set-valued vector optimization via abstract subdifferential. We introduce a generalized augmented Lagrangian function induced by a coupling vector-valued function for primal optimization problem and construct related set-valued dual map and dual optimization problem on the basis of weak efficiency, which used by the concepts of supremum and infimum of a set. Then we establish weak and strong duality results under this augmented Lagrangian and present sufficient conditions for exact penalization via an abstract subdifferential of the object map. This abstract subdifferential (see Definition 2.2) which develops from scalar situation, recently became a natural language to investigate duality schemes via augmented Lagrangian function. We also define the sub-optimal path related to the dual problem and show that every cluster point of this sub-optimal path is a primal optimal solution of the object optimization problem. Our these results extend the corresponding theorems existing in scalar optimization. As an application of abstract subdifferential mentioned above, we construct in the final section a generalized vector variational inequality problem and investigate some properties of this variational inequality.

The paper is structured as following. In Section 2, we present basic definitions and assumptions, and also state our primal-dual scheme. In Section 3, we first show that our duality results including weak and strong duality, and then establish a criterion to exact penalty representation. In Section 4, we define the sub-optimal path related to the dual problem and examine the convergence properties of this sub-optimal path. The final Section 5, as an application of abstract subdifferential, we consider a generalized vector variational inequality defined by means of abstract subdifferential and its gap function.

2 Preliminaries

Let Y be a real linear topological space which is partially ordered by a pointed closed convex cone K with a nonempty interior $\operatorname{int} K$ in Y. We use the following notations:

$$y \leq_K y' \quad \text{iff } y' - y \in K,$$

$$y <_K y' \quad \text{iff } y' - y \in \operatorname{int}K,$$

and

$$y \nleq_K y' \quad \text{iff } y' - y \notin \mathrm{K},$$

$$y \nless_K y' \quad \text{iff } y' - y \notin \operatorname{int}K.$$

The largest element (not belonging to Y) of Y with respect to partial order $\leq_K$ is denoted as $+\infty_Y$. In the same way, the smallest element (not belonging to Y) of Y related to partial order $\leq_K$ is denoted as $-\infty_Y$. Let $\overline{Y} = Y \cup \{\pm\infty_Y\}$.

For simplicity, we use "$\leq$" instead of "$\leq_K$", "$<$" instead of "$<_K$" and also "$+\infty$" instead of "$+\infty_Y$", "$-\infty$" instead of "$-\infty_Y$", "0" instead of the origin of all space.

Given a set $M \subset \overline{Y}$, we define the set of elements above M and the set of elements below M by $A(M) = \{y \in \overline{Y} \mid \; y' < y \;\text{ for some } y' \in M\},\; B(M) = \{y \in \overline{Y} \mid \; y < y' \;\text{ for some } y' \in M\},$ respectively. Clearly,

$$A(M) = (M + \operatorname{int}K) \cup \{+\infty\}, \quad B(M) = (M - \operatorname{int}K) \cup \{-\infty\}, \qquad \forall\, M \subset Y, \;\; M \neq \emptyset,$$

and $A(\emptyset) = B(\emptyset) = \emptyset$. Here, $A(\{y\})$ and $B(\{y\})$ are simply denoted by $A(y)$ and $B(y)$, respectively.

Definition 2.1[15]**.** *Let $M \subset Y$ be a given set.*

(i) An element $\hat{y} \in Y$ is said to be a weakly infimal element of M if $\hat{y} \notin A(M)$ and $A(\hat{y}) \subset A(M)$, that is, there is no $y \in M$ such that $y < \hat{y}$ and if the relation $\hat{y} < y'$ implies the existence of some $y \in M$ such that $y < y'$. The set of all weakly infimal elements of M is called the weak infimum of M and is denoted by $\inf M$. The weak supremum of M, $\sup M$, is defined analogously.

(ii) An element $\hat{y} \in M$ is said to be a weakly minimal element of M if $\hat{y} \notin A(M)$, that is, there is no $y \in M$ such that $y < \hat{y}$. The set of all weakly minimal elements of M is called the weak minimal of M and is denoted by $\min M$. The weak maximum of M, $\max M$, is defined analogously.

Let us summarize some properties of the supremum and infimum of a set which is useful in next section.

Lemma 2.1[15]**.** *Let $M \subset \overline{Y}$ be a given set. Then, one has:*

(i) $A(M) = A(\inf M)$ *and* $B(M) = B(\sup M)$.

(ii) $M \subset \inf M \cup A(M)$ *and* $M \subset \sup M \cup B(M)$.

(iii) $\overline{Y} = \inf M \cup A(\inf M) \cup B(\inf M)$ *and the three sets on the right-hand side are disjoint.*

(iv) $B(\bigcup\limits_{i\in I} M_i) = \bigcup\limits_{i\in I} B(M_i)$ *and* $A(\bigcup\limits_{i\in I} M_i) = \bigcup\limits_{i\in I} A(M_i)$, *where $M_i \subset \overline{Y}$, $i \in I$ and I is an arbitrary index set.*

(v) $\sup(\inf M) = \inf M$, $\inf(\sup M) = \sup M$, $\sup(\sup M) = \sup M$, $\inf(\inf M) = \inf M$.

(vi) $\inf M_1 \subset (\inf M_2) \cup A(\inf M_2)$ *and* $\sup M_1 \subset (\sup M_2) \cup B(\sup M_2)$ *for all* $M_1 \subset M_2 \subset \overline{Y}$.

(vii) *For two arbitrary sets M_1 and M_2 in $\overline{Y}$ it holds: $M_1 \cap A(M_2) = \emptyset$ if and only if $B(M_1) \cap M_2 = \emptyset$.*

Proposition 2.2. *If $M \subset \inf N$, then $B(M) \cap N = \emptyset$.*

Proof. It follows immediately from the definition of $\inf N$, we know $A(N) \cap \inf N = \emptyset$. Considering $M \subset \inf N$, we obtain $A(N) \cap M = \emptyset$, this together with Lemma (vii) yield $B(M) \cap N = \emptyset$.

Let $F : X \to \overline{Y}$ be a set-valued map with its domain given by

$$\text{dom}F = \{x \in X \mid F(x) \cap Y \neq \emptyset\}.$$

Define

$$\operatorname*{Argmin}_{x\in X} F(x) = \Big\{\overline{x} \in \text{dom}F \mid \exists\ \overline{z} \in F(\overline{x}) \setminus \{\pm\infty\} \text{ and } \overline{z} \in \inf_{x\in X} F(x)\Big\},$$

and

$$\text{Argmin}F = \text{Argmin}_x F(x) = \operatorname*{Argmin}_{x\in X} F(x).$$

Obviously, if $\overline{x} \in \operatorname*{Argmin}\limits_{x\in X}$ and $\overline{z} \in F(\overline{x}) \cap \inf\limits_{x\in X} F(x)$, then $\overline{z} \in \inf F(\overline{x})$.

Lemma 2.3[14]**.** *Let $F : X \to \overline{Y}$ be a set-valued map. Then one has*

$$\sup\Big(\bigcup_{x\in X} F(x)\Big) = \sup\Big(\bigcup_{x\in X} \sup F(x)\Big),$$

$$\inf\Big(\bigcup_{x\in X} F(x)\Big) = \inf\Big(\bigcup_{x\in X} \inf F(x)\Big)$$

Lemma 2.4[4]**.** *For $F : X \times Y \to \overline{Z}$ one has in terms of $P(y) = \inf_{x\in X} F(x,y)$ and $Q(x) = \inf_{y\in Y} F(x,y)$ that*

$$\inf_{(x,y)\in X\times Y} F(x,y) = \inf_{y\in Y} P(y) = \inf_{x\in X} Q(x),$$

$$\begin{aligned} Argmin_{(x,y)} F(x,y) =& \Big\{(\overline{x},\overline{y}) \in X\times Y \mid (F(\overline{x},\overline{y})\setminus\{\pm\infty\}) \cap \Big(\inf_x F(x,\overline{y})\Big) \cap \Big(\inf_y P(y)\Big) \neq \emptyset\Big\} \\ =& \Big\{(\overline{x},\overline{y}) \in X\times Y \mid (F(\overline{x},\overline{y})\setminus\{\pm\infty\}) \cap \Big(\inf_y F(\overline{x},y)\Big) \cap \Big(\inf_x Q(x)\Big) \neq \emptyset\Big\}. \end{aligned}$$

Lemma 2.5[4]**.** *Let $M \subset \overline{Y}$, and let $K \subset Y$ be a pointed closed convex cone with int$K \neq \emptyset$ and $0 \in K$. Then,*

(i) $\sup(M - K) = \sup M$, $\inf(M + K) = \inf M$;

(ii) $\inf M = \min(M + K) \subset M + K$, *if M is a nonempty compact set in Y.*

In the sequel, we assume that Y be a real linear topological space, X, Z be two Hausdorff topological spaces and order cone $R^m_+ \subset R^m$ is a positive quadrant. We consider the optimization problem,

$$\text{(P)} \quad \inf_{x\in X} F(x) = \inf F(X) = \inf\Big(\bigcup_{x\in X} F(x)\Big),$$

where $F : X \to R^m \cup \{+\infty\} = R^m_{+\infty}$ is a set-valued map with dom$F \neq \emptyset$ and $F(x) \neq -\infty$, $\forall x \in X$. This is equivalent to that F is proper on X, i.e., $F(x) \neq -\infty$ for all $x \in X$, and there exist $\overline{x} \in X$ and $\overline{v} \in F(\overline{x})$ such that $\overline{v} < +\infty$. We fix a base point in Z and denote it by 0. In order to introduce our duality scheme, we consider a duality parameterization for F, which is a set-valued map $\phi : X \times Z \to R^m_{+\infty}$ satisfying $\phi(x,0) = F(x)$ for all $x \in X$. We also consider a perturbation map $H : Z \to R^m \cup \{\pm\infty\} = \overline{R^m}$, related to this duality parameterization, given by

$$H(z) = \inf_{x\in X} \phi(x,z).$$

Since F is proper, $H(0) \neq +\infty$. For a set $V \subset Z$, we use the notation $V^C = Z \setminus V$.

In what follows, we consider a coupling vector-valued function (for similar scalar function see [2]) $\rho : Z \times Y \times R_+ \to R^m$, where $R_+ = [0,+\infty)$, that satisfies the following basic assumptions:

(C_1) For any $(y,r) \in Y \times R_+$, $\rho(0,y,r) = 0$.

(C_2) For every neighborhood $V \subset Z$ of 0, and for every $(y,\overline{r}) \in Y \times R_+$, it holds that

(i) $A^V_{y,\overline{r}}(r) = \inf_{z\in V^C}\{\rho(z,y,\overline{r}) - \rho(z,y,r)\} \subset \text{int}R^m_+,\ \forall\ r > \overline{r}$;

(ii) $\lim_{r\to\infty} A^V_{y,\overline{r}}(r) = +\infty$.

Remark 2.6. It follows immediately from C_2 that

$$\rho(z,y,\overline{r}) - \rho(z,y,r) \in R^m_+,$$

and

$$\lim_{r\to+\infty} \min_{1\le i\le \text{m}} (\rho_1,\rho_2,\cdots,\rho_m) = +\infty, \qquad \forall\, \rho \in A^V_{y,\overline{r}}(r) \subset \text{int}R^m_+,$$

for every neighborhood $V \subset Z$ of 0 and for all $(y,\overline{r}) \in Y \times R_+$, $r > \overline{r}$.

The augmented Lagrangian function $L : X \times Y \times R_+ \to \overline{R^m}$ for (P), induced by the coupling function ρ, is defined as

$$L(x, y, r) = \inf_{z \in Z} \{\phi(x, z) - \rho(z, y, r)\}$$

The dual map $\psi : Y \times R_{++} \to \overline{R^m}$ is defined as

$$\psi(y, r) = \inf_{x \in X} L(x, y, r),$$

and therefore the dual problem is stated as

$$(\mathrm{P}^*) \qquad \sup_{(y,r) \in (Y \times R_+)} \psi(y, r).$$

We denote $\inf P = \inf_{x \in X} F(x)$ and $\sup P^* = \sup_{(y,r) \in Y \times R_+} \psi(y, r)$.

From definition in [2] and [9], we consider an extension of the concept of generalized abstract subgradient to the case of set-valued map.

Definition 2.2. *Let $G : Z \to \overline{R^m}$ be a set-valued map, $\overline{z} \in Z$ and $\overline{\lambda} \in G(\overline{z})$, we say that (y, r) is a generalized abstract subgradient of G at $(\overline{z}, \overline{\lambda})$ (with respect to ρ) iff*

$$\overline{\lambda} - \rho(\overline{z}, y, r) \in \inf_{z \in Z} \{G(z) - \rho(z, y, r)\}.$$

The set of generalized abstract subgradient of G at $(\overline{z}, \overline{\lambda})$, denoted by $\partial_\rho G(\overline{z}, \overline{\lambda})$, is called generalized abstract subdifferential of G at $(\overline{z}, \overline{\lambda})$ with respect to the coupling function ρ. Moreover, we let $\partial_\rho G(\overline{z}) = \bigcup_{\overline{\lambda} \in G(\overline{z})} \partial_\rho G(\overline{z}, \overline{\lambda})$. If $\partial_\rho G(\overline{z}, \overline{\lambda}) \neq \emptyset$ for every $\overline{\lambda} \in G(\overline{z})$, then G is said to be abstract subdifferentiable at $\overline{z}$.

Remark 2.7. It follows from C_1, the definition of $\partial_\rho G(0)$ and Remark 2.6 that, if $(y, r_0) \in \partial_\rho G(0)$, then $(y, r) \in \partial_\rho G(0)$ for all $r \geq r_0$.

Proposition 2.8. *Take $(\overline{y}, \overline{r}) \in Y \times R_+$ and $\lambda_0 \in H(0)$. Then*

(i) $(\overline{y}, \overline{r}) \in \partial_\rho H(0, \lambda_0)$ if and only if $\lambda_0 \in \psi(\overline{y}, \overline{r})$;

(ii) $(\overline{y}, \overline{r}) \in dom(-\psi)$ if and only if there exists $\overline{c} \in R^m$ such that $\overline{c} \in \psi(\overline{y}, \overline{r})$.

Proof. (i) follows from the following equivalences:

$$\begin{aligned}(\overline{y}, \overline{r}) \in \partial_\rho H(0, \lambda_0) \Leftrightarrow & \lambda_0 \in \inf_{z \in Z} \{H(z) - \rho(z, \overline{y}, \overline{r})\} \Leftrightarrow \lambda_0 \in \inf_{z \in Z} \inf_{x \in X} \{\phi(x, z) - \rho(z, \overline{y}, \overline{r})\} \\ \Leftrightarrow & \lambda_0 \in \inf_{x \in X} \inf_{z \in Z} \{\phi(x, z) - \rho(z, \overline{y}, \overline{r})\} \Leftrightarrow \lambda_0 \in \inf_{x \in X} L(x, \overline{y}, \overline{r}) \Leftrightarrow \lambda_0 \in \psi(\overline{y}, \overline{r}).\end{aligned}$$

The third "⇔" is from Lemma 2.4. Next, (ii) follows from the following

$$\begin{aligned}(\overline{y}, \overline{r}) \in \mathrm{dom}(-\psi) \Leftrightarrow & -\psi(\overline{y}, \overline{r}) \neq +\infty \Leftrightarrow \psi(\overline{y}, \overline{r}) \neq -\infty \\ \Leftrightarrow & \exists\ \overline{c} \in R^m \text{ such that } \overline{c} \in \psi(\overline{y}, \overline{r}).\end{aligned}$$

□

The following Theorem shows that the perturbation map H associated with (P) is abstract subdifferential at 0.

Theorem 2.9. *Assume that C_1, C_2 hold and there exist $(\overline{y},\overline{r}) \in Y \times R_+$ such that $(\overline{y},\overline{r}) \in dom(-\psi)$. If there also exists a neighborhood $V \subset Z$ of 0 such that $B(H(0)) \cap (H(z) - \rho(z,\overline{y},\overline{r})) = \emptyset$ for all $z \in V$. Then there exists r_0 such that $(\overline{y},r) \in \partial_\rho H(0)$ for all $r \geq r_0$.*

Proof. Since F is proper on X, then $H(0) \neq +\infty$. Since $(\overline{y},\overline{r}) \in \mathrm{dom}(-\psi)$, then by Proposition 2.8 (ii) that there exists $\overline{c} \in R^m$ such that $\overline{c} \in \psi(\overline{y},\overline{r}) = \inf\limits_{z\in Z}\{H(z) - \rho(z,\overline{y},\overline{r})\}$, so for all $\lambda_0 \in H(0)$, $\lambda_0 \not< \overline{c}$, that is $H(0) \neq -\infty$. As a consequence, $H(0) \subset R^m$. Suppose by contradiction that for all $\lambda_0 \in H(0)$, $k > 0$, there exist $r_k > k$, $z_k \in Z$ and $\lambda_k \in H(z_k)$ satisfying

$$\lambda_0 > \lambda_k - \rho(z_k,\overline{y},r_k). \tag{2.1}$$

Assume that $\{z_k\}_{k\in N}$ converges to 0. Then there exists k_0 such that $z_k \in V$ for all $k \geq k_0 \geq \overline{r}$ and so $\lambda_k - \rho(z_k,\overline{y},r_k) \in B(H(0))$, which contradict assumption $B(H(0)) \cap (H(z) - \rho(z,\overline{y},\overline{r})) = \emptyset$ for all $z \in V$ and $r \geq \overline{r}$. Therefore, $\{z_k\}_{k\in N}$ does not converge to 0, which implies that there exists some open neighborhood $W \subset Z$ of 0, we can take $W = V$, and a subsequence $\{z_{k_j}\}_{j\in N} \subset \{z_k\}_{k\in N}$ such that $\{z_{k_j}\}_{j\in N} \subset V^C$. Now using (2.1) and the fact there exists $\overline{c} \in R^m$ such that $\overline{c} \in \psi(\overline{y},\overline{r})$, so we have

$$\begin{aligned}\lambda_0 > \lambda_{k_j} - \rho(z_{k_j},\overline{y},r_{k_j}) =& \lambda_{k_j} - \rho(z_{k_j},\overline{y},\overline{r}) + \rho(z_{k_j},\overline{y},\overline{r}) - \rho(z_{k_j},\overline{y},r_{k_j}) \\ \not<& \overline{c} + (\rho(z_{k_j},\overline{y},\overline{r}) - \rho(z_{k_j},\overline{y},r_{k_j})).\end{aligned}$$

Furthermore,

$$\lambda_0 \not< \overline{c} + (\rho(z_{k_j},\overline{y},\overline{r}) - \rho(z_{k_j},\overline{y},r_{k_j})),$$

which contradicts C_2 (ii), because $\lim\limits_{j\to\infty}(\rho(z_{k_j},\overline{y},\overline{r}) - \rho(z_{k_j},\overline{y},r_{k_j})) = +\infty$. So we conclude that there exists $r_0 \geq \overline{r}$ such that $\lambda_0 \not> \lambda - \rho(z,\overline{y},r_0)$ for all $z \in Z$ and $\lambda \in H(z)$ and $A(\lambda_0) \subset A(\bigcup\limits_{z\in Z} H(z) - \rho(z,\overline{y}),r_0)$ with $\lambda_0 \in H(0)$, which means that $(\overline{y},r_0) \in \partial_\rho H(0)$. The result follows immediately from Remark 2.6. □

Remark 2.10. It follows from Theorem 2.9 that, if $(\overline{y},\overline{r}) \in Y \times R_+$ such that $H(0) \subset \inf\limits_{z\in Z}\{H(z) - \rho(z,\overline{y},\overline{r})\}$, then $H(0) \subset \inf\limits_{z\in Z}\{H(z) - \rho(z,\overline{y},r)\}$ for all $r \geq \overline{r}$. Obviously, for all $\lambda_1 \in \inf\limits_{z\in Z}\{H(z) - \rho(z,\overline{y},\overline{r})\}$ and $\lambda_2 \in \inf\limits_{z\in Z}\{H(z) - \rho(z,\overline{y},r)\}$, we have $\lambda_2 \not< \lambda_1$, where $r \geq \overline{r}$.

3 Duality and Exact Penalty Representation

In this section we establish sufficient conditions for weak and strong duality as well as penalty representations of the primal problem (P).

The following result gives a weak duality theorem.

Theorem 3.1 (Weak duality). *One has*

$$A(\inf P) \cap \sup P^* = \emptyset.$$

Proof. First we need to prove

$$F(x) \cap B(\psi(y,r)) = \emptyset,$$

for all $x \in X$, $y \in Y$ and $r \in R_+$. Suppose on the contrary that there exist $\overline{x} \in X$, $\overline{v} \in F(\overline{x})$ and $(\overline{y},\overline{r}) \in Y \times R_+$ such that $\overline{v} \in B(\psi(\overline{y},\overline{r}))$. Then there are some $\overline{q} \in \psi(\overline{y},\overline{r})$ such that $\overline{q} > \overline{v}$.

Since $\psi(\overline{y},\overline{r}) = \inf\limits_{x\in X}\inf\limits_{z\in Z}\{\phi(x,z)-\rho(z,\overline{y},\overline{r})\} = \inf\{\bigcup\limits_{x\in x}\bigcup\limits_{z\in Z}\{\phi(x,z)-\rho(z,\overline{y},\overline{r})\}\}$, thus, for all $v\in\bigcup\limits_{x\in x}\bigcup\limits_{z\in Z}(\phi(x,z)-\rho(z,\overline{y},\overline{r}))$, we have $v\not<\overline{q}$. However, $\overline{v}\in F(\overline{x}) = \phi(\overline{x},0)-\rho(0,\overline{y},\overline{r})$, hence, $\overline{v\not<\overline{q}}$, it is a contradiction. Which means $F(x)\cap B(\psi(y,r))=\emptyset$, $\forall\ x\in X$, $y\in Y$, $r\in R_+$.

It follows that

$$F(x)\cap B\Big(\inf_{x\in X}L(x,y,r)\Big)=\emptyset,\qquad \forall\, x\in X,\quad y\in Y,\quad r\in R_+.$$

Hence,

$$\Big(\bigcup_{x\in X}F(x)\Big)\cap B\Big(\inf_{x\in X}L(x,y,r)\Big)=\emptyset,\qquad y\in Y,\quad r\in R_+.$$

Since r is arbitrary, we can assert from Lemma 2.1(iv) that

$$\Big(\bigcup_{x\in X}F(x)\Big)\cap B\Big(\bigcup_{r\in R_+}\bigcup_{y\in Y}\inf_{x\in X}L(x,y,r)\Big)=\emptyset.$$

Combining this above with Lemma 2.1(i), we have

$$\Big(\bigcup_{x\in X}F(x)\Big)\cap B(\sup P^*)=\emptyset.$$

By Lemma 2.1(vii),

$$A\Big(\bigcup_{x\in X}F(x)\Big)\cap \sup P^*=\emptyset.$$

This and Lemma 2.1(i) imply that

$$A\Big(\inf\bigcup_{x\in X}F(x)\Big)\cap \sup P^*=\emptyset.$$

That is,

$$A(\inf P)\cap\sup P^*=\emptyset.$$

The proof is complete. □

Now we give a sufficient condition which is related to abstract subdifferentiable of perturbation map H at 0 ensuring strong duality for the primal problem.

Theorem 3.2 (Strong duality)**.** *If H is abstract subdifferentiable at 0, then $H(0)\subset\sup P^*$. Consequently, $H(0)=\max P^*$.*

Proof. We note that

$$\begin{aligned}\sup P^* &= \sup_{(y,r)\in Y\times R_+}\psi(y,r)\\ &= \sup_{(y,r)\in Y\times R_+}\inf_{x\in X}\inf_{z\in Z}(\phi(x,z)-\rho(z,y,r)\})\\ &= \sup_{(y,r)\in Y\times R_+}\inf_{z\in Z}\inf_{x\in X}(\phi(x,z)-\rho(z,y,r)\})\\ &= \sup_{(y,r)\in Y\times R_+}\{\inf_{z\in Z}\{H(z)-\rho(z,y,r)\}\},\end{aligned}$$

where the third "=" above is from Lemma 2.4. Since H is abstract subdifferentiable at 0, so for every $\lambda_0 \in H(0)$, there exist $(y_0, r_0) \in Y \times R_+$ such that

$$\lambda_0 \in \inf_{z \in Z}\{H(z) - \rho(z, y_0, r_0)\}.$$

That is,

$$\begin{aligned} H(0) &\subset \bigcup_{(y,r)\in Y\times R_+} \inf_{z\in Z}\{H(z) - \rho(z,y,r)\} \\ &\subset \sup_{(y,r)\in Y\times R_{++}} \inf_{z\in Z}\{H(z) - \rho(z,y,r)\} \cup B(\sup P^*), \end{aligned}$$

the last inclusion follows from Lemma 2.1 (ii). By the weak duality and Lemma 2.1 (vii), we know that $H(0) \cap B(\sup P^*) = \emptyset$, therefore, $H(0) \subset \sup P^*$.

Now, we prove $H(0) \supset \sup P^*$. Take any $\lambda_0 \in \sup P^*$, from the weak duality, we know that $\lambda_0 \notin A(H(0))$. If $\lambda_0 \in \inf(H(0))$, by Lemma 2.1 (v) and the definition of $H(0)$, we have $H(0) = \inf(H(0))$, so $\lambda_0 \in H(0)$; otherwise, if $\lambda_0 \notin \inf(H(0))$, then by Lemma 2.1 (iii), $\lambda_0 \in B(\inf(H(0)))$, i.e., there exists $\overline{\lambda} \in \inf(H(0))$ such that

$$\lambda_0 < \overline{\lambda}. \tag{3.1}$$

By $H(0) = \inf(H(0))$, so $\overline{\lambda} \in H(0)$. Since H is abstract subdifferentiable at 0, thus, there exist $(\overline{y}, \overline{r}) \in Y \times R_+$ such that $\overline{\lambda} \in \inf\limits_{z\in Z}\{H(z) - \rho(z, \overline{y}, \overline{r})\}$, consequently,

$$\lambda_0 \not< \overline{\lambda}$$

which contradict (3.1), so $H(0) \supset \sup P^*$ holds. In the following, we just need to prove $H(0) \subset \max P^*$. Let $\lambda_0 \in H(0)$, since H is abstract subdifferentiable at 0, so there exist (y_0, r_0) such that $\lambda_0 \in \inf\limits_{z\in Z}\{H(z) - \rho(z, y_0, r_0)\}$. Considering $\lambda_0 \in H(0)$, so we have $\lambda_0 \in \max P^*$. The proof is complete. □

Theorem 3.3. *If $H(0) = \max P^*$, then H is abstract subdifferentiable at 0.*

Proof. It follows immediately from the definition of abstract subdifferential. □

Exact penalty representation for augmented Lagrangian function was defined and studied by Rockafellar and Wets ([12], Chap.11). A criterion for such a representation was also presented in ([12], Theorem 11.61), and this criterion has been studied for more generalized augmented Lagrangians, for instance, by Burachik and Rubinov[3], Huang and Yang[6] and, Zhou and Yang[17]. In next theorem we extend this criterion to our more general setting.

Recall the definitions about exact penalty representation in [2] and [4], we introduce the following concept.

Definition 3.1. *Consider the primal and dual problems (P) and (P^*). An element $\overline{y} \in Y$ is said to support an exact penalty representation for problem(P) iff there exists $r_0 \in R_+$, such that for any $r \geq r_0$,*

(i) $H(0) \subset \psi(\overline{y}, r)$;

(ii) $Argmin_x F(x) \subset Argmin_x L(x, \overline{y}, r)$.

Theorem 3.4. *Assume that assumptions of Theorem 2.9 hold, if $(\overline{y}, r_0) \in Y \times R_+$ such that $(\overline{y}, r_0) \in \partial_\rho H(0, \lambda_0)$ for all $\lambda_0 \in H(0)$, then $\overline{y}$ supports an exact penalty representation for problem (P).*

Proof. By assumption $(\overline{y}, r_0) \in \partial_\rho H(0, \lambda_0)$ for all $\lambda_0 \in H(0)$, we can conclude that

$$H(0) \subset \inf_{z\in Z}\{H(z) - \rho(z, \overline{y}, r_0)\} = \inf_{z\in Z}\inf_{x\in X}\{\phi(x,z) - \rho(z,\overline{y}, r_0)\}$$
$$= \inf_{x\in X}\inf_{z\in Z}\{\phi(x,z) - \rho(z,\overline{y}, r_0)\} = \inf_{x\in X} L(x, \overline{y}, r_0),$$

the second "=" above being from Lemma 2.4. By Remark 2.10, we have

$$H(0) \subset \inf_{z\in Z}\{H(z) - \rho(z, \overline{y}, r)\} = \inf_{x\in X} L(x, \overline{y}, r) = \psi(\overline{y}, r)$$

for all $r \geq r_0$.

We now prove that (ii) holds. Let $\overline{x} \in \mathrm{Argmin}_x F(x)$. It remains to prove that $\overline{x} \in \mathrm{Argmin}_x L(x, \overline{y}, r)$ for r large enough. Since $\overline{x} \in \mathrm{Argmin}_x F(x)$, it follows F being proper on X that there exists $v \in F(\overline{x}) \setminus \{\pm\infty\}$ and $v \in \inf_{x\in X} F(x)$. We can assert from $(\overline{y}, \overline{r}) \in \mathrm{dom}(-\psi)$ that $H(0) \cap F(\overline{x}) \subset R^m$. By (i) and Lemma 2.4, one has

$$v \in \inf_{(x,z)\in X\times Z}[\phi(x,z) - \rho(z, \overline{y}, r)],$$

when r large enough. Clearly, $v \in F(\overline{x}) = \phi(\overline{x}, 0) - \rho(0, \overline{y}, r)$. Therefore,

$$(\overline{x}, 0) \in \mathop{\mathrm{Argmin}}_{(x,z)\in X\times Y} \big[\phi(x,z) - \rho(z, \overline{y}, r)\big].$$

Consider the following functions $M(x,z) = \phi(x,z) - \rho(z, \overline{y}, r)$, $J(z) = \inf_{x\in X} M(x,z)$ and $K(x) = \inf_{z\in Z} M(x,z)$. Then $J(z) = H(z) - \rho(z, \overline{y}, r)$ and $K(x) = L(x, \overline{y}, r)$. By Lemma 2.4, we have

$$\overline{v} \in \Big(\inf_{z\in Z} M(\overline{x}, z)\Big) \cap \Big(\inf_{x\in X} K(x)\Big).$$

This means that there exists $\overline{v} \in \phi(\overline{x}, 0) - \rho(0, \overline{y}, r) = F(\overline{x}) \subset R^m$ such that

$$\overline{v} \in L(\overline{x}, \overline{y}, r) \text{ and } \overline{v} \in \inf_{x\in X} L(x, \overline{y}, r).$$

Hence, $\overline{x} \in \mathop{\mathrm{Argmin}}_{x\in X} L(x, \overline{y}, r)$, which shows $\overline{y}$ supports an exact penalty representation for problem (P). □

Remark 3.5. It follows immediately from the Theorem 3.4 that if $H(z)$ is subdifferential at 0 and $(\overline{y}, r_0) \in Y \times R_+$ such that $(\overline{y}, r_0) \in \partial_\rho H(0)$, then $\overline{y}$ supports an exact penalty representation for problem (P).

In the special case $X = R$ and F is a vector-valued function and proper on X, we can assert that y in Theorem 3.4 supports an exact penalty representation for (P).

Example 1. Consider the following optimization:

$$\begin{array}{ll} \text{(VO)} & \inf_{x\in X} f(x) \\ \text{s.t.} & g(x) \leq 0, \end{array}$$

where $f(x) = (x, -x)$ and $g(x) = x^2 - 1,\ x \in X$. Let $S = \{x \in X \mid \ g(x) \leq 0\}$. Define

$$F(x) = \begin{cases} f(x), & x \in S, \\ \infty, & \text{otherwise.} \end{cases}$$

Then (VO) is equivalent to (P) when $X = Z = R$ and $m = 2$. Let $z \in R$, define

$$\phi(x,z) = \begin{cases} f(x), & \text{if } g(x) \leq z, \\ +\infty, & \text{otherwise.} \end{cases}$$

We see that ϕ is a perturbed function of F and $\phi(x,0) = F(x)$ for all $x \in X$. Let $\sigma : Z \to R$ satisfy $\sigma(0) = 0$, $\inf_{V^C} \sigma(z) > 0$ and $\sigma(z) \geq |z|$, where V is a neighborhood of 0. It can be computed that

$$H(0) = \{(x, -x) | \; x \in [-1, 1]\},$$

$$H(z) = \begin{cases} \{(x, -x) \mid x \in [\sqrt{z+1}, -\sqrt{z+1}]\}, & \text{if} \quad z \geq -1 \\ +\infty, & \text{if} \quad z < -1. \end{cases}$$

Let $\rho(z, y, r) = y \cdot z - r\sigma(z)$. Take any $y \in R$ and $\overline{r} > |y|$, we can conclude that y supports a penalty representation for (VO), which means that there exist $\overline{r} \geq 0$ such that for any $r \geq \overline{r}$,

$$\inf_{x \in S} f(x) \subset \inf_{x \in X} L(x, y, r), \qquad \operatorname*{Argmin}_{x \in S} f(x) \subset \operatorname*{Argmin}_{x \in X} L(x, y, r),$$

Now take any $\overline{r} > |y|$, we can assert that

$$L(x, y, r) = \begin{cases} (x, -x), & x \in [-1, 1] \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$\inf_{x \in X} L(x, y, \overline{r}) = \{(x, -x), \; x \in [-1, 1]\}, \qquad \operatorname*{Argmin}_{x \in X} L(x, y, \overline{r}) = [-1, 1].$$

Actually, we have

$$\inf_{x \in S} f(x) = \inf_{x \in X} L(x, y, \overline{r}), \qquad \operatorname*{Argmin}_{x \in S} f(x) = \operatorname*{Argmin}_{x \in X} L(x, y, r), \qquad \forall \, r > |y|,$$

and so y support an exact penalty representation for (VO).

Theorem 3.6. *If $\overline{y}$ supports an exact penalty representation for problem (P) and $H(0) \neq -\infty$, then there exist $\overline{r} \in R_+$ such that $(\overline{y}, \overline{r}) \in dom(-\psi)$ and there also exists a neighborhood $V \subset Z$ of 0 such that $B(H(0)) \cap (H(z) - \rho(z, \overline{y}, r)) = \emptyset$ for all $z \in V$.*

Proof. Let $\overline{y}$ supports an exact penalty representation for problem (P). Then there exist $\overline{r} \in R_+$ such that Definition 3.1 (i) holds, that is

$$H(0) \subset \psi(\overline{y}, r) = \inf_{x \in X} \{L(x, \overline{y}, r)\} = \inf_{z \in Z} \{H(z) - \rho(z, \overline{y}, r)\},$$

for all $\overline{r} \geq r$. Thus,

$$B(H(0)) \cap (H(z) - \rho(z, \overline{y}, r)) = \emptyset,$$

for all $\overline{r} \geq r$ and $z \in Z$. Furthermore, $H(0) \neq -\infty$, then there exist $-\infty \neq \lambda_0 \in H(0)$ such that

$$\lambda_0 \in \psi(\overline{y}, r) \tag{3.2}$$

for all $\overline{r} \geq r$. In particular, when $r = \overline{r}$, (3.2) implies $(\overline{y}, \overline{r}) \in dom(-\psi)$. The proof is complete. □

Corollary 3.7. *Assume that C_1, C_2 hold and $H(0) \neq -\infty$, then $\overline{y}$ supports a penalty representation for problem (P) if and only if $H(z)$ is subdifferential at 0.*

Proof. The result follows immediately from Theorem 2.9, Theorem 3.4 and Theorem 3.6. □

4 Sub-optimal Path

To obtain an exact solution of an optimization problem may, in general, be very hard or even impossible. However, when the optimal value of the problem is finite, approximate solutions always exist and they are, in principle, easier to find than exact solutions. In [16], the authors defined a sub-optimal path related with the dual problem and established some convergence results in finite dimensional spaces. In [2], the authors considered an optimal path related to the duality scheme and analyze its convergence properties. This result is related to Burachik and Rubinov[3], where the authors consider an optimal path in the sense that all the subproblems are supposed to be solved exactly. Motivated by the references above, in this section, we consider the sub-optimal path related to our duality scheme and analyze its convergence properties.

Recall that the calculation of the dual map leads to the following problem:

$$\inf\{\phi(x,z)-\rho(z,y,r):(x,z)\in X\times Z)\}. \tag{4.1}$$

Next we define the sub-optimal path related to our duality scheme.

Definition 4.1. *Let $I\in R_+$ be unbounded above, and for each $r\in I$ take $\varepsilon_r\in intK$. The set $\{(x_r,z_r)\}_r\subset X\times Z$ is called a sub-optimal path of problem (4.1) if for each $q_r\in\psi(y,r)$ with $q_r\neq-\infty$, there exists $h_r\in\phi(x_r,z_r)$ such that*

$$h_r-\rho(z_r,y,r)\leq q_r+\varepsilon_r. \tag{4.2}$$

Definition 4.2 ([7], Definition 3.1.7)**.** *Set-valued map $F:X\to R^m$ is said to be closed at $\overline{x}$ if for every net $(x_k,y_k)_{k\in I}$ with $y_k\in F(x_k)$ converging to $(\overline{x},\overline{y})$, we have $\overline{y}\in F(\overline{x})$.*

We denote by $\mathbb{N}(z_0)$ the collection of all neighborhoods of $z_0\in Z$.

Definition 4.3[4]**.** *Set-valued map $H:Z\to R^m$ is said to be R^m_+-lower bounded at z_0 by a vector $b\in R^m$, if there exists a set $V\in\mathbb{N}(z_0)$ such that $H(V)\subset(b+R^m_+)\cup\{+\infty\}$.*

In the following, we denote $e=(1,1,\cdots,1)\in R^m$ and assume

(C_3) for each $(y,r)\in Y\times R_+$, $\rho(\cdot,y,r)$ is continuous at 0.

Theorem 4.1. *Assume that*

(a) there exists $(\overline{y},\overline{r})\in dom(-\psi)$, and conditions C_1, C_2 and C_3 hold;

(b) H(z) is R^m_+-lower bounded at 0 by a vector b and there exists a neighborhood $V\subset Z$ of 0 such that $B(H(0))\cap(H(z)-\rho(z,\overline{y},r))=\emptyset$ for all $z\in V$ and $r\geq\overline{r}$. If there also exist a neighborhood $W\subset Z$ of 0, a vector $\alpha\in R^m$ and a compact subset $B\subset X$ such that $H(0)\subset(\alpha-\operatorname{int}R^m_+)$ and

$$L_{\phi,W}(\alpha)=\{x\in X:\ \exists\ h_x\in\phi(x,z)\text{ s.t. }h_x\leq\alpha\}\subset B,\qquad \textit{for all } z\in W, \tag{4.3}$$

whenever the parameterization function $\phi(x,z)$ is closed at $(x,0)$ for each $x\in X$. Then

(i) there exists a sub-optimal path $\{(x_r,z_r)\}_{r\geq r_0}$.

(ii) Take a set $I\subset R_+$ unbounded above and consider a sub-optimal path $\{(x_r,z_r)\}_{r\geq r_0}$ satisfying $\lim\limits_{r\in I,r\to\infty}\varepsilon_r=0$. Then $\{z_r\}_{r\in I}$ converges to 0, and the set of cluster points of $\{x_r\}_{r\in I}$ is a nonempty set contained in the primal optimal solution set.

Proof. Considering Remark 2.6 and Remark 2.10, if $r>\overline{r}$, then $\psi(\overline{y},r)\neq-\infty$ and $q_r\not\leq\overline{c}$ for all $q_r\in\psi(\overline{y},r)$ by item (a) and Proposition 2.2 (ii), consequently, $H(0)\neq-\infty$. From Theorem

2.9, there exists $r_0 \geq \overline{r}$ and $\lambda_0 \in H(0)$ such that $(\overline{y}, r) \in \partial_\rho H(0, \lambda_0)$ for all $r \geq r_0$, so we can conclude $\lambda_0 \in \psi(\overline{y}, r)$ for all $r \geq r_0$. For $\lambda_0 = q_r \in \psi(\overline{y}, r)$ and $\varepsilon_r \in \operatorname{int} K$, we have

$$q_r < q_r + \varepsilon_r.$$

Thus the existence of a sub-optimal path is trivially ensured by the definition of weak infimal element , which proves (i).

For proving (ii), let $(x_r, z_r)\}_{r\in I}$ be a sub-optimal path. Assume that $\lim_{r\in I, r\to\infty} \varepsilon_r = 0$. Suppose by contradiction that $\{z_r\}_r$ does not converge to 0 when $r \to \infty$. Thus there exist an open neighborhood $V \subset Z$ of 0 and $J \subset I$, unbounded above, such that $\{z_{r_j}\}_{r\in J} \subset V^C$. Therefore, there exists $h_{r_j} \in \phi(x_{r_j}, z_{r_j})$ and $q_{r_j} \in \psi(\overline{y}, r_j)$ such that

$$\begin{aligned}\lambda_0 + \varepsilon_{r_j} = q_{r_j} + \varepsilon_{r_j} \geq & h_{r_j} - \rho(z_{r_j}, \overline{y}, r_j)\\ = & h_{r_j} - \rho(z_{r_j}, \overline{y}, \overline{r}) + \rho(z_{r_j}, \overline{y}, \overline{r}) - \rho(z_{r_j}, \overline{y}, r_j)\\ \not< & \overline{c} + \inf_{z\in V^C}\{\rho(z, \overline{y}, \overline{r}) - \rho(z, \overline{y}, r_j)\}.\end{aligned}$$

Since $\lim_{r_j\to+\infty} \varepsilon_{r_j} = 0$, we conclude that

$$\lambda_0 + e - \overline{c} \not< \inf_{z\in V^C}\{\rho(z, \overline{y}, \overline{r}) - \rho(z, \overline{y}, r)\},$$

which contradicts to the condition C_2 (ii). It follows that $z_r \to 0$ as r large enough.

Consider an open neighborhood $W \subset Z$ of 0 and $\alpha > \lambda_0$ as in assumption (b). Since $\{z_r\}_{r\in I}$ converges to 0, there exists $\tilde{r} \in I$ such that $\{z_r\}_{r\geq\tilde{r}, r\in I} \subset W$. Take $\delta = \min_{1\leq i\leq m}\{\alpha^i - \lambda_0^i\} > 0$. The function $\rho(., \overline{y}, r)$ is continuous at 0 by condition C_3, so there exist some $r_1 > \max\{r_0, \tilde{r}\}$ such that $\rho(z_r, \overline{y}, r) < \frac{\delta}{2}e$ and $\varepsilon_r < \frac{\delta}{2}e$ for all $r > r_1$. Thus, we have

$$\lambda_0 + \frac{\delta}{2}e > q_r + \varepsilon_r > h_r - \rho(z_r, \overline{y}, r), \qquad \text{for all } r > r_1,\ r \in I,$$

which implies

$$h_r < \lambda_0 + \delta e < \alpha, \qquad \text{for all } r > r_1, \quad r \in I.$$

that is to say $\{x_r\}_{r\geq r_1} \subset L_{\phi,W}(\alpha)$. Assumption (b) implies that $\{x_r\}_{r\geq r_1} \subset B$, where B is a compact set. In particular, since $\{z_r\}_{r\in I}$ converges to 0, the set of cluster points of the sub-optimal path $\{(x_r, z_r) : r \in I\}$ is nonempty. Moreover, every cluster point has the form $(x_0, 0)$. Let us prove that x_0 is a primal optimal solution, where x_0 is an arbitrary cluster point of $\{x_r\}_{r\in I}$. Take a subnet $\{x_{r_j}\}_{j\in J}$ converging to x_0, and $j_0 \in J$ satisfying $r_j \geq r_1$ for all $j \geq j_0$, $j \in J$. Observe that $\{z_{r_j}\}_{j\in J}$ converges to 0. Thus

$$\lambda_0 + \varepsilon_{r_j} = q_{r_j} + \varepsilon_{r_j} > h_{r_j} - \rho(z_{r_j}, \overline{y}, r_j) > h_{r_j} - \rho(z_{r_j}, \overline{y}, \overline{r}),$$

for all $j \geq j_0$, $j \in J$. Since $H(z)$ is R^m_+−lower bounded by a vector b, we can conclude that $\{h_{r_j}\} \subset R^m$ is bounded, so it has a converged subsequence. Without loss of generality, we take the $\lim_{j\in I, r_j}$ in these inequalities, we obtain

$$\lambda_0 \geq \overline{\lambda} - 0 = \overline{\lambda} \in \phi(x_0, 0) = F(x_0),$$

where $\overline{\lambda} \in \phi(x_0, 0)$ follows from ϕ being closed at $(x_0, 0)$. For $\overline{\lambda} \leq \lambda_0$ and $\lambda_0 \in H(0)$, we obtain $\overline{\lambda} \in H(0) = \inf_{x\in X} \phi(x, 0)$, that is $\overline{\lambda} \in \inf_{x\in X} F(x)$. The proof is complete. □

The results of Theorem 4.1 is illustrated by the following simple counterexample.

Example 2. Consider the following optimization:

$$\begin{array}{lll} \text{(VO)} & \inf_{x\in S} f(x) & \\ \text{s.t.} & f_i(x) \le 0, & i=1,2, \end{array}$$

where $x=(x_1,x_2,x_3)\in R^3$, $f=x_3$, $f_1=x_1$ and $f_2=x_2$. We let

$$S=\{x\in R^3 \mid \ x_1y_1+x_2y_2-x_3\le 0, \ \text{ for all } \ (y_1,y_2)\in C\},$$

where

$$C=\{y\in R^2 \mid \ y_1\le 0, \ y_1^2+2y_2\le 0\}.$$

Define

$$F(x)=\begin{cases} f(x), & x\in S,\\ +\infty, & \text{otherwise,}\end{cases}$$

and

$$\phi(x,z)=\begin{cases} f(x), & \text{if } f_i\le z_i \text{ and } x\in S\\ +\infty, & \text{otherwise.}\end{cases}$$

where $z\in R^2$.

We see that ϕ is perturbed function of F and $\phi(x,0)=F(x)$ for all $x\in S$. Let $\rho(z,y,r)=\langle y,z\rangle - r\|z\|^2$ with $y\in R^2$. It can be computed that

$$H(z)=\begin{cases} \dfrac{z_1^2}{2z_2}, & \text{if } z_1\le 0 \text{ and } z_2>0,\\ 0, & \text{if } z_1\ge 0, \ z_2\ge 0,\\ +\infty, & \text{otherwise.}\end{cases}$$

So it can be shown that

$$\psi(y,r)=-\frac{1}{4r}\|y\|^2.$$

Let $x_r=(\frac{1}{r^2},\frac{1}{r^3},\frac{1}{r^5})$, $z_r=(\frac{1}{r^2},\frac{1}{r^3})$ and $y_r=(\frac{2}{r},\frac{2}{r^2})$, then

$$\psi(y_r,r)=-\frac{1}{4r}\|y_r\|^2=-\frac{1}{r^3}-\frac{1}{r^5}\to 0.$$

It is easy to see that

$$\phi(x_r,z_r)-\langle y_r,z_r\rangle + r\|z_r\|^2 = f_0(x_r)-\langle y_r,z_r\rangle + r\|z_r\|^2 = -\frac{1}{r^3}.$$

Now, take $\varepsilon_r=\frac{1}{r^2}$, then all the assumptions in Theorem 4.1 are satisfied and

$$\phi(x_r,z_r)-\langle y_r,z_r\rangle + r\|z_r\|^2 < \psi(y_r,r)+\varepsilon_r,$$

where $r\to+\infty$ and $\varepsilon_r\to 0$. Thus $\{(x_r,z_r)\}_{r>0}$ is a sub-optimal path, $(x_r,z_r)\to(0,0)$ as $r\to+\infty$ and 0 is the optimal solution of (VO).

5 Generalized Vector Variational Inequality

Let $(y,r) \in Y \times R_+$ and set-valued map $G : Z \times Y \times R_{++} \to R^m$, we define $(y,r) : Z \to R^m$ as following:

For any fixed point $\overline{z} \in Z$ there exists $\overline{g} \in G(\overline{z}, y, r)$ such that

$$\langle (y,r), z - \overline{z} \rangle \leq g - \overline{g}, \qquad \text{for all } g \in G(z,y,r) \text{ and } z \in Z. \tag{5.1}$$

The vector variational inequality problem defined by pairs (y,r) consists in finding $\overline{z} \in Z$ and $(y,r)_{\overline{z}}$ (related to $\overline{z}$) such that

$$(\text{VVI}) \quad \langle (y,r)_{\overline{z}}, z - \overline{z} \rangle \not< 0, \qquad \forall\, z \in Z. \tag{5.2}$$

Definition 5.1[1]**.** *A set-valued map $\gamma : Z \to R^m$ is said to be a gap function for* (VVI) *if it satisfies the following conditions:*

(i) $0 \not< \gamma(z)$, $\forall\ z \in Z$ ($0 \not< \gamma(z)$ means that $0 \not< \nu$ for all $\nu \in \gamma(z)$);

(ii) $0 \in \gamma(z)$ if and only if $z \in Z$ solves the problem (VVI).

If we let $G(z,y,r) = H(z) - \rho(z,y,r)$, where $H(z)$ and $\rho(z,y,r)$ defined in Section 2, then the generalized duality results investigated in Section 3 allow us to introduce a gap function for (VVI). We define the following map for all $z \in Z$,

$$\begin{aligned} \gamma(z) &= \bigcup_{(y,r)\in Y\times R_{++}} \{\psi(y,r) - (\lambda - \rho(z,y,r))\} \\ &= \bigcup_{(y,r)\in Y\times R_{++}} \Big\{ \inf_{u\in Z} \{H(u) - \rho(u,y,r)\} - (\lambda - \rho(z,y,r)) \Big\}, \end{aligned}$$

where $\lambda \in H(z)$, and $\lambda - \rho(u,y,r)$ is similar to $\bar{g}$ in (5.1).

In order to prove function $\gamma(z)$ is a gap function for (VVI), we need the following assumption:

(C_4) $\langle (y,r)_{\overline{z}}, z - \overline{z} \rangle \not< 0$ whenever $g - \bar{g} \not< 0$ for all $g \in G(z,y,r)$, $z \in Z$ and $\bar{g} \in G(\bar{z},y,r)$, $\bar{z} \in Z$.

Theorem 5.1. *Let assumption (C_4) hold, then the above function $\gamma(z)$ is a gap function for* (VVI).

Proof. By the definition of $\gamma(z)$, we obtain $0 \not< \gamma(z)$ for all $z \in Z$. This completes the proof of Definition 5.1(i). Let us prove the Definition 5.1(ii). For sufficiency, if there exist $\overline{z} \in Z$ solves the (VVI), then $\exists\ (\overline{y}, \overline{r})_{\bar{z}}$ such that (5.2) holds, that is

$$\langle (\overline{y}, \overline{r}), z - \overline{z} \rangle_{\bar{z}} \not< 0, \qquad \forall\, z \in Z,$$

this together with (5.1), we derive there exist $\overline{\lambda} \in H(\overline{z})$ such that

$$0 \not> \lambda - \rho(z, \overline{y}, \overline{r}) - (\overline{\lambda} - \rho(\overline{z}, \overline{y}, \overline{r})), \qquad \text{for all } \lambda \in H(z) \text{ and } z \in Z. \tag{5.3}$$

That is

$$\lambda - \rho(z, \overline{y}, \overline{r}) \not< (\overline{\lambda} - \rho(\overline{z}, \overline{y}, \overline{r})), \qquad \text{for all } \lambda \in H(z) \text{ and } z \in Z.$$

Hence

$$(\overline{\lambda} - \rho(\overline{z}, \overline{y}, \overline{r})) \in \inf_{z\in Z} \{H(z) - \rho(z, \overline{y}, \overline{r})\}, \tag{5.4}$$

this yields,

$$0 \in \gamma(\overline{z}).$$

For necessity, by the converse process of proof of sufficiency, we derive (5.3) is true. This combining with assumption (C_4) yield (5.2) holds. The proof is complete. □

Theorem 5.2. *If $\overline{z} \in Z$ solves the* (VVI)*, then $\partial_\rho H(\overline{z}) \neq \emptyset$.*

Proof. Since $\overline{z}$ solves the (VVI), so there exist $(\overline{y}, \overline{r})_{\overline{z}}$ such that (5.2) holds. Following from the proof of sufficiency in Theorem 5.1, we obtain (5.4) is true, which implies $\partial_\rho H(\overline{z}, \overline{\lambda}) \neq \emptyset$, the proof is complete. □

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