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Weighted Estimates for a Class of Global Maximal Operators Associated with Dispersive Equation

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Abstract For a function ϕ satisfying some suitable growth conditions, consider the following general dispersive equation defined by

$$\begin{cases} i\partial_t u + \phi(\sqrt{-\Delta})u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x), & f \in \mathcal{S}(\mathbb{R}^n), \end{cases} \qquad (*)$$

where $\phi(\sqrt{-\Delta})$ is a pseudo-differential operator with symbol $\phi(|\xi|)$. In the present paper, when the initial data f belongs to Sobolev space, we give the local and global weighted L^q estimate for the global maximal operator S_ϕ^{**} defined by $S_\phi^{**} f(x) = \sup_{t\in\mathbb{R}} |S_{t,\phi} f(x)|$, where

$$S_{t,\phi} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi + it\phi(|\xi|)} \hat{f}(\xi) d\xi$$

is a formal solution of the equation $(*)$.

Keywords global maximal operator; weighted estimate; pseudo-differential operator; dispersive equation
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1 Introduction

For $t \in \mathbb{R}$ and $a > 1$, defined the operator $S_{t,a}$ by

$$S_{t,a} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it|\xi|^a} \hat{f}(\xi) d\xi, \qquad f \in \mathcal{S}(\mathbb{R}^n),$$

where $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x) dx$. Then the local and global maximal operators S_a^* and S_a^{**} associated with the family of operators $\{S_{t,a}\}_{0<t<1}$ and $\{S_{t,a}\}_{t\in\mathbb{R}}$ respectively, are defined by

$$S_a^* f(x) = \sup_{0<t<1} |S_{t,a} f(x)|, \qquad x \in \mathbb{R}^n$$

and

$$S_a^{**} f(x) = \sup_{t\in\mathbb{R}} |S_{t,a} f(x)|, \qquad x \in \mathbb{R}^n.$$

In 1995, Sjölin[18] gave the following local weighted estimate of the local maximal operator S_a^*:

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Theorem A[18]. *Assume that $a > 1$, $n \geq 2$ and f is radial. If $2 \leq q \leq 4$, $\alpha = \frac{q(2n-1)}{4} - n$, then*

$$\Big(\int_{B(0,R)} |S_a^* f(x)|^q |x|^\alpha dx\Big)^{1/q} \leq C_R \|f\|_{H^{\frac{1}{4}}(\mathbb{R}^n)}, \tag{1.1}$$

where $B(0;R) := \{x \in \mathbb{R}^n; |x| \leq R\}$. If $\alpha < \frac{q(2n-1)}{4} - n$, then the estimate (1.1) does not hold for all radial functions f.

Here and in the sequel, $H^s(\mathbb{R}^n)\,(s \in \mathbb{R})$ denotes the non-homogeneous Sobolev space defined by

$$H^s(\mathbb{R}^n) = \Big\{ f \in \mathcal{S}' : \|f\|_{H^s} = \Big(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi\Big)^{1/2} < \infty \Big\}.$$

It is well-known that $u(x,t) := S_{t,a}f(x)$ is the solution of the fractional Schrödinger equation:

$$\begin{cases} i\partial_t u + (-\Delta)^{a/2} u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x). \end{cases} \tag{1.2}$$

The estimate (1.1) implies that for suitable index s, the solution of the equation (1.2) converges to its initial date f almost everywhere, that is

$$\lim_{t\to 0} u(x,t) = f(x), \qquad \text{a.e. } x \in \mathbb{R}^n. \tag{1.3}$$

See [1–3, 6, 9, 15, 17, 24, 25] for example.

In 2001, Walther[26] obtain the following global weighted estimate for the global maximal operator S_a^{**}:

Theorem B[26]. *Assume that $n = 1$, $a > 1$, $2 \leq q \leq 4$, for f in Schwartz class $\mathcal{S}(\mathbb{R})$, then*

$$\Big(\int_{\mathbb{R}} |S_a^{**} f(x)|^q |x|^{\frac{q}{4}-1} dx\Big)^{1/q} \leq C \|f\|_{\dot{H}^{\frac{1}{4}}(\mathbb{R})}, \tag{1.4}$$

where $\dot{H}^s(\mathbb{R}^n)\,(s \in \mathbb{R})$ denotes the homogeneous Sobolev space, which is defined by

$$\dot{H}^s(\mathbb{R}^n) = \Big\{ f \in \mathcal{S}' : \|f\|_{\dot{H}^s} = \Big(\int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi\Big)^{1/2} < \infty \Big\}.$$

Recently, the authors of [11, 12] and [4] gave some Strichartz estimates for a class of generalized dispersive equation defined by

$$\begin{cases} i\partial_t u + \phi(\sqrt{-\Delta}) u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x), & f \in \mathcal{S}(\mathbb{R}^n), \end{cases} \tag{1.5}$$

where $\phi(\sqrt{-\Delta})$ is a pseudo-differential operator with symbol $\phi(|\xi|)$. The equation (1.5) includes many well-known equations. For instance, the half-wave equation ($\phi(r) = r$), the fractional Schrödinger equation ($\phi(r) = r^a\ (0 < a, a \neq 1)$), the Beam equation ($\phi(r) = \sqrt{1+r^4}$), Klein-Gordon or semirelativistic equation ($\phi(r) = \sqrt{1+r^2}$), iBq ($\phi(r) = r\sqrt{1+r^2}$), imBq ($\phi(r) = \frac{r}{\sqrt{1+r^2}}$) and the fourth-order Schrödinger equation ($\phi(r) = r^2 + r^4$) (see [5, 10, 13, 14] and references therein). Noting that

$$u(x,t) = e^{it\phi(\sqrt{-\Delta})} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi + it\phi(|\xi|)} \hat{f}(\xi) d\xi =: S_{t,\phi} f(x)$$

is the formal solution of the equation (1.5). Then the local maximal operator S_ϕ^* defined by

$$S_\phi^* f(x) = \sup_{0<t<1} |S_{t,\phi} f(x)|, \qquad x \in \mathbb{R}^n$$

and the global maximal operator S_ϕ^{**} of the family of operators $\{S_{t,\phi}\}_{t\in\mathbb{R}}$ defined by

$$S_\phi^{**} f(x) = \sup_{t\in\mathbb{R}} |S_{t,\phi} f(x)|, \qquad x \in \mathbb{R}^n.$$

On the other hand, the authors in [7] gave some global L^2 estimate for the maximal operator S_ϕ^* under symbol ϕ satisfying some growth conditions. Moreover, in one dimension, under symbol ϕ satisfying the conditions (H1)–(H3), and curve satisfying some suitable growth conditions, in [8] we also obtained some weighted L^q maximal estimate along curve associated with solution to dispersive equation (1.5). The main purpose of the present paper is to give a local and global weighted estimates for the global maximal operator S_ϕ^{**} with $\phi : \mathbb{R}^+ \to \mathbb{R}$, satisfying some suitable conditions. We first give our main result in this paper for the dimension $n = 1$.

Theorem 1.1. *Assume that $n = 1$ and ϕ satisfies the following conditions*:

(H1) *There exists $m_1 > 1$, such that $|\phi'(r)| \sim r^{m_1-1}$ and $|\phi''(r)| \gtrsim r^{m_1-2}$ for all $0 < r < 1$;*

(H2) *There exists $m_2 > 1$, such that $|\phi'(r)| \sim r^{m_2-1}$ and $|\phi''(r)| \gtrsim r^{m_2-2}$ for all $r \geq 1$;*

(H3) *Either $\phi''(r) > 0$ or $\phi''(r) < 0$ for all $r > 0$.*

If $\frac{1}{4} \leq s < \frac{1}{2}$, $2 \leq q \leq \frac{2}{1-2s}$ and $\alpha = q(\frac{1}{2} - s) - 1$, then

$$\Big(\int_{\mathbb{R}} |S_\phi^{**} f(x)|^q |x|^\alpha dx \Big)^{1/q} \leq C \|f\|_{\dot{H}^s(\mathbb{R})}. \tag{1.6}$$

Moreover, only if $\alpha \geq q(\frac{1}{2} - s) - 1$, the local estimate

$$\Big(\int_{B} |S_\phi^{**} f(x)|^q |x|^\alpha dx \Big)^{1/q} \leq C \|f\|_{\dot{H}^s(\mathbb{R})} \tag{1.7}$$

holds for all f, where B is an arbitrary ball in $\mathbb{R}$.

Remark 1.2. There are many elements ϕ satisfying the conditions (H1)–(H3), for instance, r^a $(a > 1)$, $\sqrt{1 + r^4}$ and $r^2 + r^4$ and so on. However, the aforementioned $\sqrt{1 + r^2}$, $r\sqrt{1 + r^2}$ and $\frac{r}{\sqrt{1+r^2}}$ do not satisfy the condition (H1) or (H2).

Remark 1.3. Obviously, in case $s = \frac{1}{4}$, Theorem 1.1 implies Theorem B. Noting that the fact $H^s(\mathbb{R}) \subset \dot{H}^s(\mathbb{R})$ if $s > 0$, one may see that the conclusions of Theorem 1.1 also hold if replacing $\dot{H}^s(\mathbb{R})$ by non-homogeneous Sobolev space $H^s(\mathbb{R})$ in estimate (1.6) and (1.7). Hence, the following consequence of Theorem 1.1 is immediately when $\alpha = 0$.

Corollary 1.4. *Assume that $n = 1$, ϕ satisfies the conditions in Theorem 1.1. If $\frac{1}{4} \leq s < \frac{1}{2}$, $q = \frac{2}{1-2s}$, then*

$$\Big(\int_{\mathbb{R}} |S_\phi^{**} f(x)|^q dx \Big)^{1/q} \leq C \|f\|_{H^s(\mathbb{R})}. \tag{1.8}$$

If $\frac{1}{4} \leq s < \frac{1}{2}$, then the local estimate

$$\Big(\int_{B} |S_\phi^{**} f(x)|^q dx \Big)^{1/q} \leq C \|f\|_{H^s(\mathbb{R})} \tag{1.9}$$

holds if and only if $q \leq \frac{2}{1-2s}$, where B is an arbitrary ball in $\mathbb{R}$.

In 1997, Sjölin[19] gave the following global and local estimates for the local maximal operator S_a^*, which implies the convergence almost everywhere of the solution for the equation (1.2) with initial date:

Theorem C[19]. *Assume that $n = 1$, $a > 1$, $\frac{1}{4} \le s < \frac{1}{2}$, then the global estimate*

$$\|S_a^* f\|_{L^q(\mathbb{R})} \le C\|f\|_{H^s(\mathbb{R})} \tag{1.10}$$

holds for $q = \frac{2}{1-2s}$, and the local estimate

$$\|S_a^* f\|_{L^q(B)} \le C_B\|f\|_{H^s(\mathbb{R})} \tag{1.11}$$

holds if and only if $q \le \frac{2}{1-2s}$, where B is an arbitrary ball in $\mathbb{R}$.

Remark 1.5. Clearly, Corollary 1.4 improves and extends Theorem C.

Now let us turn to the case of the dimension $n \ge 2$.

Theorem 1.6. *Assume that $n \ge 2$ and f is radial, ϕ is radial and satisfies* (H1)–(H3). *If $\frac{1}{4} \le s < \frac{1}{2}$, $2 \le q \le \frac{2}{1-2s}$ and $\alpha = q(\frac{n}{2} - s) - n$, then*

$$\Big(\int_{\mathbb{R}^n} |S_\phi^{**} f(x)|^q |x|^\alpha dx\Big)^{1/q} \le C\|f\|_{H^s(\mathbb{R}^n)}. \tag{1.12}$$

Moreover, only if $\alpha \ge q(\frac{n}{2} - s) - n$, the local estimate

$$\Big(\int_{B} |S_\phi^{**} f(x)|^q |x|^\alpha dx\Big)^{1/q} \le C\|f\|_{H^s(\mathbb{R}^n)} \tag{1.13}$$

holds for all radial functions f, where B is an arbitrary ball in $\mathbb{R}^n$.

Remark 1.7. Obviously, Theorem 1.6 is an improvement and extension of Theorem A for the case $s = \frac{1}{4}$. The following consequence of Theorem 1.6 is also obvious when $\alpha = 0$.

Corollary 1.8. *Assume that $n \ge 2$ and f is radial, ϕ satisfies the conditions in Theorem 1.6. If $\frac{1}{4} \le s < \frac{1}{2}$, $q = \frac{2n}{n-2s}$, then*

$$\Big(\int_{\mathbb{R}^n} |S_\phi^{**} f(x)|^q dx\Big)^{1/q} \le C\|f\|_{H^s(\mathbb{R}^n)}. \tag{1.14}$$

If $\frac{1}{4} \le s < \frac{1}{2}$, then the local estimate

$$\Big(\int_{B} |S_\phi^{**} f(x)|^q dx\Big)^{1/q} \le C\|f\|_{H^s(\mathbb{R}^n)} \tag{1.15}$$

holds if and only if $q \le \frac{2n}{n-2s}$, where B is an arbitrary ball in $\mathbb{R}^n$.

Remark 1.9. When ϕ satisfies the conditions (H1)–(H3), Corollary 1.4 and Corollary 1.8 implies the a.e. convergence of the solution of the general dispersive equation (1.5) for for $s \ge \frac{1}{4}$ when the initial data f in $H^s(\mathbb{R})$ and the radial initial data f in $H^s(\mathbb{R}^n)\,(n \ge 2)$, respectively.

This paper is organized as follows. The proofs of Theorem 1.1 and Theorem 1.6 are given in Section 2 and Section 3, respectively. In the proofs of above main conclusions, Lemma 2.1 plays an important role, which will be proved in Section 4. In final section, we give the weighted maximal estimates for the functions in $\mathcal{H}_k$, which is the set of all combination by the radial functions and solid spherical harmonics.

2 The Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. To do this, we need to present two lemma (i.e., Lemma 2.1 and Lemma 2.3 below), which play a key role in proving Theorem 1.1 and Theorem 1.6. The proof of Lemma 2.1 will be given in Section 4.

2.1 Key Lemma and Proof of the Sufficiency in Theorem 1.1

Lemma 2.1. *Assume that ϕ satisfies* (H1)–(H3). *If $\frac{1}{2} \le s < 1$, and $\mu \in C_0^\infty(\mathbb{R})$, then*

$$\Big| \int_{\mathbb{R}} e^{ix\xi + it\phi(|\xi|)} |\xi|^{-s} \mu\Big(\frac{\xi}{N}\Big) d\xi \Big| \le C \frac{1}{|x|^{1-s}},$$

for $x \in \mathbb{R} \setminus \{0\}$, $t \in \mathbb{R}$ and $N = 1, 2, 3, \cdots$. Here the constant C may depend on s and m_1, m_2 and μ but not on x, t and N.

Remark 2.2. In case $\phi(|\xi|) = |\xi|^a (a > 1)$, Lemma 2.1 was showed by Sjölin in [21].

Lemma 2.3[16]**.** *If $r \ge p$, $0 \le \alpha < 1 - \frac{1}{p}$, $0 \le \gamma < \frac{1}{r}$ and $\gamma = \alpha + \frac{1}{p} + \frac{1}{r} - 1$, then*

$$\Big(\int_R |\hat{f}(\xi)|^r |\xi|^{-\gamma r} d\xi \Big)^{1/r} \le \Big(\int_R |f(x)|^p |x|^{\alpha p} dx \Big)^{1/p}.$$

Applying Lemma 2.3, we have the following estimate:

$$\Big(\int_R |\hat{f}(\xi)|^2 |\xi|^{-2s} d\xi \Big)^{1/2} \le \Big(\int_R |f(x)|^p |x|^{(s+\frac{1}{2})p-1} dx \Big)^{1/p}, \tag{2.1}$$

where $\frac{1}{4} \le s < \frac{1}{2}$ and $\frac{2}{1+2s} \le p \le 2$. In fact, taking $r = 2$, $\gamma = s$, it follows that

$$\alpha = \gamma - \frac{1}{p} - \frac{1}{r} + 1 = s - \frac{1}{p} - \frac{1}{2} + 1 = s + \frac{1}{2} - \frac{1}{p}.$$

Note that $\frac{2}{1+2s} \le p \le 2 = r$ and $0 \le \alpha < 1 - \frac{1}{p}$, $0 \le \gamma = s < \frac{1}{2} = \frac{1}{r}$ by $\alpha = s + \frac{1}{2} - \frac{1}{p}$ and $\frac{1}{4} \le s < \frac{1}{2}$. Thus, the estimate (2.1) follows by Lemma 2.3.

Let us turn to the proof of sufficiency in Theorem 1.1. That is, it only needs to show that the following estimate :

$$\Big(\int_{\mathbb{R}} |S_\phi^{**} f(x)|^q |x|^\alpha dx \Big)^{1/q} \le C \|f\|_{\dot{H}^s(\mathbb{R})}, \tag{2.2}$$

where $\frac{1}{4} \le s < \frac{1}{2}$, $2 \le q \le \frac{2}{1-2s}$ and $\alpha = q(\frac{1}{2} - s) - 1$. Let $t(x) : \mathbb{R} \to \mathbb{R}$ be a measurable function. Denote

$$Tf(x) = |x|^{\frac{\alpha}{q}} \int_{\mathbb{R}} e^{ix\cdot\xi} e^{it(x)\phi(|\xi|)} \widehat{f}(\xi) d\xi, \qquad f \in \mathcal{S}(\mathbb{R}).$$

By linearizing the maximal operator (see [17, p. 707]), to prove (2.2) it needs to show that

$$\Big(\int_{\mathbb{R}} |Tf(x)|^q dx \Big)^{1/q} \le C \|f\|_{\dot{H}^s(\mathbb{R})}, \tag{2.3}$$

where $\frac{1}{4} \le s < \frac{1}{2}$ and $2 \le q \le \frac{2}{1-2s}$. Taking function $\rho \in C_0^\infty(\mathbb{R})$ such that $\rho(x) = 1$ if $|x| < 1$, and $\rho(x) = 0$ if $|x| \ge 2$. Denote

$$Rg(x) = |x|^{\frac{\alpha}{q}} \int_{\mathbb{R}} e^{ix\cdot\xi} e^{it(x)\phi(|\xi|)} |\xi|^{-s} g(\xi) d\xi, \qquad g \in \mathcal{S}(\mathbb{R}).$$

We claim

$$\|Rg\|_{L^q(\mathbb{R})} \le C\|g\|_{L^2(\mathbb{R})}. \tag{2.4}$$

Note that $Tf(x) = R\big(|\cdot|^s\hat{f}(\cdot)\big)(x)$, we have by (2.4)

$$\begin{aligned}\Big(\int_{\mathbb{R}}|Tf(x)|^q dx\Big)^{1/q} &\le C\Big(\int_{\mathbb{R}}\Big|R\big(|\cdot|^s\hat{f}(\cdot)\big)(x)\Big|^q dx\Big)^{1/q}\\ &\le C\Big(\int_{\mathbb{R}}|\xi|^{2s}|\hat{f}(\xi)|^2 d\xi\Big)^{1/2}\\ &= C\|f\|_{\dot{H}^s(\mathbb{R})},\end{aligned}$$

which implies (2.3). Now we prove (2.4). Denote

$$R_N g(x) = \rho\Big(\frac{x}{N}\Big)\,|x|^{\frac{\alpha}{q}}\int_{\mathbb{R}} e^{ix\cdot\xi}e^{it(x)\phi(|\xi|)}\rho\Big(\frac{\xi}{N}\Big)|\xi|^{-s}g(\xi)d\xi, \qquad N>2.$$

On the other hand, it is easy to see that the adjoint operator R_N' of R_N is given by

$$R_N' h(\xi) = |\xi|^{-s}\rho\Big(\frac{\xi}{N}\Big)\int_{\mathbb{R}}|x|^{\frac{\alpha}{q}}\rho\Big(\frac{x}{N}\Big)e^{-ix\cdot\xi}e^{-it(x)\phi(|\xi|)}h(x)dx, \qquad N>2.$$

To prove (2.4) it suffices to prove that

$$\|R_N g\|_{L^q(\mathbb{R})} \le C\|g\|_{L^2(\mathbb{R})}. \tag{2.5}$$

By duality, we turn to prove that

$$\|R_N' h\|_{L^2(\mathbb{R})} \le C\|h\|_{L^p(\mathbb{R})}, \tag{2.6}$$

where $\frac{2}{1+2s} \le p \le 2$ by $2 \le q \le \frac{2}{1-2s}$ and $\frac{1}{p}+\frac{1}{q}=1$. Since

$$\begin{aligned}\|R_N'h\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}}|R_N'h(\xi)|^2 d\xi\\ &= \int_{\mathbb{R}}\Big(|\xi|^{-s}\rho(\frac{\xi}{N})\int_{\mathbb{R}}|x|^{\frac{\alpha}{q}}\rho(\frac{x}{N})e^{-ix\cdot\xi}e^{-it(x)\phi(|\xi|)}h(x)dx\Big)\\ &\quad\times\overline{\Big(|\xi|^{-s}\rho(\frac{\xi}{N})\int_{\mathbb{R}}|y|^{\frac{\alpha}{q}}\rho(\frac{y}{N})e^{-iy\cdot\xi}e^{-it(y)\phi(|\xi|)}h(y)dy\Big)}d\xi\\ &=: \rho\Big(\frac{x}{N}\Big)\rho\Big(\frac{y}{N}\Big)\int_{\mathbb{R}}\int_{\mathbb{R}}K_N(x,y)|x|^{\frac{\alpha}{q}}h(x)|y|^{\frac{\alpha}{q}}h(y)dxdy,\end{aligned} \tag{2.7}$$

where

$$K_N(x,y) := \int_{\mathbb{R}} e^{i[(y-x)\xi+(t(y)-t(x))\phi(|\xi|)]}|\xi|^{-2s}\rho\Big(\frac{\xi}{N}\Big)^2 d\xi.$$

Note that ϕ satisfies the conditions in Lemma 2.1, and $\frac{1}{2} \le 2s < 1$ by $\frac{1}{4} \le s < \frac{1}{2}$, thus by Lemma 2.1, we obtain

$$|K_N(x,y)| \le C\frac{1}{|x-y|^{1-2s}}. \tag{2.8}$$

Thus, by (2.7) and (2.8), Parseval's equality and (2.1), combining with the fact $\frac{\alpha p}{q}+(s+\frac{1}{2})p-1=0$ by $\alpha = q(\frac{1}{2}-s)-1$ and $\frac{1}{p}+\frac{1}{q}=1$, we obtain

$$\int |R_N'h(\xi)|^2 d\xi \le C\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\frac{1}{|x-y|^{1-2s}}|y|^{\frac{\alpha}{q}}|h(y)|dy\Big)|x|^{\frac{\alpha}{q}}|h(x)|dx$$

$$
\begin{aligned}
&= C\int_{\mathbb{R}} I_{2s}(|\cdot|^{\frac{\alpha}{q}}|h(\cdot)|)(x)(|x|^{\frac{\alpha}{q}}|h(x)|)dx\\
&= C\int_{\mathbb{R}} \left|\left(|h(\cdot)||\cdot|^{\frac{\alpha}{q}}\right)\hat{}(\xi)\right|^2|\xi|^{-2s}d\xi\\
&\le C\Big(\int_{\mathbb{R}} \left(|h(x)||x|^{\frac{\alpha}{q}}\right)^p|x|^{(s+\frac12)p-1}dx\Big)^{2/p}\\
&= C\Big(\int_{\mathbb{R}} |h(x)|^p|x|^{\frac{\alpha p}{q}+(s+\frac12)p-1}dx\Big)^{2/p} = C\|h\|_{L^p(\mathbb{R})},
\end{aligned}
$$

where and in the sequel, I_α denotes the Riesz potential of order α, which is defined by

$$I_\alpha(f)(u) = \int_{\mathbb{R}} \frac{f(v)}{|u-v|^{1-\alpha}}dv.$$

It follows that (2.6) holds.

Summing up above estimates, we complete the proof of the sufficiency part of Theorem 1.1.

2.2 Proof of the Necessity of Theorem 1.1

We choose an even and non-negative $\varphi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\varphi \subset \{\xi; \frac12 < |\xi| < 2\}$ satisfying $\varphi(\xi) = 1$ if $\frac54 \le |\xi| \le \frac74$. Denote $\hat{f} := \varphi(\frac{\xi}{\lambda})$ for $\lambda > 1$. By simple calculation, we have

$$\|f\|_{\dot{H}^s(\mathbb{R})} \le C\lambda^{\frac12+s}, \tag{2.9}$$

where C is independent of λ. On the other hand,

$$S_{t,\phi}f(x) = (2\pi)^{-1}\int_{\mathbb{R}} e^{ix\cdot\xi}e^{it\phi(|\xi|)}\varphi\Big(\frac{\xi}{\lambda}\Big)d\xi = (2\pi)^{-1}\lambda\int_{\mathbb{R}} e^{i\lambda x\cdot\eta}e^{it\phi(|\lambda\eta|)}\varphi(\eta)d\eta. \tag{2.10}$$

Thus, taking $t = 0$ in (2.10), we get

$$S_{0,\phi}f(x) = (2\pi)^{-1}\lambda\int_{\mathbb{R}} e^{i\lambda x\cdot\eta}\varphi(\eta)d\eta = (2\pi)^{-1}\lambda\hat{\varphi}(\lambda x). \tag{2.11}$$

Note that

$$\hat{\varphi}(0) = \int_{\mathbb{R}}\varphi(x)dx \ge \int_{\frac54\le|\xi|\le\frac74}\varphi(x)dx > 1,$$

there exists $0 < \delta < \frac{\lambda}{2}$, when $|x| \le \frac{\delta}{\lambda}$, we have $\hat{\varphi}(\lambda x) > \frac12$. Hence, when $|x| \le \frac{\delta}{\lambda}$, we obtain

$$S_\phi^{**}f(x) \ge |S_{0,\phi}f(x)| \ge c_0\lambda, \tag{2.12}$$

where $c_0 = \frac{1}{4\pi}$. Since S_ϕ^{**} satisfies the local estimate (1.7), so if take $B_0 := B(0,1)$, the unit ball in $\mathbb{R}$, then by (2.9) and (2.12), we obtain

$$C_{B_0}C\lambda^{\frac12+s} \ge \Big(\int_{B_0}|S_\phi^{**}f(x)|^q|x|^\alpha dx\Big)^{1/q} \ge \Big(\int_{|x|\le\frac{\delta}{\lambda}}(c_0\lambda)^q|x|^\alpha dx\Big)^{1/q} \ge c_0\Big(\frac{\delta^{\alpha+1}}{\alpha+1}\Big)^{\frac1q}\lambda^{1-\frac{\alpha+1}{q}},$$

which follows that

$$\lambda^{1-\frac{\alpha+1}{q}} \le C\lambda^{\frac12+s}, \tag{2.13}$$

where C only depends on δ and q, not depend on λ. Taking λ large in (2.13), then $\alpha \ge q(\frac12-s)-1$ is necessary for the inequality (2.13), i.e., $\alpha \ge q(\frac12 - s) - 1$ is necessary for the local estimate (1.7) holds when $\frac14 \le s < \frac12$ and $2 \le q \le \frac{2}{1-2s}$.

3 The Proof of Theorem 1.6

3.1 Proof of the Sufficiency in Theorem 1.6

We prove the sufficiency part of Theorem 1.6. That is, when $n \geq 2$ and f is radial, ϕ is radial and satisfies (H1)–(H3), it only needs to prove that the following estimate:

$$\Big(\int_{\mathbb{R}^n} |S_\phi^{**} f(x)|^q |x|^\alpha dx\Big)^{1/q} \leq C\|f\|_{H^s(\mathbb{R}^n)}, \tag{3.1}$$

where $\frac{1}{4} \leq s < \frac{1}{2}$, $2 \leq q \leq \frac{2}{1-2s}$ and $\alpha = q(\frac{n}{2} - s) - n$. Assume $f \in \mathcal{S}(\mathbb{R}^n)$ is a radial function, then (see [22, p. 155])

$$\hat{f}(\xi) = (2\pi)^{\frac{n}{2}} |\xi|^{1-\frac{n}{2}} \int_0^\infty f(s) J_{\frac{n}{2}-1}(s|\xi|) s^{\frac{n}{2}} ds,$$

where and in the sequel, $J_m(r)$ denotes the Bessel function defined by

$$J_m(r) = \frac{(\frac{r}{2})^m}{\Gamma(m+\frac{1}{2})\pi^{\frac{1}{2}}} \int_{-1}^1 e^{irt}(1-t^2)^{m-\frac{1}{2}} dt, \qquad m > -\frac{1}{2}.$$

Let $t(x) : \mathbb{R}^n \to \mathbb{R}$ be a measurable and radial function. Denote

$$Tf(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi + it(x)\phi(|\xi|)} \hat{f}(\xi) d\xi.$$

Therefore, we get

$$Tf(u) = (2\pi)^{\frac{n}{2}-n} u^{1-\frac{n}{2}} \int_0^\infty J_{\frac{n}{2}-1}(ru) e^{it(u)\phi(r)} \hat{f}(r) r^{\frac{n}{2}} dr, \qquad u > 0, \tag{3.2}$$

where $Tf(u) = Tf(x)$ with $u = |x|$ and $\hat{f}(r) = \hat{f}(\xi)$ with $r = |\xi|$. By linearizing the maximal operator and using polar coordinates, to prove (3.1) it suffices to prove that

$$\Big(\int_0^\infty |Tf(u)|^q u^{q(\frac{n}{2}-s)-1} du\Big)^{1/q} \leq \Big(\int_0^\infty |\hat{f}(r)|^2 (1+r^2)^s r^{n-1} dr\Big)^{1/2}, \tag{3.3}$$

where $\frac{1}{4} \leq s < \frac{1}{2}$, $2 \leq q \leq \frac{2}{1-2s}$. Denote

$$g(r) = \hat{f}(r)(1+r^2)^{\frac{s}{2}} r^{\frac{n}{2}-\frac{1}{2}}, \qquad r > 0. \tag{3.4}$$

By (3.2) and (3.4), we have

$$\begin{aligned} Tf(u) u^{\frac{n}{2}-s-\frac{1}{q}} &= (2\pi)^{-\frac{n}{2}} u^{1-\frac{n}{2}} u^{\frac{n}{2}-s-\frac{1}{q}} \int_0^\infty J_{\frac{n}{2}-1}(ru) e^{it(u)\phi(r)} \hat{f}(r) r^{\frac{n}{2}} dr \\ &= (2\pi)^{-\frac{n}{2}} u^{1-s-\frac{1}{q}} \int_0^\infty J_{\frac{n}{2}-1}(ru) e^{it(u)\phi(r)} g(r)(1+r^2)^{-\frac{s}{2}} r^{\frac{1}{2}} dr \\ &=: (2\pi)^{-\frac{n}{2}} Pg(u), \end{aligned}$$

where the operator P is defined by

$$Pg(u) := u^{1-s-\frac{1}{q}} \int_0^\infty J_{\frac{n}{2}-1}(ru) e^{it(u)\phi(r)} g(r)(1+r^2)^{-\frac{s}{2}} r^{\frac{1}{2}} dr, \qquad r > 0.$$

Thus, to get (3.3) it only to show that

$$\Big(\int_0^\infty |Pg(u)|^q du\Big)^{1/q} \le C\Big(\int_0^\infty |g(r)|^2 dr\Big)^{1/2}. \tag{3.5}$$

Denote by P' the adjoint operator of P, that is

$$\int_0^\infty Pg(r)\overline{h(r)}du = \int_0^\infty g(r)\overline{P'h(r)}dr. \tag{3.6}$$

It is easy to check that

$$P'h(r) = (1+r^2)^{-\frac{s}{2}} r^{\frac{1}{2}} \int_0^\infty J_{\frac{n}{2}-1}(ru)e^{-it(u)\phi(r)}u^{1-s-\frac{1}{q}}h(u)du, \qquad r>0.$$

Therefore, by (3.5) and (3.6), to obtain (3.3) we only need to verify that

$$\|P'h\|_{L^2(0,\infty)} \le C\|h\|_{L^p(0,\infty)}, \tag{3.7}$$

where p satisfies $\frac{1}{p}+\frac{1}{q}=1$ with $2\le q\le \frac{2}{1-2s}$ and $\frac{1}{4}\le s<\frac{1}{2}$. In fact, it follows that $1<\frac{2}{1+2s}\le p\le 2$. Denote $\sigma=\frac{1}{q}+s-\frac{1}{2}$, then

$$P'h(r) = (1+r^2)^{-\frac{s}{2}} \int_0^\infty (ru)^{\frac{1}{2}} J_{\frac{n}{2}-1}(ru)e^{-it(u)\phi(r)}u^{-\sigma}h(u)du.$$

Lemma 3.1[22]**.** *$J_m(r)=\sqrt{\frac{2}{\pi r}}\cos(r-\frac{\pi m}{2}-\frac{\pi}{4})+\mathrm{O}(r^{-\frac{3}{2}})$ as $r\to\infty$. In particular, $J_m(r)=\mathrm{O}(r^{-\frac{1}{2}})$ as $r\to\infty$.*

Applying Lemma 3.1, we shall prove the following estimates:

$$\big|t^{\frac{1}{2}}J_{\frac{n}{2}-1}(t)-(b_1e^{it}+b_2e^{-it})\big| \le \frac{C}{t}, \qquad t>1, \tag{3.8}$$

and

$$\big|t^{\frac{1}{2}}J_{\frac{n}{2}-1}(t)-(b_1e^{it}+b_2e^{-it})\big| \le C, \qquad 0<t\le 1, \tag{3.9}$$

where b_1 and b_2 are the constants depending on n. In fact, by Lemma 3.1, when $t\to\infty$, we get

$$J_{\frac{n}{2}-1}(t) = \sqrt{\frac{2}{\pi t}}\cos\Big(t-\frac{\pi(n-1)}{4}\Big)+O(t^{-\frac{3}{2}}).$$

It follows that when $t\to\infty$, we get

$$\begin{aligned} t^{\frac{1}{2}}J_{\frac{n}{2}-1}(t) &= \sqrt{\frac{2}{\pi}}\cos\Big(\frac{\pi(n-1)}{4}\Big)\cos t+\sqrt{\frac{2}{\pi}}\sin\Big(\frac{\pi(n-1)}{4}\Big)\sin t+O(t^{-1}) \\ &= (b_1+b_2)\cos t+i(b_1-b_2)\sin t+O(t^{-1}) \\ &= b_1e^{it}+b_2e^{-it}+O(t^{-1}), \end{aligned}$$

where

$$b_1=\frac{1}{2}\sqrt{\frac{2}{\pi}}\Big(\cos\Big(\frac{\pi(n-1)}{4}\Big)+i\sin\Big(\frac{\pi(n-1)}{4}\Big)\Big)$$

and

$$b_2=\frac{1}{2}\sqrt{\frac{2}{\pi}}\Big(\cos\Big(\frac{\pi(n-1)}{4}\Big)-i\sin\Big(\frac{\pi(n-1)}{4}\Big)\Big).$$

It follows that (3.8) holds when $t>1$. On the other hand, by the definition of Bessel function

$$J_m(t)=\frac{(\frac{t}{2})^m}{\Gamma(m+\frac{1}{2})\pi^{\frac{1}{2}}}\int_{-1}^{1}e^{its}(1-s^2)^{m-\frac{1}{2}}ds, \qquad m>-\frac{1}{2},$$

we have $|J_m(t)|\le Ct^m$ for $m>-\frac{1}{2}$ and $t>0$. Since $n\ge 2$, so $|J_{\frac{n}{2}-1}(t)|\le Ct^{\frac{n}{2}-1}$ for $0<t<1$. Therefore, when $0<t<1$, we have

$$|t^{1/2}J_{\frac{n}{2}-1}(t)-(b_1e^{it}+b_2e^{-it})|\le Ct^{\frac{1}{2}}t^{\frac{n}{2}-1}+|b_1e^{it}|+|b_2e^{-it}|\le Ct^{\frac{n}{2}-\frac{1}{2}}+|b_1|+|b_2|\le C, \tag{3.10}$$

which implies the estimate (3.9). Invoking (3.8) and (3.9), we have

$$P'h(r)=:b_1A_1(r)+b_2A_2(r)+B(r), \tag{3.11}$$

where

$$A_1(r)=(1+r^2)^{-\frac{s}{2}}\int_0^\infty e^{iru}e^{-it(u)\phi(r)}u^{-\sigma}h(u)du,$$

$$A_2(r)=(1+r^2)^{-\frac{s}{2}}\int_0^\infty e^{-iru}e^{-it(u)\phi(r)}u^{-\sigma}h(u)du,$$

and

$$|B(r)|\le C(1+r^2)^{-\frac{s}{2}}\int_0^\infty \min\Big\{1,\frac{1}{ru}\Big\}u^{-\sigma}|h(u)|du. \tag{3.12}$$

We first estimate A_1 and A_2. Denote

$$A(r)=(1+r^2)^{-\frac{s}{2}}\int_0^\infty e^{iru}e^{-it(u)\phi(|r|)}u^{-\sigma}h(u)du, \qquad r\in\mathbb{R}.$$

We claim that

$$\Big(\int_{\mathbb{R}}|A(r)|^2dr\Big)^{1/2}\le C\|h\|_{L^p(0,\infty)}, \tag{3.13}$$

and then

$$\Big(\int_0^\infty |A_i(r)|^2dr\Big)^{1/2}\le C\Big(\int_{\mathbb{R}}|A(r)|^2dr\Big)^{1/2}, \qquad i=1,2, \tag{3.14}$$

by (3.13) and (3.14), we have

$$\Big(\int_0^\infty |A_i(r)|^2dr\Big)^{1/2}\le C\|h\|_{L^p(0,\infty)}, \qquad i=1,2. \tag{3.15}$$

Now we prove (3.13) holds. We take a real-valued function $\rho\in C_0^\infty(\mathbb{R})$ such that $\rho(r)=1$ if $|r|<1$, and $\rho(r)=0$ if $|r|\ge 2$, and for $N>1$, set $\rho_N(r)=\rho(\frac{r}{N})$. And define

$$A_N(r)=\rho_N(r)|r|^{-s}\int_0^\infty e^{iru}e^{-it(u)\phi(|r|)}u^{-\sigma}h(u)du.$$

We first assume that the following estimate holds:

$$\Big(\int_{\mathbb{R}}|A_N(r)|^2dr\Big)^{1/2}\le C\|h\|_{L^p(0,\infty)}. \tag{3.16}$$

Thus, let $N\to\infty$ in (3.16) and by Fatou' Lemma, it follows (3.13) holds. Now we prove (3.16). By Fubini's Theorem, we have

$$\int_{\mathbb{R}}|A_N(r)|^2dr=:\int_0^\infty\int_0^\infty I(u,v)u^{-\sigma}h(u)v^{-\sigma}\overline{h(v)}dudv, \tag{3.17}$$

where

$$I(u,v) := \int_{\mathbb{R}} e^{i[(u-v)r-(t(u)-t(v))\phi(|r|)]}|r|^{-2s}\rho_N(r)^2 dr.$$

Note that ϕ satisfies the conditions in Lemma 2.1, and $\frac{1}{2} \leq 2s < 1$ by $\frac{1}{4} \leq s < \frac{1}{2}$, thus by Lemma 2.1, we get

$$I(u,v) \leq C\frac{1}{|u-v|^{1-2s}}. \tag{3.18}$$

By (3.17), (3.18) and Parseval's equality, we obtain

$$\begin{aligned}
\|A_N\|^2_{L^2(\mathbb{R})} &\leq C\int_0^\infty \int_0^\infty \frac{1}{|u-v|^{1-2s}} u^{-\sigma} h(u) v^{-\sigma}|h(v)|dudv \\
&= C\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{|u-v|^{1-2s}} u^{-\sigma} h_1(u) v^{-\sigma}|h_1(v)|dudv \\
&= C\int_{\mathbb{R}} I_{2s}(t^{-\sigma}|h_1|)(u) u^{-\sigma}|h_1(u)|du \\
&= C\int_{\mathbb{R}} |\xi|^{-2s}(u^{-\sigma}|h_1|)\hat{\ }(\xi)\overline{(u^{-\sigma}|h_1|)\hat{\ }(\xi)}d\xi \\
&= C\int_{\mathbb{R}} |(u^{-\sigma}|h_1|)\hat{\ }(\xi)|^2|\xi|^{-2s}d\xi, \qquad (3.19)
\end{aligned}$$

where

$$h_1(u) = \begin{cases} h(u), & u \geq 0, \\ 0, & u < 0. \end{cases} \tag{3.20}$$

Thus by (2.1), we get

$$\begin{aligned}
\Big(\int_{\mathbb{R}} |(u^{-\sigma}|h_1|)\hat{\ }(\xi)|^2|\xi|^{-2s}d\xi\Big)^{1/2} &\leq C\Big(\int_{\mathbb{R}} |u^{-\sigma}h_1|^p|u|^{(s+\frac{1}{2})p-1}du\Big)^{1/p} \\
&= C\Big(\int_{\mathbb{R}} u^{-\sigma p+sp+\frac{p}{2}-1}|h_1|^p du\Big)^{1/p} \qquad (3.21) \\
&= C\|h_1\|_{\mathbb{R}} = C\|h\|_{L^p(0,\infty)},
\end{aligned}$$

where we invoking the fact $-\sigma p + sp + \frac{p}{2} - 1 = -\frac{p}{q} - sp + \frac{p}{2} + sp + \frac{p}{2} - 1 = -\frac{p}{q} + p - 1 = 0$ by $\sigma = \frac{1}{q} + s - \frac{1}{2}$ and $\frac{1}{p} + \frac{1}{q}$. Thus (3.16) holds from (3.19) and (3.21).

Next, we will show the estimate

$$\Big(\int_0^\infty |B(r)|^2 dr\Big)^{1/2} \leq C\|h\|_{L^p(0,\infty)} \tag{3.22}$$

holds. The proof for (3.22) will be split into the following two cases.

Case I. $0 < r < 1$. we have

$$\Big(\int_0^1 |B(r)|^2 dr\Big)^{1/2} \leq C\|h\|_{L^p(0,\infty)}. \tag{3.23}$$

In fact, by (3.12), when $0 < r < 1$, we have

$$|B(r)| \leq C\int_0^\infty \min\Big\{1, \frac{1}{ru}\Big\} u^{-\sigma}|h(u)|du \leq C\Big(\int_0^{\frac{1}{r}} u^{-\sigma}|h(u)|du + \frac{1}{r}\int_{\frac{1}{r}}^\infty u^{-1-\sigma}|h(u)|du\Big),$$

where $\sigma = \frac{1}{q} + s - \frac{1}{2}$. By Hölder's inequality, we get

$$\begin{aligned}|B(r)| &\leq C\Big(\int_0^{\frac{1}{r}} u^{-\sigma q} du\Big)^{1/q} \|h\|_{L^p(0,\infty)} + C\frac{1}{r}\Big(\int_{\frac{1}{r}}^{\infty} u^{(-1-\sigma)q} du\Big)^{1/q} \|h\|_{L^p(0,\infty)} \\ &= C(\frac{1}{r})^{\frac{1}{q}-\sigma}\|h\|_{L^p(0,\infty)} + C\frac{1}{r}(\frac{1}{r})^{\frac{1}{q}-\sigma-1}\|h\|_{L^p(0,\infty)} \\ &= Cr^{\sigma-\frac{1}{q}}\|h\|_{L^p(0,\infty)} = Cr^{s-\frac{1}{2}}\|h\|_{L^p(0,\infty)},\end{aligned}$$

where we using the fact $\sigma - \frac{1}{q} = s - \frac{1}{2}$. Thus, when $\frac{1}{4} \leq s < \frac{1}{2}$, we have

$$\Big(\int_0^1 |B(r)|^2 dr\Big)^{1/2} \leq \Big(\int_0^1 r^{2s-1} dr\Big)^{1/2} \|h\|_{L^p(0,\infty)} \leq C\|h\|_{L^p(0,\infty)},$$

which implies (3.22).

Case II. $r \geq 1$. In this case,

$$\Big(\int_1^{\infty} |B(r)|^2 dr\Big)^{1/2} \leq C\|h\|_{L^p(0,\infty)}. \tag{3.24}$$

In fact, by (3.12), when $r \geq 1$, we have

$$\begin{aligned}|B(r)| &\leq Cr^{-s}\int_0^{\frac{1}{r}} \min\Big\{1, \frac{1}{ru}\Big\} u^{-\sigma}|h(u)|du + Cr^{-s}\int_{\frac{1}{r}}^{\infty} \min\{1, \frac{1}{ru}\} u^{-\sigma}|h(u)|du \\ &= Cr^{-s}\int_0^{\frac{1}{r}} u^{-\sigma}|h(u)|du + Cr^{-1-s}\int_{\frac{1}{r}}^{\infty} u^{-\sigma-1}|h(u)|du \\ &=: CQ_1(r) + CQ_2(r).\end{aligned} \tag{3.25}$$

Denote $M_1(t) = \frac{1}{t}Q_1(\frac{1}{t}),\ 0 < t < 1$. Note that $t^{-1+s} \leq (t-u)^{-1+s}$ when $\frac{1}{4} \leq s < \frac{1}{2}$ and $u \leq t$. It follows that

$$\begin{aligned}M_1(t) &= t^{-1+s}\int_0^t u^{-\sigma}|h(u)|du \\ &\leq \int_0^t (t-u)^{-1+s}u^{-\sigma}|h(u)|du \\ &\leq \int_{\mathbb{R}} (t-u)^{-1+s}u^{-\sigma}|h_1(u)|du \\ &=: I_s(u^{-\sigma}|h_1|)(t),\end{aligned}$$

where h_1 defined as above. Thus, by Parseval's equality and (3.21), we obtain

$$\begin{aligned}\Big(\int_1^{\infty} |Q_1(r)|^2 dr\Big)^{1/2} &= \Big(\int_0^1 |M_1(t)|^2 dt)^{1/2} \\ &\leq C\Big(\int_{\mathbb{R}} |I_s(u^{-\sigma}|h_1|)(t)|^2 dt\Big)^{1/2} \\ &= C\Big(\int_{\mathbb{R}} |(u^{-\sigma}|h_1|)\hat{}(\xi)|^2|\xi|^{-2s} d\xi\Big)^{1/2} \\ &\leq C\|h_1\|_{L^p(\mathbb{R})} = C\|h\|_{L^p(0,\infty)},\end{aligned}$$

which follows that

$$\Big(\int_1^\infty |Q_1(r)|^2 dr\Big)^{1/2} \le C\|h\|_{L^p(0,\infty)}. \tag{3.26}$$

Next, we estimate Q_2. Denote $M_2(t) = \frac{1}{t}Q_2(\frac{1}{t})$,$0 < t < 1$. Note that $u^{-1+s} \le (u-t)^{-1+s}$ when $\frac{1}{4} \le s < \frac{1}{2}$ and $u \ge t$. It follows that

$$\begin{aligned} M_2(t) &= t^s \int_t^\infty u^{-1-\sigma}|h(u)|du \\ &\le \int_t^\infty u^s u^{-1} u^{-\sigma}|h(u)|du \\ &\le \int_t^\infty u^{-1+s} u^{-\sigma}|h(u)|du \\ &\le \int_t^\infty (u-t)^{-1+s} u^{-\sigma}|h(u)|du \\ &\le \int_{\mathbb{R}} |u-t|^{-1+s} u^{-\sigma}|h_1(u)|du \\ &=: I_s(u^{-\sigma}|h_1|)(t), \end{aligned}$$

where h_1 defined as above. Thus, by Parseval's equality and (3.21), we obtain

$$\begin{aligned} \Big(\int_1^\infty |Q_2(r)|^2 dr\Big)^{1/2} &= \Big(\int_0^1 |M_2(t)|^2 dt\Big)^{1/2} \\ &\le C\Big(\int_{\mathbb{R}} |I_s(u^{-\sigma}|h|)(t)|^2 dt\Big)^{1/2} \\ &= C\Big(\int_{\mathbb{R}} |(u^{-\sigma}|h_1|)\hat{}(\xi)|^2 |\xi|^{-2s} d\xi\Big)^{1/2} \\ &\le C\|h_1\|_{L^p(\mathbb{R})} = C\|h\|_{L^p(0,\infty)}, \end{aligned}$$

which follows that

$$\Big(\int_1^\infty |Q_2(r)|^2 dr\Big)^{1/2} \le C\|h\|_{L^p(0,\infty)}. \tag{3.27}$$

Thus (3.24) holds from (3.25), (3.26) and (3.27). It follows that the estimate (3.22) from (3.23) and (3.24). Hence, the estimate (3.7) holds from (3.11), (3.15) and (3.22).

Summing up above estimates, we finish the proof of the sufficiency part of Theorem 1.6.

3.2 Proof of the Necessity Part of Theorem 1.6

Taking a radial nonnegative function φ in $C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subset \{\xi; 1 < |\xi| < 2\}$, which satisfies $\varphi(\xi) = 1$ if $\frac{5}{4} \le |\xi| \le \frac{7}{4}$. Denote $\hat{f}(\xi) := \varphi(\frac{\xi}{\lambda})$ for $\lambda > 1$. By simple calculation, we have

$$\|f\|_{H^s(\mathbb{R}^n)} \le C\lambda^{\frac{n}{2}+s}, \tag{3.28}$$

where C is independent of λ. Since

$$S_{t,\phi}f(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{it\phi(|\xi|)}\varphi\Big(\frac{\xi}{\lambda}\Big)d\xi = (2\pi)^{-n}\lambda^n\int_{\mathbb{R}^n} e^{i\lambda x\cdot\eta}e^{it\phi(|\lambda\eta|)}\varphi(\eta)d\eta, \tag{3.29}$$

if taking $t = 0$ in (3.29), we get

$$S_{0,\phi}f(x) = (2\pi)^{-n}\lambda^n\int_{\mathbb{R}^n} e^{i\lambda x\cdot\eta}\varphi(\eta)d\eta = (2\pi)^{-n}\lambda^n\hat{\varphi}(\lambda x). \tag{3.30}$$

Note that

$$\hat{\varphi}(0) = \int_{\mathbb{R}^n} \varphi(x)dx \geq \int_{\frac{5}{4}\leq|\xi|\leq\frac{7}{4}} \varphi(x)dx > 1,$$

there exists $0 < \delta < \frac{\lambda}{2}$, when $|x| \leq \frac{\delta}{\lambda}$, we get $\hat{\varphi}(\lambda x) > \frac{1}{2}$. Thus, when $|x| \leq \frac{\delta}{\lambda}$, we obtain

$$S_\phi^{**} f(x) \geq |S_{0,\phi} f(x)| \geq c_0 \lambda^n, \tag{3.31}$$

where $c_0 = \frac{1}{2(2\pi)^n}$. Assume the local estimate (1.13) holds, so if choose $B_0 := B(0,1)$, the unit ball in $\mathbb{R}^n$, then by (3.28) and (3.31), we get

$$C_{B_0} C\lambda^{\frac{n}{2}+s} \geq \Big(\int_{B_0} |S_\phi^{**} f(x)|^q |x|^\alpha dx\Big)^{1/q} \geq c_0\Big(\int_{|x|\leq\frac{\delta}{\lambda}} \lambda^{nq}|x|^\alpha dx\Big)^{1/q} =: c_1\lambda^{n-\frac{\alpha+n}{q}},$$

where $c_1 = c_0\big(\frac{\omega_{n-1}\delta^{\alpha+n}}{\alpha+n}\big)^{\frac{1}{q}}$ and ω_{n-1} denote the area of the unit sphere in $\mathbb{R}^n$. It follows that

$$\lambda^{n-\frac{\alpha+n}{q}} \leq C\lambda^{\frac{n}{2}+s}, \tag{3.32}$$

where C depends on δ, n and q only, not depend on λ. Taking λ large enough in (3.32), then $\alpha \geq q(\frac{n}{2} - s) - n$ is necessary for the inequality (3.32), i.e., $\alpha \geq q(\frac{n}{2} - s) - n$ is necessary for the weight local estimate (1.13) holds when $\frac{1}{4} \leq s < \frac{1}{2}$ and $2 \leq q \leq \frac{2}{1-2s}$.

4 The Proof of Lemma 2.1

Now we prove Lemma 2.1, we need the following variant of van der Corput's lemma:

Lemma 4.1[23]**.** *Assume that $a < b$ and set $I = [a,b]$. Let $F \in C^\infty(I)$ be real-valued and assume that $\psi \in C^\infty(I)$.*

(i) Assume that $|F'(x)| \geq \lambda > 0$ for $x \in I$ and that F' is monotonic on I. Then

$$\Big|\int_a^b e^{iF(x)}\psi(x)dx\Big| \leq C\frac{1}{\lambda}\Big\{|\psi(b)| + \int_a^b |\psi'(x)|dx\Big\},$$

where C does not depend on F, ψ or I.

(ii) Assume that $|F''(x)| \geq \lambda > 0$ for $x \in I$. Then

$$\Big|\int_a^b e^{iF(x)}\psi(x)dx\Big| \leq C\frac{1}{\lambda^{1/2}}\Big\{|\psi(b)| + \int_a^b |\psi'(x)|dx\Big\},$$

where C does not depend on F, ψ or I.

We now return to the proof of Lemma 2.1. By the conditions (H1) and (H2), there exist positive constants C_i $(i = 1, 2, \cdots, 6)$ so that for $r \geq 1$ and $m_2 > 1$ such that

$$C_1 r^{m_2-1} \leq |\phi'(r)| \leq C_2 r^{m_2-1} \qquad \text{and} \qquad |\phi''(r)| \geq C_3 r^{m_2-2}, \tag{4.1}$$

and for $0 < r < 1$ and $m_1 > 1$ such that

$$C_4 r^{m_1-1} \leq |\phi'(r)| \leq C_5 r^{m_1-1} \qquad \text{and} \qquad |\phi''(r)| \geq C_6 r^{m_1-2}. \tag{4.2}$$

Without loss of generality, we may assume $t > 0$ and $\xi > 0$. Denote

$$I = \int_0^\infty e^{ix\xi+it\phi(\xi)}\xi^{-s}\mu\Big(\frac{\xi}{N}\Big)d\xi.$$

To prove Lemma 2.1 it suffices to show that

$$|I| \le C\frac{1}{|x|^{1-s}}, \tag{4.3}$$

where the constant C may depend on s and m_1, m_2, C_i $(i = 1, 2, \cdots, 6)$ and μ but not on x, t and N. Write

$$I = \int_{\xi \le |x|^{-1}} e^{ix\xi + it\phi(\xi)}\xi^{-s}\mu\Big(\frac{\xi}{N}\Big)d\xi + \int_{\xi \ge |x|^{-1}} e^{ix\xi + it\phi(\xi)}\xi^{-s}\mu\Big(\frac{\xi}{N}\Big)d\xi =: I_1 + I_2.$$

Thus, to get (4.3) it suffices to give the following estimates:

$$|I_1| \le C\frac{1}{|x|^{1-s}}, \tag{4.4}$$

and

$$|I_2| \le C\frac{1}{|x|^{1-s}}, \tag{4.5}$$

where the constant C may depend on s and m_1, m_2, C_i $(i = 1, 2, \cdots, 6)$ and μ but not on x, t and N. The estimate of (4.4) is simple. Since $\mu \in C_0^\infty(\mathbb{R})$ and $s < 1$, we have

$$|I_1| \le C\int_{\xi \le |x|^{-1}} \xi^{-s} d\xi = C\frac{1}{|x|^{1-s}},$$

which follows (4.4) holds. As for (4.5), denote $\psi(\xi) = \xi^{-s}\mu(\frac{\xi}{N})$. We first show the following estimate holds:

$$\max_{\xi \ge |x|^{-1}} |\psi(\xi)| + \int_{|x|^{-1}}^{\infty} |\psi'(\xi)| d\xi \le C|x|^s. \tag{4.6}$$

In fact, by $\mu \in C_0^\infty(\mathbb{R})$ and $\frac{1}{2} \le s < 1$, we obtain

$$\max_{\xi \ge |x|^{-1}} |\psi| \le C|x|^s. \tag{4.7}$$

Since

$$\psi'(\xi) = \xi^{-s}\frac{1}{N}\mu'\Big(\frac{\xi}{N}\Big) - s\xi^{-s-1}\mu\Big(\frac{\xi}{N}\Big),$$

and

$$\int_{|x|^{-1}}^{\infty} \frac{1}{N}\Big|\mu'\Big(\frac{\xi}{N}\Big)\Big| d\xi \le C,$$

it follows that

$$\int_{|x|^{-1}}^{\infty} |\psi'(\xi)| d\xi \le |x|^s \int_{|x|^{-1}}^{\infty} \frac{1}{N}\Big|\mu'\Big(\frac{\xi}{N}\Big)\Big| d\xi + C\int_{|x|^{-1}}^{\infty} \xi^{-s-1} d\xi \le C|x|^s. \tag{4.8}$$

Therefore, (4.6) follows from (4.7) and (4.8). Now we split the proof for (4.5) into two cases according to the value of $|x|$.

Case I. $|x| \le 1$. We choose C_7 such that $C_7 \ge \frac{2}{C_1}$.

(I-a). $|x|^{m_2} \le \frac{t}{C_7}$. Denote $F(\xi) = x\xi + t\phi(\xi)$, we have

$$F'(\xi) = x + t\phi'(\xi) \qquad \text{and} \qquad F''(\xi) = t\phi''(\xi).$$

Note that $\xi \geq |x|^{-1} \geq 1$ by $|x| \leq 1$, by (4.1), we have

$$\Big|\frac{t}{x}\phi'(\xi)\Big| \geq \frac{C_1C_7|x|^{m_2}}{|x|}\xi^{m_2-1} \geq \frac{C_1C_7|x|^{m_2}}{|x|}|x|^{1-m_2} = C_1C_7 \geq 2. \tag{4.9}$$

Note that $F'(\xi) = x(1+\frac{t}{x}\phi'(\xi))$ and by (4.9), we get

$$|F'(\xi)| \geq |x|\Big(\Big|\frac{t}{x}\phi'(\xi)\Big| - 1\Big) \geq |x|. \tag{4.10}$$

Note that ϕ' is monotonic on $\mathbb{R}^+$ by the condition (H3), it follows that F' is monotonic for $\xi \geq |x|^{-1}$. Thus, using (i) of Lemma 4.1 and by estimates (4.10) and (4.6), we obtain

$$|I_2| = \Big|\int_{|x|^{-1}}^{\infty} e^{iF(\xi)}\psi(\xi)d\xi\Big| \leq C|x|^{-1}|x|^s = C\frac{1}{|x|^{1-s}},$$

it follows that (4.5) holds.

(I-b). $|x|^{m_2} > \frac{t}{C_7}$. We choose $\delta_1 > 0$ is small enough such that $\delta_1 \leq (\frac{1}{2C_2})^{\frac{1}{m_2-1}}$ and $\lambda_1 > 0$ is large enough such that $\lambda_1 \geq (\frac{2}{C_1})^{\frac{1}{m_2-1}}$. Note that $\lambda_1 > \delta_1$ by $C_1 \leq C_2$ and $m_2 > 1$. Denote

$$A_1 = \Big\{\xi \geq |x|^{-1} :\ \xi \leq \delta_1\Big(\frac{|x|}{t}\Big)^{\frac{1}{m_2-1}}\Big\},$$
$$A_2 = \Big\{\xi \geq |x|^{-1} :\ \delta_1\Big(\frac{|x|}{t}\Big)^{\frac{1}{m_2-1}} \leq \xi \leq \lambda_1\Big(\frac{|x|}{t}\Big)^{\frac{1}{m_2-1}}\Big\},$$
$$A_3 = \Big\{\xi \geq |x|^{-1} :\ \xi \geq \lambda_1\Big(\frac{|x|}{t}\Big)^{\frac{1}{m_2-1}}\Big\}.$$

Hence we may write

$$I_2 = \int_{A_1} e^{iF(\xi)}\psi(\xi)d\xi + \int_{A_2} e^{iF(\xi)}\psi(\xi)d\xi + \int_{A_3} e^{iF(\xi)}\psi(\xi)d\xi := I_{2,1} + I_{2,2} + I_{2,3}. \tag{4.11}$$

Now we give the estimates of $I_{2,j}$ $(j = 1,2,3)$, respectively. First, we consider $I_{2,1}$. For $\xi \in A_1$, since $m_2 > 1$ and by (4.1) we get

$$t|\phi'(\xi)| \leq C_2t\xi^{m_2-1} \leq t\delta_1^{m_2-1}\Big(\frac{|x|}{t}\Big)^{\frac{m_2-1}{m_2-1}} = C_2\delta_1^{m_2-1}|x| \leq \frac{|x|}{2},$$

which follows that

$$|F'(\xi)| \geq |x| - t|\phi'(\xi)| \geq \frac{|x|}{2}. \tag{4.12}$$

Applying (i) of Lemma 4.1, by (4.12) and (4.6) we obtain

$$|I_{2,1}| \leq C|x|^{-1}|x|^s = C\frac{1}{|x|^{1-s}}. \tag{4.13}$$

Next, we estimate $I_{2,3}$. For $\xi \in A_3$, since $m_2 > 1$ and by (4.1) we have

$$t|\phi'(\xi)| \geq C_1t\xi^{m_2-1} \geq C_1t\lambda_1^{m_2-1}\Big(\frac{|x|}{t}\Big)^{\frac{m_2-1}{m_2-1}} = C_1\lambda_1^{m_2-1}|x| \geq 2|x|.$$

From this we have

$$|F'(\xi)| \geq t|\phi'(\xi)| - |x| \geq |x|. \tag{4.14}$$

Using (i) of Lemma 4.1, (4.14) and (4.6), we get

$$|I_{2,3}| \le C|x|^{-1}|x|^s = C\frac{1}{|x|^{1-s}}. \tag{4.15}$$

Finally, we estimate $I_{2,2}$. Note that for $\xi \in A_2$, $|F''(\xi)| \ge C_3 t\xi^{m_2-2}$ by $F''(\xi) = t\phi''(\xi)$ and (4.1). Thus, we have

$$|F''(\xi)| \ge Ct\Big(\frac{|x|}{t}\Big)^{\frac{m_2-2}{m_2-1}} = Ct^{\frac{1}{m_2-1}}|x|^{\frac{m_2-2}{m_2-1}} \tag{4.16}$$

and

$$\max_{\xi\in A_2}|\psi| + \int_{|x|^{-1}}^{\infty}|\psi'|d\xi \le C\Big(\frac{|x|}{t}\Big)^{\frac{-s}{m_2-1}}. \tag{4.17}$$

In fact, since

$$|\psi(\xi)| \le \xi^{-s} \le \delta_1^{-s}\Big(\frac{|x|}{t}\Big)^{\frac{-s}{m_2-1}} \le C\Big(\frac{|x|}{t}\Big)^{\frac{-s}{m_2-1}} \tag{4.18}$$

and

$$\begin{aligned}\int_{A_2}|\psi'(\xi)|d\xi &\le \int_{A_2}\xi^{-s}\frac{1}{N}\Big|\mu'\Big(\frac{\xi}{N}\Big)\Big|d\xi + C\int_{A_2}\xi^{-s-1}d\xi \\ &\le \delta^{-s}\Big(\frac{|x|}{t}\Big)^{\frac{-s}{m_2-1}}\int_{A_2}\frac{1}{N}\Big|\mu'\Big(\frac{\xi}{N}\Big)\Big|d\xi + C\Big(\frac{|x|}{t}\Big)^{\frac{-s}{m_2-1}} \\ &\le C\Big(\frac{|x|}{t}\Big)^{\frac{-s}{m_2-1}},\end{aligned} \tag{4.19}$$

which imply (4.17). Applying (ii) of Lemma 4.1, by (4.16) and (4.17) we get

$$|I_{2,2}| \le C\big(t^{\frac{1}{m_2-1}}|x|^{\frac{m_2-2}{m_2-1}}\big)^{-\frac{1}{2}}\Big(\frac{|x|}{t}\Big)^{\frac{-s}{m_2-1}} \le Ct^{\frac{2s-1}{2(m_2-1)}}|x|^{\frac{2-m_2-2s}{2(m_2-1)}}. \tag{4.20}$$

Since $t < C_7|x|^{m_2}$ by $|x|^{m_2} > \frac{t}{C_7}$ and $s \ge \frac{1}{2}$, $m_2 > 1$, thus by (4.20), we obtain

$$|I_{2,2}| \le C|x|^{\frac{m_2(2s-1)}{2(m_2-1)}}|x|^{\frac{2-m_2-2s}{2(m_2-1)}} = C|x|^{\frac{(m_2-1)(2s-2)}{2(m_2-1)}} = C\frac{1}{|x|^{1-s}}. \tag{4.21}$$

Thus, the estimate (4.5) follows from estimate (4.11), (4.13), (4.15) and (4.21).

Case II. $|x| > 1$. Denote

$$I_2 = \int_1^{\infty} e^{ix\xi+it\phi(\xi)}\xi^{-s}\mu\Big(\frac{\xi}{N}\Big)d\xi + \int_{|x|^{-1}\le\xi\le1} e^{ix\xi+it\phi(\xi)}\xi^{-s}\mu\Big(\frac{\xi}{N}\Big)d\xi =: I_{2,4} + I_{2,5}. \tag{4.22}$$

We first estimate $I_{2,4}$. Note that $\xi \ge 1$, similar to estimating I_2 in Case I. We may estimate $I_{2,4}$ for the cases $|x|^{m_2} \le \frac{t}{C_7}$ and $|x|^{m_2} > \frac{t}{C_7}$, respectively. Hence, we obtain

$$|I_{2,4}| = \Big|\int_1^{\infty} e^{iF(\xi)}\psi(\xi)d\xi\Big| \le C\frac{1}{|x|^{1-s}}. \tag{4.23}$$

Next, we consider $I_{2,5}$. Since $|x|^{-1} \le \xi < 1$, we choose C_8 such that $C_8 \ge \frac{2}{C_4}$.

(II-a) If $|x|^{m_1} \le \frac{t}{C_8}$. Similar to the estimate in (I-a) of Case (I), we may get

$$|I_{2,5}| = \Big|\int_{|x|^{-1}\le\xi<1} e^{iF(\xi)}\psi(\xi)d\xi\Big| \le C|x|^{-1}|x|^s = C|x|^{s-1}.$$

(II-b) If $|x|^{m_1} > \frac{t}{C_8}$. We choose $\delta_2, \lambda_2 > 0$ such that $\delta_2 \leq (\frac{1}{2C_5})^{\frac{1}{m_1-1}}$ and $\lambda_2 \geq (\frac{2}{C_4})^{\frac{1}{m_1-1}}$. Then $\lambda_2 > \delta_2$ by $C_4 \leq C_5$ and $m_1 > 1$. Denote

$$B_1 = \Big\{1 > \xi \geq |x|^{-1} : \xi \leq \delta_2\Big(\frac{|x|}{t}\Big)^{\frac{1}{m_1-1}}\Big\},$$
$$B_2 = \Big\{1 > \xi \geq |x|^{-1} : \delta_2\Big(\frac{|x|}{t}\Big)^{\frac{1}{m_1-1}} \leq \xi \leq \lambda_2\Big(\frac{|x|}{t}\Big)^{\frac{1}{m_1-1}}\Big\},$$
$$B_3 = \Big\{1 > \xi \geq |x|^{-1} : \xi \geq \lambda_2\Big(\frac{|x|}{t}\Big)^{\frac{1}{m_1-1}}\Big\}.$$

Therefore,

$$I_{2,5} = \int_{B_1} e^{iF(\xi)}\psi(\xi)d\xi + \int_{B_2} e^{iF(\xi)}\psi(\xi)d\xi + \int_{B_3} e^{iF(\xi)}\psi(\xi)d\xi. \tag{4.24}$$

Similar to estimating $I_{2,j}$ $(j = 1, 2, 3)$ above, we may get

$$\Big|\int_{B_1} e^{iF(\xi)}\psi(\xi)d\xi\Big| \leq C|x|^{-1}|x|^s = C\frac{1}{|x|^{1-s}}, \tag{4.25}$$

$$\Big|\int_{B_3} e^{iF(\xi)}\psi(\xi)d\xi\Big| \leq C|x|^{-1}|x|^s = C\frac{1}{|x|^{1-s}} \tag{4.26}$$

and

$$\Big|\int_{B_2} e^{iF(\xi)}\psi(\xi)d\xi\Big| \leq Ct^{\frac{2s-1}{2(m_1-1)}}|x|^{\frac{2-m_1-2s}{2(m_1-1)}}. \tag{4.27}$$

Since $|x|^{m_1} > \frac{t}{C_8}$ and $s \geq \frac{1}{2}$, $m_1 > 1$, by (4.27) we have

$$\Big|\int_{B_2} e^{iF(\xi)}\psi(\xi)d\xi\Big| \leq C|x|^{\frac{m_1(2s-1)}{2(m_1-1)}}|x|^{\frac{2-m_1-2s}{2(m_1-1)}} = C|x|^{\frac{(m_1-1)(2s-1)}{2(m_1-1)}} = C\frac{1}{|x|^{1-s}}. \tag{4.28}$$

Thus, by (4.24), (4.25), (4.26) and (4.28), we get

$$|I_{2,5}| \leq C\frac{1}{|x|^{1-s}}, \tag{4.29}$$

and (4.5) holds from (4.23) and (4.29).

Summing up above all estimates, we show (4.3) and complete the proof of Lemma 2.1.

5 Estimate for Combination by Radial Functions and Solid Spherical Harmonics

Let $\mathcal{A}_k$ be the set of all solid spherical harmonics of degree k. It is well-known (see [22, p. 151]) that there exists a direct decomposition

$$L^2(\mathbb{R}^n) = \sum_{k=0}^{\infty} \oplus \mathfrak{D}_k.$$

The subspace $\mathfrak{D}_k$ is of all finite linear combinations of functions of the form $f(|x|)P(x)$, where f ranges over the radial functions and P over $\mathcal{A}_k$ such that $f(|\cdot|)P(\cdot) \in L^2(\mathbb{R}^n)$.

Fix $k \geq 0$ and let $P_1, P_2, \cdots, P_{a_k}$ denote an orthonormal basis in $\mathcal{A}_k$. Every element in $\mathfrak{D}_k$ can be written in the following form

$$f(x) = \sum_{j=1}^{a_k} f_j(|x|)P_j(x), \tag{5.1}$$

and

$$\int_{\mathbb{R}^n} |f(x)|^2 dx = \sum_{j=1}^{a_k} \int_0^\infty |f_j(r)|^2 r^{n+2k-1} dr.$$

Let $\mathcal{H}_0(\mathbb{R}^n)$ be the class of all radial functions in $\mathcal{S}(\mathbb{R}^n)$, and $\mathcal{H}_k$ $(k \in \mathbb{N})$ be the set of functions defined by (5.1) with $f_j \in \mathcal{H}_0(\mathbb{R}^n)$ and $P_j \in \mathcal{A}_k$ for $j = 1, 2, \cdots, a_k$. From the proof in [20, p. 399–400], in fact, Sjölin obtained the following result:

Theorem D[20]. Assume that $a > 1$, $n \geq 2$ and $f \in \mathcal{H}_k$ $(k \geq 0)$. If $2 \leq q \leq 4$, $\alpha = \frac{q(2n-1)}{4} - n$, then

$$\Big(\int_{\mathbb{R}^n} |S_a^{**} f(x)|^q |x|^\alpha dx \Big)^{1/q} \leq C \|f\|_{H^{\frac{1}{4}}(\mathbb{R}^n)}. \tag{5.2}$$

We give the global weighted L^q estimate of the maximal operator S_ϕ^{**} for $f \in \mathcal{H}_k$.

Theorem 5.1. *Assume that $n \geq 2$ and ϕ satisfies the conditions in Theorem 1.6, $f \in \mathcal{H}_k$ $(k \geq 0)$. If $\frac{1}{4} \leq s < \frac{1}{2}$, $2 \leq q \leq \frac{2}{1-2s}$ and $\alpha = q(\frac{n}{2} - s) - n$, then*

$$\Big(\int_{\mathbb{R}^n} |S_\phi^{**} f(x)|^q |x|^\alpha dx \Big)^{1/q} \leq C \|f\|_{H^s(\mathbb{R}^n)}. \tag{5.3}$$

Remark 5.2. Obviously, Theorem 5.1 is an improvement and extension of Theorem D for the case $s = \frac{1}{4}$.

Proof of Theorem 5.1. When $k = 0$, Theorem 5.1 follows from Theorem 1.6. Hence we need only to give the proof of Theorem 5.1 for $k \geq 1$. To do this, we need the following a well-known fact:

Lemma 5.3[22]**.** *Suppose $n \geq 2$ and $f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ has the form $f(x) = f_0(|x|)P(x)$, where $P(x)$ is a solid spherical harmonic of degree k, then $\hat{f}$ has the form $\hat{f}(x) = F_0(|x|)P(x)$, where*

$$F_0(r) = (2\pi)^{\frac{n}{2}} i^{-k} r^{-\frac{n}{2}-k+1} \int_0^\infty f_0(s) J_{\frac{n}{2}+k-1}(rs) s^{\frac{n}{2}+k} ds,$$

where J_m denotes the Bessel function.

Let us return to the proof of Theorem 5.1. By [20, p. 396] we know that for $f \in \mathcal{H}_k (k \geq 1)$,

$$\|f\|_{H^s(\mathbb{R}^n)} = \Big(\sum_{j=1}^{a_k} \int_0^\infty |F_j(r)|^2 (1+r^2)^s r^{n+2k-1} dr \Big)^{1/2}. \tag{5.4}$$

On the other hand, by Lemma 5.3 we have

$$\hat{f}(x) = \sum_{j=1}^{a_k} F_j(|x|) P_j(x), \tag{5.5}$$

where

$$F_j(r) = (2\pi)^{\frac{n}{2}} i^{-k} r^{1-\frac{n}{2}-k} \int_0^\infty f_j(s) J_{\frac{n}{2}+k-1}(rs) s^{\frac{n}{2}+k} ds, \qquad r > 0.$$

Thus, by (5.5) we get

$$S_{t,\phi} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it\phi(|\xi|)} \hat{f}(\xi) d\xi = \sum_{j=1}^{a_k} (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \Big(e^{it\phi(|\xi|)} F_j(|\xi|) P_j(\xi) \Big) d\xi.$$

Applying Lemma 5.3, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n} e^{ix\cdot\xi}\big(e^{it\phi(|\xi|)}F_j(|\xi|)P_j(\xi)\big)d\xi\\ =&\big(e^{it\phi(|\cdot|)}F_j(|\cdot|)P_j(-\cdot)\big)^{\wedge}(x)\\ =&(2\pi)^{\frac{n}{2}}i^{-k}s^{1-\frac{n}{2}-k}\Big(\int_0^\infty J_{\frac{n}{2}+k-1}(rs)e^{it\phi(r)}F_j(r)r^{\frac{n}{2}+k}dr\Big)P_j(-x),\end{aligned}$$

where $s=|x|>0$. Thus, we have

$$S_{t,\phi}f(x)=\sum_{j=1}^{a_k}(2\pi)^{-\frac{n}{2}}i^{-k}|x|^{1-\frac{n}{2}-k}\Big(\int_0^\infty J_{\frac{n}{2}+k-1}(r|x|)e^{it\phi(r)}F_j(r)r^{\frac{n}{2}+k}dr\Big)P_j(-x). \tag{5.6}$$

Denote by $\mathcal{F}_n$ the Fourier transform in $\mathbb{R}^n$, then $F_j=i^{-k}\mathcal{F}_{n+2k}f_j$. Note that for a radial function $h\in\mathcal{S}(\mathbb{R}^{n+2k})$, we have

$$\mathcal{F}_{n+2k}h(x)=(2\pi)^{\frac{n}{2}}|x|^{1-\frac{n}{2}-k}\int_0^\infty h(r)J_{\frac{n}{2}+k-1}(r|x|)r^{\frac{n}{2}+k}dr.$$

Define the operator $S^{n+2k}_{t,\phi}$ on the set of all radial function in $\mathcal{S}(\mathbb{R}^{n+2k})$ by

$$S^{n+2k}_{t,\phi}h(x):=(2\pi)^{-n-2k}\int_{\mathbb{R}^{n+2k}}e^{ix\cdot\xi}e^{it\phi(|\xi|)}\mathcal{F}_{n+2k}h(|\xi|)d\xi.$$

Obviously, $S^{n+2k}_{t,\phi}h$ is still a radial function. Then

$$\begin{aligned}S^{n+2k}_{t,\phi}f_j(|x|)&=i^k(2\pi)^{-n-2k}\int_{\mathbb{R}^{n+2k}}e^{ix\cdot\xi}\big(e^{it\phi(|\xi|)}F_j(\xi)\big)d\xi\\ &=i^k(2\pi)^{-\frac{n}{2}-2k}|x|^{1-\frac{n}{2}-k}\int_0^\infty J_{\frac{n}{2}+k-1}(r|x|)e^{it\phi(r)}F_j(r)r^{\frac{n}{2}+k}dr.\end{aligned} \tag{5.7}$$

By (5.6) and (5.7), we get

$$S_{t,\phi}f(x)=i^{-2k}(2\pi)^{2k}\sum_j S^{n+2k}_{t,\phi}f_j(|x|)\cdot P_j(-x),\qquad x\in\mathbb{R}^n, \tag{5.8}$$

where we may regard $S^{n+2k}_{t,\phi}f_j(|x|)$ as a function on $\mathbb{R}^n$ since $S^{n+2k}_{t,\phi}f_j$ is a radial function. Denote

$$S^{n+2k,**}_{\phi}f_j(|y|)=\sup_{t\in\mathbb{R}}|S^{n+2k}_{t,\phi}f_j(|y|)|,\qquad y\in\mathbb{R}^{n+2k}\quad\text{or}\quad y\in\mathbb{R}^n. \tag{5.9}$$

Then by (5.8) and (5.9), we obtain

$$S^{**}_{\phi}f(x)\le C_{n,k}\sum_j(S^{n+2k,**}_{\phi}f_j(|x|))|x|^k. \tag{5.10}$$

Using the notation $v=|x|$ and $r=|\xi|$. By (5.10), we get

$$\Big(\int_{\mathbb{R}^n}|S^{n,**}_{\phi}f(x)|^q v^\alpha dx\Big)^{1/q}\le C\sum_{j=1}^{a_k}\Big(\int_{\mathbb{R}^n}|S^{n+2k,**}_{\phi}f_j(v)|^q|x|^\alpha v^{kq}dx\Big)^{1/q}, \tag{5.11}$$

where $\frac{1}{4} \le s < \frac{1}{2}$, $2 \le q \le \frac{2}{1-2s}$ and $\alpha = q(\frac{n}{2} - s) - n$. Denote $\beta = q(\frac{n+2k}{2} - s) - (n+2k)$, it follows that $kq + \alpha + n - 1 = \beta + n + 2k - 1$. Using representation of polar coordinates, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |S_\phi^{n+2k,**} f_j(v)|^q v^\alpha v^{kq} dx &= \omega_{n-1} \int_0^\infty |S_\phi^{n+2k,**} f_j(v)|^q v^{kq+\alpha+n-1} dv \\ &= \frac{\omega_{n-1}}{\omega_{n+2k-1}} \int_{\mathbb{R}^{n+2k}} |S_\phi^{n+2k,**} f_j(v)|^q v^\beta dx,\end{aligned} \tag{5.12}$$

where ω_{n-1} and ω_{n+2k-1} denote areas of the unit sphere in $\mathbb{R}^n$ and $\mathbb{R}^{n+2k}$, respectively. Since f_j is a radial function in $\mathbb{R}^{n+2k}$, $\frac{1}{4} \le s < \frac{1}{2}$, $2 \le q \le \frac{2}{1-2s}$ and $\beta = q(\frac{n+2k}{2} - s) - (n+2k)$, thus applying Theorem 1.6, we obtain

$$\Big(\int_{\mathbb{R}^{n+2k}} |S_\phi^{n+2k,**} f_j(x)|^q |x|^\beta dx \Big)^{1/q} \le C \|f_j\|_{H^s(\mathbb{R}^{n+2k})}. \tag{5.13}$$

Note that $\mathcal{F}_{n+2k} f_j = i^k F_j$, we get

$$\begin{aligned}\|f_j\|^2_{H^s(\mathbb{R}^{n+2k})} &= \int_{\mathbb{R}^{n+2k}} |F_j(|\xi|)|^2 (1+|\xi|^2)^s d\xi \\ &= \omega_{n+2k-1} \int_0^\infty |F_j(r)|^2 (1+r^2)^s r^{n+2k-1} dr.\end{aligned} \tag{5.14}$$

Therefore, by (5.11), (5.12), (5.13), (5.14) and (5.4), we obtain

$$\begin{aligned}\Big(\int_{\mathbb{R}^n} |S_\phi^{n,**} f(x)|^q |x|^\alpha dx \Big)^{1/q} &\le C \Big(\sum_{j=1}^{a_k} \int_0^\infty |F_j(r)|^2 (1+r^2)^s r^{n+2k-1} dr \Big)^{1/2} \\ &= C \|f\|_{H^s(\mathbb{R}^n)}.\end{aligned} \tag{5.15}$$

Thus, we complete the proof of Theorem 5.1.

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