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A Note on Harnack Type Inequality for the Gaussian Curvature Flow

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Abstract In this short note we present a new Harnack expression for the Gaussian curvature flow, which is modeled from the shrinking self similiar solutions. As applications we give alternate proofs of Chow's Harnack inequality and entropy estimate.

Keywords Gaussian Curvature flow; Harnack inequality
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1 introduction

Entropy monotonicity and Harnack inequality are two powerful tools in the studies of both intrinsic and extrinsic geometric flows. In the Ricci flow, the important $\mathcal{W}$ entropy and its corresponding Harnack were introduced in the seminal work of Perelman[17]. These were considered in a general setting in [2, 6, 8]. In the Gaussian curvatue flow, Chow[3] first studied the entropy and Harnack type inequalities by deriving a surprising and interesting identity along the flow, see Eq.(1.2) below. The entropy and Harnack were studied deeply and systematically in a series of articles of Li-Li[10–13] and Li[14, 15]. See also a survey article by Ni[16].

In the Gaussian curvature flow, a new entropy was defined in [7] based on Chow's entropy and inspired by Perelman's idea, and this was discussed in [1]. Similar ideas appeared in [5, 9]. The ideas are to rewrite the formulas to fit the shrinkers' equations. In this short note we work out the Harnack expression modeled from the shrinking self similar solutions to the Gaussian curvature flow. Since we rely heavily on some deep thoerems in [3], we invite the readers to read the following together with [3].

We adapt the notations of [3]. Given a hypersurface M parameterized by a map $F : M^n \to \mathbb{R}^{n+1}$, the Gaussian curvature flow is given by

$$\frac{\partial F(x,t)}{\partial t} = -K(x,t)\nu(x,t) \tag{1.1}$$

for $x \in M, t \geq 0$. Here K denotes the Gaussian curvature and ν denotes the outward unit normal vector field. Let h_{ij} be the second fundamental form. Assume $h_{ij} > 0$. Define

$$P_{ij} = \nabla_i \nabla_j K - h_{kl}^{-1} \nabla_k h_{ij} \nabla_l K + K h_{ij}^2$$

and

$$P = h_{ij}^{-1} P_{ij} = \Box K - \frac{1}{K} h_{ij}^{-1} \nabla_i K \nabla_j K + HK$$

where $\Box = h_{ij}^{-1} \nabla_i \nabla_j$.

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Theorem 1.1[3]**.** *Along the Gaussian curvature flow* (1.1) *for convex hypersurface, there holds*

$$\frac{\partial P}{\partial t} = K\Box P + 2\langle \nabla K, \nabla P\rangle_h + |P_{ij}|_h^2 + P^2 \tag{1.2}$$

Eq. (1.2) is trivial only on the steady self similar solutions which satisfy $P_{ij} = 0$. As in other curvature flows, a natural question can be asked is whether there is a Harnack expression which is trivial only on the shrinkers satisfying

$$P_{ij} - \frac{h_{ij}}{(n+1)(T-t)} = 0$$

where T is the extinction time of the flow.

The purpose of this note is to illustrate the following: in the studies of curvature flows, whenever there is a quantity associated to the steadier one can construct a quantity associated to the shrinker.

Theorem 1.2. *Let* $\frac{d\tau}{dt} = -(n+1)$ *and* $Q = \tau^2 P - n\tau$*. Along the Gaussian curvature flow one has*

$$\frac{\partial Q}{\partial t} = K\Box Q + 2\langle \nabla K, \nabla Q\rangle_h + \tau^2\Big|P_{ij} - \frac{h_{ij}}{\tau}\Big|_h^2 + \tau^2\Big(P - \frac{n}{\tau}\Big)^2 \tag{1.3}$$

In the next section we will prove Eq. (1.3) and show two applications of it.

2 Proof of the Identity and Its Applications

Proof of Therem 1.2. Starting from (1.2) we have

$$\begin{aligned}
\frac{\partial P}{\partial t} &= K\Box P + 2\langle \nabla K, \nabla P\rangle_h + |P_{ij}|_h^2 + P^2 \\
&= K\Box P + 2\langle \nabla K, \nabla P\rangle_h + \Big(\Big|P_{ij} - \frac{h_{ij}}{\tau}\Big|_h^2 + 2\langle P_{ij}, \frac{h_{ij}}{\tau}\rangle_h - \frac{n}{\tau^2}\Big) \\
&\quad + \Big(\Big(P - \frac{n}{\tau}\Big)^2 + \frac{2n}{\tau}P - \frac{n^2}{\tau^2}\Big) \\
&= K\Box P + 2\langle \nabla K, \nabla P\rangle_h + \Big|P_{ij} - \frac{h_{ij}}{\tau}\Big|_h^2 + \Big(P - \frac{n}{\tau}\Big)^2 \\
&\quad + \frac{2(n+1)}{\tau}P - \frac{n(n+1)}{\tau^2}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial Q}{\partial t} &= \tau^2\frac{\partial P}{\partial t} - 2(n+1)\tau P + n(n+1) \\
&= \tau^2 K\Box P + 2\tau^2\langle \nabla K, \nabla P\rangle_h + \tau^2\Big|P_{ij} - \frac{h_{ij}}{\tau}\Big|_h^2 + \tau^2\Big(P - \frac{n}{\tau}\Big)^2.
\end{aligned}$$

Note that $\Box Q = \tau^2\Box P, \nabla Q = \tau^2\nabla P$ and this completes the proof. □

2.1 Application to the Harnack Inequality

As an application of Eq. (1.3) we give an alternate proof of Chow's Harnack inequality[3].

Corollary 2.1. *Along the Gaussian curvature flow on convex hypersurfaces there hold*

$$\frac{\partial K}{\partial t} + \frac{nK}{(n+1)t} - |\nabla K|_h^2 \geq 0. \tag{2.1}$$

Proof. Let $\tau = -(n+1)t$. By Eq. (1.3) we have $\frac{\partial Q}{\partial t} \geq K\Box Q + 2\langle \nabla K, \nabla Q\rangle_h$. Combining with $Q = 0$ at $t = 0$ and applying the maximum principle we have $Q(x,t) \geq 0$ for all $t \geq 0$, namely $\tau^2 P - n\tau \geq 0$ or

$$P + \frac{n}{(n+1)t} \geq 0. \tag{2.2}$$

Since along the Gaussian curvature flow (1.1) it holds

$$\frac{\partial K}{\partial t} = K\Box K + HK^2$$

Inequality (2.2) can be rewritten as

$$\frac{\partial K}{\partial t} + \frac{nK}{(n+1)t} - |\nabla K|_h^2 \geq 0.$$

□

Remark 2.2. Chow's original proof of Inequality (2.2) is based on the identity (1.2). Using $|P_{ij}|_h^2 \geq P^2/n$ one gets

$$\frac{\partial P}{\partial t} \geq K\Box P + 2\langle \nabla K, \nabla P\rangle_h + \frac{n+1}{n}P^2.$$

Solving the corresponding ordinary differential equation to the partial differential inequality above

$$\frac{dp}{dt} = \frac{n+1}{n}p^2$$

whose general solution satisfies $p(t) \geq -\frac{n+1}{nt}$, and applying the maximum principle one gets Inequality (2.2).

2.2 Revisit to the Entropy Estimate

In this subsection we let $\tau = (n+1)(T-t)$ with $[0,T)$ being the maximal time interval of the Gaussian curvature flow. In [3] Chow first considered the entropy

$$N(t) = \int_M K \log K d\mu \tag{2.3}$$

and he proved the entropy estimate that

$$N(t) \leq N(0) + \frac{n\sigma_n}{n+1} \log \frac{T}{T-t} \tag{2.4}$$

with σ_n being volume of the unit sphere. His proof is based on Eq. (1.2). Now we revisit the entropy estimate based on Eq. (1.3).

By (1.3) one has

$$\frac{d}{dt}\int_M (\tau^2 P - n\tau)K d\mu = \int_M \tau^2 \left(\left|P_{ij} - \frac{h_{ij}}{\tau}\right|_h^2 + \left(P - \frac{n}{\tau}\right)^2\right) K d\mu.$$

In particular $\int_M (\tau^2 P - n\tau)K d\mu$ is increasing for $t \in [0,T)$.

Since $\lim\limits_{\tau\to 0} \tau^2 \int_M PK d\mu = 0$ and thus

$$\lim_{t\to T} \int_M \left(\tau^2 P - n\tau\right) K d\mu = 0$$

we have $\int_M \left(\tau^2 P - n\tau\right) K d\mu \leq 0$, namely

$$\int_M (\tau P - n) K d\mu \leq 0, \qquad t \in [0, T)$$

or

$$\int_M PK d\mu \leq \frac{n\sigma_n}{(n+1)(T-t)}.$$

Since $\frac{dN}{dt} = \int_M PK d\mu$ and thus

$$\frac{dN}{dt} \leq \frac{n\sigma_n}{(n+1)(T-t)}.$$

Integrating the above inequality on $[0, t]$ gives

$$N(t) \leq N(0) + \frac{n\sigma_n}{n+1} \log \frac{T}{T-t}$$

and this is the entropy estimate (2.4).

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