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The Characterization of p-factor-critical Graphs

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Abstract A graph G is said to be p-factor-critical if $G - u_1 - u_2 - \cdots - u_p$ has a perfect matching for any $u_1, u_2, \cdots, u_p \in V(G)$. The concept of p-factor-critical is a generalization of the concepts of factor-critical and bicritical for $p = 1$ and $p = 2$, respectively. Heping Zhang and Fuji Zhang[*Construction for bicritical graphs and k-extendable bipartite graphs*, Discrete Math., 306(2006) 1415–1423] gave a concise structure characterization of bicritical graphs. In this paper, we present the characterizations of p-factor-critical graphs and minimal p-factor-critical graphs for $p \geq 2$. As an application, we also obtain a class of graphs which are minimal p-factor-critical for $p \geq 1$.

Keywords perfect matching, p-factor-critical graph; transversal
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1 Introduction

All graphs considered in this paper are finite, connected, loopless and have no multiple edges. Our notation and terminology in graph theory follows Bondy and Murty[3] and Lovász and Plummer[10].

Let G be a graph. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$, respectively. A *matching* M in G is a subset of $E(G)$ in which no two edges have a vertex in common. A vertex v is *covered* by M if some edge of M is incident to v, otherwise, it is said to be *uncovered* (or *missed*) by M. A matching M is *perfect* if it covers every vertex of G and near perfect if it covers all but one vertex of G. Denote $\Gamma_G(v)$ the neighbor set of v in G. Let A and B be two sets, then $A \setminus B$ denotes A minus B. If M is a matching and P is a path in G such that the edges on P appear in M and $E(G) \setminus M$ alternately, then P is an *M-alternating path*.

Let G be a graph. Denote $D(G)$ the set of all vertices in G which are not covered by at least one maximum matching of G. Let $A(G)$ be the set of vertices in $V(G) - D(G)$ which are adjacent to at least one vertex in $D(G)$. Finally, let $C(G) = V(G) - D(G) - A(G)$.

A graph G of order n is *p-factor-critical*, where n and p are positive integers with the same parity, if the deletion of any set of p vertices results in a graph with a perfect matching. The concept of p-factor-critical is a generalization of the concepts of factor-critical and bicritical for $p = 1$ and $p = 2$, respectively. Let G be a p-factor-critical graph, G is called minimal if $G - e$ is not p-factor-critical for any $e \in E(G)$. An edge $e \in E(G)$ is called deletable if $G - e$ is still p-factor-critical. Favaron[4, 5] and Favaron et al.[6] gave some properties of p-factor-critical graphs. Favaron and Shi[7, 8] obtained some properties of minimally p-factor-critical graphs and characterized $(n-4)$-factor-critical graphs and minimally $(n-4)$-factor-critical graphs, where n

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is the order of the graph. Aldred et al.[1], and Wen and Lou[12] characterized $2k$-factor-critical graphs and $(2k+1)$-factor-critical graphs using M-alternating path theory, respectively.

Let X be a finite set and $\mathcal{F}$ a family of non-empty subsets of X. A subset $S \subseteq X$ is called a *transversal* of $\mathcal{F}$ if $S \cap F_i \neq \emptyset$ for each $F_i \in \mathcal{F}$. A transversal S of $\mathcal{F}$ is minimal if it does not exist $S' \subset S$ such that S' is also a transversal of $\mathcal{F}$.

Heping Zhang and Fuji Zhang[13] gave the following theorem and characterized the structure of bicritical graphs.

Theorem 1.1[13]**.** *A graph G is bicritical if and only if for any vertex w of G, $H = G - w$ is factor-critical and $\Gamma_G(w) \subseteq V(H)$ is a transversal of $\mathcal{D}_2(H)$, where $\mathcal{D}_2(H)= \big\{D(H-u_1-u_2) : u_1, u_2 \in V(H)\big\}$.*

In this paper, we present the characterizations of p-factor-critical graphs and minimal p-factor-critical graphs for $p \geq 2$. As an application, we also obtain a class of graphs which are minimal p-factor-critical for $p \geq 1$.

In order to obtain the main results in this paper, we need to recall the Gallai-Edmonds Structure Theorem.

Theorem 1.2 (The Gallai-Edmonds Structure Theorem[10])**.** *If G is a graph and $D(G)$, $A(G)$, $C(G)$ are defined as above, then*
(1) the components of the subgraph induced by $D(G)$ are factor-critical,
(2) the subgraph induced by $C(G)$ has a perfect matching,
(3) the bipartite graph, say $B(G)$, obtained from G by deleting the vertices of $C(G)$ and the edges spanned by $A(G)$ and contracting each component of $D(G)$ to a single vertex has positive surplus (as viewed from $A(G)$),
(4) if M is any maximum matching of G, then it contains a near perfect matching of each component of $D(G)$, a perfect matching of each component of $C(G)$ and a maximum matching of $B(G)$.

2 Main Results

In this section, we will characterize p-factor-critical graphs and minimal p-factor-critical graphs for $p \geq 2$. In addition, we will also get a class of minimal p-factor-critical graphs for $p \geq 1$.

Theorem 2.1. *Let $p \geq 2$ be a positive integer. A graph G is p-factor-critical if and only if for any vertex w of G, $H = G - w$ is $(p-1)$-factor-critical and $\Gamma_G(w) \subseteq V(H)$ is a transversal of $\mathcal{D}_p(H)$, where $\mathcal{D}_p(H)= \big\{D(H-u_1-\cdots-u_p) : u_i \in V(H),\ i = 1, \cdots, p\big\}$.*

Proof. Suppose that G is a p-factor-critical graph. Then, for any vertex $w \in V(G)$, we have $H = G - w$ is $(p-1)$-factor-critical. Let M be a perfect matching of $G - u_1 - \cdots - u_p$ where $u_1, \cdots, u_p \in V(H)$ and sw be the edge covering vertex w in M. Obviously, $M - sw$ is a near perfect matching of $H - u_1 - \cdots - u_p$ and misses s. Then $s \in \Gamma_G(w) \cap D(H - u_1 - \cdots - u_p)$ and $\Gamma_G(w)$ is a transversal of $\mathcal{D}_p(H)$.

Conversely, suppose that H is a $(p-1)$-factor-critical graph and $S \subseteq V(H)$ is a transversal of $\mathcal{D}_p(H)$. Let G be the graph obtained from H by adding a new vertex w and all the edges connecting w to each vertex of S. Now, we shall prove that G is p-factor-critical, that is, for any $u_i \in V(G),\ i = 1, \cdots, p$, we need to prove $G - u_1 - \cdots - u_p$ has a perfect matching.

Case 1. $w \in \{u_1, \cdots, u_p\}$.

Without loss of generality, let $w = u_1$, then $G - u_1 - \cdots - u_p = H - u_2 - \cdots - u_p$ has a perfect matching.

Case 2. $w \notin \{u_1, \cdots, u_p\}$.

Since H is $(p-1)$-factor-critical, $H - u_1 - \cdots - u_p$ has a near perfect matching and

$$\big|A(H - u_1 - \cdots - u_p)\big| = c\big(D(H - u_1 - \cdots - u_p)\big) - 1.$$

Let $k = \big|A(H - u_1 - \cdots - u_p)\big|$. Then $k \geq 0$, $D(H - u_1 - \cdots - u_p)$ exactly has $k+1$ components, denoted by $D_1, D_2, \cdots, D_{k+1}$, and for $i \in \{1, 2, \cdots, k+1\}$ each D_i is factor-critical by Gallai-Edmonds Structure Theorem.

Since S is a transversal of $\mathcal{D}_p(H)$, $S \cap D(H - u_1 - \cdots - u_p) \neq \emptyset$. Let $w_1 \in S \cap D(H - u_1 - \cdots - u_p)$. Without loss of generality, assume $w_1 \in D_1$. Let H' be the bipartite graph obtained from $H - u_1 - \cdots - u_p$ by deleting the vertices of $C\big(H - u_1 - \cdots - u_p\big)$, the edges spanned by $A\big(H - u_1 - \cdots - u_p\big)$ and contracting each component of $D\big(H - u_1 - \cdots - u_p\big)$ to a single vertex. Then by Gallai-Edmonds Structure Theorem H' has positive surplus (as viewed from $A\big(H - u_1 - \cdots - u_p\big)$). By Hall's Theorem, the resultant bipartite graph from H' by deleting the vertex corresponding to D_1 has a perfect matching $M^* = \{e_1, \cdots, e_k\}$. Let $w_{i+1} \in V(M^*) \cap D_{i+1}$ for $i = 1, \cdots, k$. Let M_i be a perfect matching of $D_i - w_i$ for $i = 1, \cdots, k+1$ and M a perfect matching of $C(H - u_1 - \cdots - u_p)$. Then

$$\bigcup_{i=1}^{k+1} M_i \cup M \cup M^* \cup \{ww_1\} \tag{2.1}$$

is a perfect matching of $G - u_1 - \cdots - u_p$. Thus, G is p-factor-critical. □

Actually Theorem 2.1 gives a method to construct p-factor-critical graphs for $p \geq 2$. That is to say, a p-factor-critical graph G can be constructed from a $(p-1)$-factor-critical graph H and a new vertex w, and connecting all vertices of a transversal of $\mathcal{D}_p(H)$ to w. A factor-critical graph H has an ear construction and $\mathcal{D}_p(H)$ can be determined in polynomial times[10]. Some approaches for computing all transversal of $\mathcal{D}_p(H)$ have been described in different ways for $p = 2$: see [2], [11] and [9]. This method can be applied on minimal p-factor-critical graphs and we can get the following corollary:

Corollary 2.2. *Let $p \geq 2$ be a positive integer. Let H be a minimal $(p-1)$-factor-critical graph and $S \subseteq V(H)$ a minimal transversal of $\mathcal{D}_p(H)$. Adding a new vertex w and connecting each vertex of S to w by edges, the resulting graph G is a minimal p-factor-critical graph.*

Proof. The proof is by contradiction. Suppose that G is not minimal p-factor-critical. Then there exists an edge $e \in E(G)$ such that $G - e$ is still p-factor-critical. If $e = wu \in E(w, S)$, then $S - u$ is not a transversal of $\mathcal{D}_p(H)$ because of the minimal property of S, where $E(w, S)$ denotes the set of edges connecting $\{w\}$ to S in G. By Theorem 2.1, $G - e$ is not p-factor-critical which is a contradiction. If $e \in E(H)$, then $H - e$ is not $(p-1)$-factor-critical because of the minimal property of H. By Theorem 2.1, $G - e$ is not p-factor-critical and thus a contradiction is obtained again. Thus, G is p-factor-critical. □

Now it is worthy to give a class of minimal p-factor-critical graphs. The join $G \vee H$ of disjoint graphs G and H is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H. By Theorem 2.1 and Corollary 2.2, we obtain the following result.

Theorem 2.3. *Let C_k be a cycle of length k and N_p an independent vertex set with order p, where $k \geq 3$ is odd and $k \geq 2p+1$. Then $C_k \vee N_p$ is minimal $(p+1)$-factor-critical.*

Proof. For convenience, denote $G_p = C_k \vee N_p$. Firstly, we shall prove that G_p is $(p+1)$-factor-critical by induction on p. When $p = 0$, then $G_0 = C_k$. Clearly, C_k is an odd cycle and also factor-critical. When $p = 1$, then G_1 is a wheel. It is easy to see that G_1 is bicritical. Suppose that G_l is $(l+1)$-factor-critical for $l < p$. Now, we will prove G_p is $(p+1)$-factor-critical.

By induction, $G_{p-1} = C_k \vee N_{p-1}$ is p-factor-critical. Let $C_k = v_1v_2\cdots v_{i-1}v_iv_{i+1}\cdots v_kv_1$ and $N_{p-1} = \{u_1, \cdots, u_{p-1}\}$.

Claim. *The set* $\{v_1, \cdots, v_k\}$ *is a transversal of* $\mathcal{D}_{p+1}(G_{p-1})$*, where* $\mathcal{D}_{p+1}(G_{p-1}) = \big\{D(G_{p-1} - w_1 - \cdots - w_{p+1}) : w_1, \cdots, w_{p+1} \in V(G_{p-1})\big\}$.

Case 1. $\{u_1, \cdots, u_{p-1}\} \subset \{w_1, \cdots, w_{p+1}\}$.

In this case, $D(G_{p-1} - w_1 - \cdots - w_{p+1}) \subseteq \{v_1, \cdots, v_k\}$. Then $\{v_1, \cdots, v_k\} \cap D(G_{p-1} - w_1 - \cdots - w_{p+1}) \neq \emptyset$. Thus, set $\{v_1, \cdots, v_k\}$ is a transversal of $\mathcal{D}_{p+1}(G_{p-1})$.

Case 2. $\{u_1, \cdots, u_{p-1}\} \cap \{w_1, \cdots, w_{p+1}\} = \emptyset$.

In this case, $\{w_1, \cdots, w_{p+1}\} \subset \{v_1, \cdots, v_k\}$. If $G_{p-1} - w_1 \cdots - w_{p+1}$ is bipartite, then any maximum perfect matching of $G_{p-1} - w_1 \cdots - w_{p+1}$ covers all vertices of $\{u_1, \cdots, u_{p-1}\}$. Then $\{u_1, \cdots, u_t\} \cap D(G_{p-1} - w_1 - \cdots - w_k) = \emptyset$, that is, $\{v_1, \cdots, v_k\} \cap D(G_{p-1} - w_1 - \cdots - w_{p+1}) \neq \emptyset$. Thus, set $\{v_1, \cdots, v_k\}$ is a transversal of $\mathcal{D}_{p+1}(G_{p-1})$.

Case 3. $1 \leq \big|\{u_1, \cdots, u_{p-1}\} \cap \{w_1, \cdots, w_{p+1}\}\big| \leq p - 2$.

Without loss of generality, suppose that $u_1, \cdots, u_t \notin \{w_1, \cdots, w_{p+1}\}$ and $u_{t+1}, \cdots, u_{p-1} \in \{w_1, \cdots, w_{p+1}\}$, where $1 \leq t \leq p - 2$. If every maximum matching of $G_{p-1} - w_1 - \cdots - w_{p+1}$ covers all the vertices in $\{u_1, \cdots, u_t\}$, then $\{u_1, \cdots, u_t\} \cap D(G_{p-1} - w_1 - \cdots - w_k) = \emptyset$. Thus, $\{v_1, \cdots, v_k\} \cap D(G_{p-1} - w_1 - \cdots - w_{p+1}) \neq \emptyset$.

Otherwise, let $u \in \{u_1, \cdots, u_t\}$ be the vertex missed by some maximum matching, say M, of $G_{p-1} - w_1 - \cdots - w_{p+1}$. Since

$$\big|\{v_1, \cdots, v_k\}\big| - \big|\{w_1, \cdots, w_{p+1}\} \cap \{v_1, \cdots, v_k\}\big| \geq k - p \geq p + 1, \tag{2.2}$$

by the structure of $G_{p-1} - w_1 - \cdots - w_{p+1}$, there exists a triangle $uv_iv_{i+1}u$ in $G_{p-1} - w_1 - \cdots - w_{p+1}$ such that $v_iv_{i+1} \in M$. Let $M' = M \setminus \{v_iv_{i+1}\} \cup \{uv_{i+1}\}$. Then M' is also a maximum matching of $G_{p-1} - w_1 - \cdots - w_{p+1}$ and does not cover v_i. At this time, $\{u, v_i\} \subset D(G_{p-1} - w_1 - \cdots - w_{p+1})$. Then $\{v_1, \cdots, v_k\} \cap D(G_{p-1} - w_1 - \cdots - w_{p+1}) \neq \emptyset$. Thus, set $\{v_1, \cdots, v_k\}$ is a transversal of $\mathcal{D}_{p+1}(G_{p-1})$.

By the above cases, set $\{v_1, \cdots, v_k\}$ is a transversal of $\mathcal{D}_{p+1}(G_{p-1})$, where $\mathcal{D}_{p+1}(G_{p-1})$= $\big\{D(G_{p-1} - w_1 - \cdots - w_{p+1}) : w_1, \cdots, w_{p+1} \in V(G_{p-1})\big\}$. Then, by Theorem 2.1, G_p is $(p+1)$-factor-critical.

Now, we prove that G_p is minimal $(p+1)$-factor-critical by induction on p. Firstly, it is easy to verify that G_0 and G_1 are minimal factor-critical and minimal bicritical, respectively. By the above induction, G_{p-1} is minimal p-factor-critical. On the other hand, $\{v_1, \cdots, v_k\}$ is minimal transversal of $\mathcal{D}_{p+1}(G_{p-1})$, where $\mathcal{D}_{p+1}(G_{p-1})$=$\big\{D(G_{p-1} - w_1 - \cdots - w_{p+1}) : w_1, \cdots, w_{p+1} \in V(G_{p-1})\big\}$. In fact, if $\{w_1, \cdots, w_{p+1}\} = \{u_1, \cdots, u_p, v_{i-1}, v_{i+1}\}$ for each $i \in \{1, \cdots, k\}$, then $\{v_i\} = D(G_{p-1} - w_1 - \cdots - w_{p+1})$. Thus, $\{v_1, \cdots, v_k\}$ is minimal transversal of $\mathcal{D}_{p+1}(G_{p-1})$. By Corolary 2.2, G_p is minimal $(p+1)$-factor-critical. □

Corollary 2.2 gives a method to construct a minimal p-factor-critical graph from a minimal $(p-1)$-factor-critical graph and a minimal transversal. But any vertex deletion of minimal p-factor-critical graph may not result in a minimal $(p-1)$-factor-critical graph. By Theorem 2.3, for any $u \in V(N_p)$ we get $C_k \vee N_p - u$ is still minimal $(p-1)$-factor-critical. However, for any $v \in V(C_k)$ we can easily prove that $C_k \vee N_p - v$ is not minimal $(p-1)$-factor-critical. Now, we will give a characterization of minimal p-factor-critical graphs.

Theorem 2.4. *Let* $p \geq 2$ *be a positive integer. A graph* G *is minimal p-factor-critical if and only if, for any* $w \in V(G)$, $H = G - w$ *is* $(p-1)$*-factor-critical and* $S = \Gamma(w)$ *satisfies that*
(i) S *is a minimal transversal of* $\mathcal{D}_p(H)$*, and*
(ii) For each deletable edge e *of* H, S *is not a transversal of* $\mathcal{D}_p(H - e)$.

Proof. Suppose G is minimal p-factor-critical. Clearly, for any vertex $w \in V(G)$, $H = G - w$ is $(p-1)$-factor-critical. In fact, both (i) and (ii) are right. Otherwise, if (i) does not hold, then it exists a set $S' \subset S$ such that S' is also a transversal of $\mathcal{D}_p(H)$. Let $v_1, \cdots, v_k \in S - S'$. By Theorem 2.1, $G - wv_1 - \cdots - wv_k$ is p-factor-critical which contracts that G is minimal. Similar to prove that (ii) holds.

Conversely, suppose that H is a $(p-1)$-factor-critical graph and $S \subseteq V(H)$ satisfies (i) and (ii). Let G be the graph from H by adding a new vertex w and all the edges connecting w to each vertex of S. Then G is p-factor-critical by Theorem 2.1. Now, it is only needed to prove that, for any $e \in E(G)$, $G - e$ is not p-factor-critical. In fact, if $e = wu \in E(w, S)$, then $S - u$ is not transversal of $\mathcal{D}_p(H)$ because of the minimal property of S. By Theorem 2.1, $G - e$ is not p-factor-critical. If $e \in E(H)$ and e is not deletable, then $H - e$ is not $(p-1)$-factor-critical. By Theorem 2.1, $G - e$ is not p-factor-critical either. Finally, if $e \in E(H)$ and e is deletable, then, by condition (ii), S is not transversal of $\mathcal{D}_p(H - e)$. By Theorem 2.1, $G - e$ is not p-factor-critical. □

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