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On $P_{\geq 3}$-factor Deleted Graphs

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Abstract A spanning subgraph F of a graph G is called a path factor of G if each component of F is a path. A $P_{\geq k}$-factor means a path factor with each component having at least k vertices, where $k \geq 2$ is an integer. Bazgan, Benhamdine, Li and Wozniak [C. Bazgan, A. H. Benhamdine, H. Li, M. Wozniak, Partitioning vertices of 1-tough graph into paths, Theoret. Comput. Sci. 263(2001)255–261.] obtained a toughness condition for a graph to have a $P_{\geq 3}$-factor. We introduce the concept of a $P_{\geq k}$-factor deleted graph, that is, if a graph G has a $P_{\geq k}$-factor excluding e for every $e \in E(G)$, then we say that G is a $P_{\geq k}$-factor deleted graph. In this paper, we show four sufficient conditions for a graph to be a $P_{\geq 3}$-factor deleted graph. Furthermore, it is shown that four results are best possible in some sense.

Keywords toughness; isolated toughness; connectivity; $P_{\geq 3}$-factor; $P_{\geq 3}$-factor deleted graph.
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1 Introduction

In this paper, all graphs considered are finite, undirected and simple. We refer the readers to [2] for the terminology and notation not given here. For a graph G, its vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. For a vertex subset S of G, $G[S]$ denotes the subgraph of G induced by S, and write $G[V(G) \setminus S]$ for $G - S$. When $S = \{x\}$, we write $G - x = G - \{x\}$. For an edge subset E' of G, $G - E'$ denotes the subgraph obtained from G by deleting E'. When $E' = \{e\}$, let $G - e = G - \{e\}$. For any $x \in V(G)$, the degree of x in G is denoted by $d_G(x)$, and $\delta(G) = \min\{d_G(x) : x \in V(G)\}$. We use $\kappa(G)$ and $\lambda(G)$ to denote the connectivity and the edge-connectivity of G, respectively.

A spanning subgraph F of G is called a path factor of G if each component of F is a path. A $P_{\geq k}$-factor means a path factor with each component having at least k vertices, where $k \geq 2$ is an integer. If a graph G has a $P_{\geq k}$-factor excluding e for every $e \in E(G)$, then we say that G is a $P_{\geq k}$-factor deleted graph.

A graph R is factor-critical if $R - x$ contains a perfect matching for any $x \in V(R)$. For a graph H, we say that H is a sun if $H = K_1$, $H = K_2$ or H is the corona of a factor-critical graph R of order n with $n \geq 3$, i.e., H is obtained from R by adding a new vertex $w = w(v)$ together with a new edge vw for every $v \in V(R)$. A big sun is a sun with at least six vertices. A component of G is called a sun component if it is isomorphic to a sun. The number of sun components of G is denoted by $\mathrm{sun}\,(G)$.

Kaneko[8] presented a necessary and sufficient condition for a graph to have a $P_{\geq 3}$-factor. Kano, Katona and Király[9] showed a simpler proof.

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Theorem 1.1[8, 9]**.** *A graph G admits a $P_{\geq 3}$-factor if and only if for every vertex subset S of G,*

$$\text{sun}\,(G-S) \leq 2|S|.$$

For a graph G, its toughness $t(G)$ was first introduced by Chvátal[4]: if G is not complete,

$$t(G) = \min\Big\{\frac{|S|}{\omega(G-S)} : S \subseteq V(G),\ \omega(G-S) \geq 2\Big\},$$

where $\omega(G-S)$ denotes the number of connected components of $G-S$; otherwise, $t(G) = +\infty$. A variation of toughness, introduced by Enomoto[5], was defined as

$$\tau(G) = \min\Big\{\frac{|S|}{\omega(G-S)-1} : S \subseteq V(G),\ \omega(G-S) \geq 2\Big\}$$

if G is not complete; otherwise, $\tau(G) = +\infty$.

Yang, Ma and Liu[17] introduced the isolated toughness $I(G)$, which was defined as

$$I(G) = \min\Big\{\frac{|S|}{i(G-S)} : S \subseteq V(G),\ i(G-S) \geq 2\Big\}$$

if G is not complete, where $i(G-S)$ denotes the number of isolated vertices of $G-S$; otherwise, $I(G) = +\infty$. A variation of isolated toughness, introduced by Ma and Liu[10], was defined as

$$I'(G) = \min\Big\{\frac{|S|}{i(G-S)-1} : S \subseteq V(G),\ i(G-S) \geq 2\Big\}$$

if G is not complete; otherwise, $I'(G) = +\infty$.

Some results on toughness, isolated toughness and graph factors see [6, 7, 12]. Other some results on graph factors see [3, 11, 14–16, 18–27]. Bazgan, Benhamdine, Li and Wozniakup[1] presented a toughness condition for a graph to have a $P_{\geq 3}$-factor.

Theorem 1.2[1]**.** *Let G be a graph with at least three vertices. If $t(G) \geq 1$, then G admits a $P_{\geq 3}$-factor.*

In this paper, we first investigate the relationship between toughness and a $P_{\geq 3}$-factor deleted graph, and obtain a toughness condition for a graph to be a $P_{\geq 3}$-factor deleted graph which is an extension of Theorem 1.2.

Theorem 1.3. *Let G be a 2-edge connected graph. If $t(G) > \frac{1}{2}$, then G is a $P_{\geq 3}$-factor deleted graph.*

Furthermore, we present three new sufficient conditions for the existence of $P_{\geq 3}$-factor deleted graphs by using variation of toughness, isolated toughness and variation of isolated toughness which are shown in the following.

Theorem 1.4. *Let G be a graph with at least three vertices. If $\tau(G) > 1$, then G is a $P_{\geq 3}$-factor deleted graph.*

Theorem 1.5. *Let G be a 2-edge connected graph. If $I(G) > \frac{3}{2}$, then G is a $P_{\geq 3}$-factor deleted graph.*

Theorem 1.6. *Let G be a 2-edge connected graph. If $I'(G) > 3$, then G is a $P_{\geq 3}$-factor deleted graph.*

We now show some lemmas which are useful in the proofs of our main results.

Lemma 1.7[5]**.** *Let G be a non-complete graph. Then $\tau(G) \leq \kappa(G)$. Especially, G is connected if and only if $\tau(G) > 0$.*

Lemma 1.8[13]**.** *Let G be a graph. Then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

2 The Proofs of Theorems 1.3 and 1.4

The proof of Theorem 1.3 is similar to that of Theorem 1.4. In the following, we only prove Theorem 1.4, and the proof of Theorem 1.3 is omitted.

Proof of Theorem 1.4. If G is a complete graph, then it is obvious that G is a $P_{\geq 3}$-factor deleted graph. In the following, we assume that G is a non-complete graph.

For any $e \in E(G)$, we write $H = G - e$. In order to prove Theorem 1.4, we only need to verify that H admits a $P_{\geq 3}$-factor. By contradiction, we assume that H has no $P_{\geq 3}$-factor. Then it follows from Theorem 1 that there exists some subset $S \subseteq V(H) = V(G)$ satisfying

$$\mathrm{sun}\,(H - S) > 2|S|. \tag{2.1}$$

Claim 1. *$S \neq \emptyset$.*
Proof. Assume that $S = \emptyset$. Then by (2.1), we have

$$\mathrm{sun}\,(H) > 0.$$

In view of the integrity of $\mathrm{sun}\,(H)$, we obtain

$$\mathrm{sun}\,(H) \geq 1. \tag{2.2}$$

On the other hand, since G is a non-complete graph, it follows from Lemmas 1.7 and 1.8 that

$$\lambda(G) \geq \kappa(G) \geq \tau(G) > 1.$$

According to the integrity of $\lambda(G)$ and $\kappa(G)$, we have

$$\lambda(G) \geq \kappa(G) \geq 2,$$

which implies that $H = G - e$ is connected, and so

$$\mathrm{sun}\,(H) \leq \omega(H) = 1. \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\mathrm{sun}\,(H) = \omega(H) = 1. \tag{2.4}$$

Note that $|V(H)| = |V(G)| \geq 3$. Combining this with (2.4) and the definition of a big sun, $H = G - e$ is a big sun. According to the definition of a big sun, there exists a factor-critical graph R in $H = G - e$ such that $d_H(x) = 1$ for any $x \in V(H) \setminus V(R)$ and $|V(R)| = \frac{|V(H)|}{2} \geq 3$. Thus, it is easy to see that there exists a vertex $v \in V(R)$ such that $G - v$ has two connected components. Combining this with the definition of $\tau(G)$, we obtain

$$\tau(G) \leq \frac{|\{v\}|}{\omega(G - v) - 1} = \frac{1}{2 - 1} = 1,$$

which contradicts that $\tau(G) > 1$. This completes the proof of Claim 1. □

We shall consider two cases by the value of $\mathrm{sun}\,(H - S)$.

Case 1. $\mathrm{sun}\,(H - S) \leq 2$.

It follows from Claim 1, (2.1) and $\mathrm{sun}\,(H - S) \leq 2$ that

$$1 \leq |S| < \frac{\mathrm{sun}\,(H - S)}{2} \leq 1,$$

which is a contradiction.

Case 2. $\mathrm{sun}\,(H-S) \geq 3$.

It is obvious that

$$3 \leq \mathrm{sun}\,(H-S) \leq \omega(H-S) = \omega(G-S-e) \leq \omega(G-S)+1, \tag{2.5}$$

and so

$$\omega(G-S) \geq 2. \tag{2.6}$$

In terms of Claim 1, (2.1) and (2.5), we obtain

$$\frac{1}{2} = \frac{|S|}{2|S|} > \frac{|S|}{\mathrm{sun}\,(H-S)} \geq \frac{|S|}{\omega(G-S)+1},$$

that is,

$$\omega(G-S) > 2|S|-1.$$

In view of the integrity of $|S|$ and $\omega(G-S)$, we have

$$\omega(G-S) \geq 2|S|. \tag{2.7}$$

According to (2.6), (2.7), $\tau(G) > 1$, Claim 1 and the definition of $\tau(G)$, we obtain

$$1 < \tau(G) \leq \frac{|S|}{\omega(G-S)-1} \leq \frac{|S|}{2|S|-1} \leq \frac{|S|}{|S|} = 1,$$

which is a contradiction. Theorem 1.4 is proved. □

Remark 2.1. The condition $t(G) > \frac{1}{2}$ in Theorem 1.3 cannot be replaced by $t(G) \geq \frac{1}{2}$, which is shown as follows.

We construct a graph $G = K_1 \vee (K_2 \cup H)$, where H is a sun. Let $S = V(K_1)$. It is obvious that $\omega(G-S) = 2$ and $t(G) = \frac{|S|}{\omega(G-S)} = \frac{1}{2}$. We choose $e \in E(K_2)$ and $H = G-e$. Thus, we have

$$\mathrm{sun}\,(H-S) = 3 > 2 = 2|S|.$$

In terms of Theorem 1.1, H has no $P_{\geq 3}$-factor, and so, G is not a $P_{\geq 3}$-factor deleted graph.

Remark 2.2. In Theorem 1.4, the bound in the condition

$$\tau(G) > 1$$

is best possible. Let H_1, H_2, H_3 be three suns. Consider a graph H constructed from H_2 and H_3 as follows: set $V(H) = V(H_2) \cup V(H_3)$ and $E(H) = E(H_2) \cup E(H_3) \cup \{e\}$, where $e = uv$ with $u \in V(H_2)$ and $v \in V(H_3)$. Let $G = K_1 \vee (H_1 \cup H)$. We choose $S = V(K_1)$. Clearly, $\omega(G-S) = 2$ and $\tau(G) = \frac{|S|}{\omega(G-S)-1} = \frac{1}{2-1} = 1$. We write $Q = G-e$. Thus, we obtain

$$\mathrm{sun}\,(Q-S) = 3 > 2 = 2|S|.$$

According to Theorem 1.1, Q has no $P_{\geq 3}$-factor, that is, G is not a $P_{\geq 3}$-factor deleted graph.

3 The Proofs of Theorems 1.5 and 1.6

The proof of Theorem 1.6 is similar to that of Theorem 1.5. Hence, we only prove Theorem 1.5, and the proof of Theorem 1.6 is omitted.

Proof of Theorem 1.5. If G is a complete graph, then it is easy to see that G is a $P_{\geq 3}$-factor deleted graph. In the following, we assume that G is a non-complete graph.

We write $H = G - e$ for every $e = uv \in E(G)$. To verify Theorem 1.5, we only need to prove that H contains a $P_{\geq 3}$-factor. By contradiction, we assume that H has no $P_{\geq 3}$-factor. Then by Theorem 1.1, we have

$$\text{sun}\,(H - S) > 2|S| \tag{3.1}$$

for some $S \subseteq V(H) = V(G)$.

Claim 2. $S \neq \emptyset$.

Proof. Assume that $S = \emptyset$. Then by (3.1), we obtain

$$\text{sun}\,(H) > 0.$$

In terms of the integrity of $\text{sun}\,(H)$, we have

$$\text{sun}\,(H) \geq 1. \tag{3.2}$$

On the other hand, since G is 2-edge connected, we have $H = G - e$ is an edge connected graph. Thus, we have

$$\text{sun}\,(H) \leq \omega(H) = 1.$$

Combining this with (3.2), we obtain

$$\text{sun}\,(H) = \omega(H) = 1.$$

Note that $|V(H)| = |V(G)| > 2$. And so, $H = G - e$ is a big sun. We use R to denote the factor-critical subgraph of $H = G - e$, and $|V(R)| = \frac{|V(H)|}{2} \geq 3$.

If $u, v \in V(R)$ or $u \in V(R), v \in V(H) \setminus V(R)$, then we choose $Y = V(R)$. It is obvious that $i(G - Y) = |V(H) \setminus V(R)| = |V(R)| = |Y| \geq 3$. By the condition of Theorem 1.5 and the definition of $I(G)$, we have

$$\frac{3}{2} < I(G) \leq \frac{|Y|}{i(G - Y)} = \frac{|Y|}{|Y|} = 1,$$

which is a contradiction. If $u, v \in V(H) \setminus V(R)$, then there exists $w \in V(R)$ with $uw \in E(G)$. We choose $Y = (V(R) \setminus \{w\}) \cup \{u\}$. Clearly, $i(G - Y) = |V(H) \setminus Y| = |V(H)| - |Y| = 2|V(R)| - |V(R)| = |V(R)| \geq 3$. Thus, we obtain

$$\frac{3}{2} < I(G) \leq \frac{|Y|}{i(G - Y)} = \frac{|V(R)|}{|V(R)|} = 1,$$

which is a contradiction. This completes the proof of Claim 2. □

Assume that there exist a isolated vertices, b K_2's and c big sun components $H_1, H_2, \cdots, H_c$, where $|V(H_i)| \geq 6$ for $1 \leq i \leq c$, in $H - S$. Clearly, $\text{sun}\,(H - S) = a + b + c$. We choose one vertex from each K_2 component of $H - S$, and denote by X the set of such vertices. We denote by R_i the factor-critical subgraph of H_i and write $Y_i = V(R_i)$, $1 \leq i \leq c$. It is easy to see that

$|X| = b$ and $i(H_i - Y_i) = |Y_i| = \frac{|V(H_i)|}{2}$ for $1 \leq i \leq c$. Set $Y = \bigcup_{i=1}^{c} Y_i$. It follows from (3.1), $|V(H_i)| \geq 6$, Claim 2 and $\text{sun}\,(H - S) = a + b + c$ that

$$i(H - S \cup X \cup Y) = a + b + \sum_{i=1}^{c} \frac{|V(H_i)|}{2} \geq a + b + 3c \geq \text{sun}\,(H - S) \geq 2|S| + 1 \geq 3. \quad (3.3)$$

We shall consider three cases.

Case 1. $u, v \in V(aK_1)$.

In this case, $a \geq 2$. We write $T = S \cup X \cup Y \cup \{u\}$. Then it is easy to see that

$$i(G - T) = (a - 1) + b + \sum_{i=1}^{c} \frac{|V(H_i)|}{2}. \quad (3.4)$$

In view of (3.1), (3.4), $|V(H_i)| \geq 6$, Claim 2 and $\text{sun}\,(H - S) = a + b + c$, we obtain

$$i(G - T) \geq a + b + 3c - 1 \geq a + b + c - 1 = \text{sun}\,(H - S) - 1 \geq 2|S| \geq 2.$$

By (3.4), the condition of Theorem 1.5 and the definition of $I(G)$, we obtain

$$\frac{3}{2} < I(G) \leq \frac{|T|}{i(G - T)} = \frac{|S| + b + \sum\limits_{i=1}^{c} \frac{|V(H_i)|}{2} + 1}{(a - 1) + b + \sum\limits_{i=1}^{c} \frac{|V(H_i)|}{2}},$$

that is,

$$2|S| > 3a + b + \sum_{i=1}^{c} \frac{|V(H_i)|}{2} - 5. \quad (3.5)$$

It follows from (3.1) and (3.5) that

$$2|S| > 3a + b + \sum_{i=1}^{c} \frac{|V(H_i)|}{2} - 5 \geq 3a + b + 3c - 5 \geq a + b + c - 1 = \text{sun}\,(H - S) - 1 \geq 2|S|,$$

which is a contradiction.

Case 2. one vertex in $\{u, v\}$ belongs to $V(aK_1)$.

Without loss of generality, let $u \in V(aK_1)$ and $v \notin V(aK_1)$. In this case, $a \geq 1$.

Claim 3. $I(G) \leq \frac{|S| + b + \sum_{i=1}^{c} \frac{|V(H_i)|}{2}}{a + b + \sum_{i=1}^{c} \frac{|V(H_i)|}{2} - 1}$.

Proof. We consider four subcases.

Subcase 2.1. $v \in V(G) \setminus (S \cup V(aK_1) \cup V(bK_2) \cup V(H_1) \cup \cdots \cup V(H_c))$.

Obviously, $i(G - S \cup X \cup Y) = i(H - S \cup X \cup Y) - 1$. According to (3.3) and the definition of $I(G)$, we obtain

$$I(G) \leq \frac{|S \cup X \cup Y|}{i(G - S \cup X \cup Y)} = \frac{|S| + b + \sum\limits_{i=1}^{c} \frac{|V(H_i)|}{2}}{i(H - S \cup X \cup Y) - 1} = \frac{|S| + b + \sum\limits_{i=1}^{c} \frac{|V(H_i)|}{2}}{a + b + \sum\limits_{i=1}^{c} \frac{|V(H_i)|}{2} - 1}.$$

Subcase 2.2. $v \in S$.

It is easy to see that $i(G-S\cup X\cup Y)=i(H-S\cup X\cup Y)$. In terms of (3.3) and the definition of $I(G)$, we have

$$I(G)\le\frac{|S\cup X\cup Y|}{i(G-S\cup X\cup Y)}=\frac{|S|+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}}{a+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}}<\frac{|S|+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}}{a+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}-1}.$$

Subcase 2.3. $v\in V(bK_2)$.

We choose such X with $v\in X$. Thus, we have $i(G-S\cup X\cup Y)=i(H-S\cup X\cup Y)$. It follows from (3.3) and the definition of $I(G)$ that

$$I(G)\le\frac{|S\cup X\cup Y|}{i(G-S\cup X\cup Y)}=\frac{|S|+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}}{a+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}}<\frac{|S|+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}}{a+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}-1}.$$

Subcase 2.4. $v\in V(H_i),\ 1\le i\le c$.

Note that R_i is the factor-critical subgraph of $H_i,\ \ i=1,2,\cdots,c$. If $v\in V(R_i)$, where $1\le i\le c$, then we obtain $i(G-S\cup X\cup Y)=i(H-S\cup X\cup Y)$. According to (3.3) and the definition of $I(G)$, we obtain

$$I(G)\le\frac{|S\cup X\cup Y|}{i(G-S\cup X\cup Y)}=\frac{|S|+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}}{a+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}}<\frac{|S|+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}}{a+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}-1}.$$

If $v\in V(H_i)\setminus V(R_i)$, where $1\le i\le c$, then there exists a vertex $w\in V(R_i)$ such that $vw\in E(G)$. We choose $Y'=Y_1\cup\cdots Y_{i-1}\cup Y_{i+1}\cup(Y_i\setminus\{w\})\cup\{v\}$. Clearly, $|Y'|=|Y|$ and $i(G-S\cup X\cup Y')=i(H-S\cup X\cup Y')=a+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}$. By the definition of $I(G)$, we have

$$I(G)\le\frac{|S\cup X\cup Y'|}{i(G-S\cup X\cup Y')}=\frac{|S|+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}}{a+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}}<\frac{|S|+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}}{a+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}-1}.$$

Claim 3 is proved. □

It follows from Claim 3 and the condition of Theorem 1.5 that

$$\frac{3}{2}<I(G)\le\frac{|S|+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}}{a+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}-1},$$

which implies

$$2|S|>3a+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}-3. \tag{3.6}$$

Note that $|V(H_i)|\ge 6,\ \ a\ge 1,\ \ c\ge 0$ and $\mathrm{sun}(H-S)=a+b+c$. Combining these with (3.6), we obtain

$$2|S|>3a+b+3c-3\ge a+b+c-1=\mathrm{sun}(H-S)-1.$$

By the integrity of $|S|$ and $\mathrm{sun}(H-S)$, we have

$$2|S| \geq \mathrm{sun}(H-S),$$

which contradicts (3.1).

Case 3. $u, v \notin V(aK_1)$.

Claim 4. $I(G) \leq \frac{|S|+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}}{a+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}}$.

Proof. The proof of Claim 4 is similar to that of Claim 3, and is omitted. □

In terms of Claim 4 and the condition of Theorem 1.5, we have

$$\frac{3}{2} < I(G) \leq \frac{|S|+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}}{a+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}},$$

which implies

$$2|S| > 3a+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2}.$$

Combining this with $\mathrm{sun}(H-S) = a+b+c$, (3.1) and $|V(H_i)| \geq 6$, we obtain

$$2|S| > 3a+b+\sum_{i=1}^{c}\frac{|V(H_i)|}{2} \geq 3a+b+3c \geq a+b+c = \mathrm{sun}(H-S) > 2|S|,$$

which is a contradiction. The proof of Theorem 1.5 is complete. □

Remark 3.1. The condition $I(G) > \frac{3}{2}$ in Theorem 1.5 cannot be replaced by $I(G) \geq \frac{3}{2}$, which is shown as follows.

We construct a graph $G = K_1 \vee (K_2 \cup K_2)$. It is easy to see that $I(G) = \frac{3}{2}$. We choose $e \in E(K_2)$ and $H = G - e$. Thus, we have

$$\mathrm{sun}(H-S) = 3 > 2 = 2|S|$$

for $S = V(K_1)$. According to Theorem 1.1, H has no $P_{\geq 3}$-factor, and so, G is not a $P_{\geq 3}$-factor deleted graph.

Remark 3.2. For a graph $G = K_1 \vee (K_2 \cup K_2)$, it is easy to see that $I'(G) = 3$. From Remark 3.1, $G = K_1 \vee (K_2 \cup K_2)$ is not a $P_{\geq 3}$-factor deleted graph. Hence, the bound of the condition $I'(G) > 3$ in Theorem 1.6 is sharp.

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