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Degree Conditions for k-Hamiltonian $[a, b]$-factors

Jie WU[1], **Si-zhong ZHOU**[2,†]

[1]School of Economic and management, Jiangsu University of Science and Technology, Zhenjiang 212100, China
[2]School of Science, Jiangsu University of Science and Technology, Zhenjiang 212100, China
(E-mail: zsz_cumt@163.com)

Abstract Let a, b, k be nonnegative integers with $2 \le a < b$. A graph G is called a k-Hamiltonian graph if $G - U$ contains a Hamiltonian cycle for any subset $U \subseteq V(G)$ with $|U| = k$. An $[a, b]$-factor F of G is called a Hamiltonian $[a, b]$-factor if F contains a Hamiltonian cycle. If $G - U$ admits a Hamiltonian $[a, b]$-factor for any subset $U \subseteq V(G)$ with $|U| = k$, then we say that G has a k-Hamiltonian $[a, b]$-factor. Suppose that G is a k-Hamiltonian graph of order n with $n \ge \frac{(a+b-4)(2a+b+k-6)}{b-2} + k$ and $\delta(G) \ge a + k$. In this paper, it is proved that G admits a k-Hamiltonian $[a, b]$-factor if $\max\{d_G(x), d_G(y)\} \ge \frac{(a-2)n+(b-2)k}{a+b-4} + 2$ for each pair of nonadjacent vertices x and y in G.

Keywords degree condition; k-Hamiltonian graph; k-Hamiltonian $[a, b]$-factor
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1 Introduction

We consider finite undirected graphs which have neither loops nor multiple edges. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For any $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G, and by $\delta(G)$ the minimum degree of G. For any $X \subseteq V(G)$, we use $G[X]$ to denote the subgraph of G induced by X, and write $G - X$ for $G[V(G) \setminus X]$. For disjoint vertex subsets S and T of G, we write $E_G(S, T) = \{xy \in E(G) : x \in S, y \in T\}$ and set $e_G(S, T) = |E_G(S, T)|$. Let λ be a real number. Recall that $\lceil \lambda \rceil$ is the least integer such that $\lceil \lambda \rceil \ge \lambda$.

Let a and b be two positive integers with $a \le b$. Then a spanning subgraph F of G satisfying $a \le d_F(x) \le b$ for any $x \in V(G)$ is called an *$[a, b]$-factor* of G. An *r-factor* is an $[r, r]$-factor. A graph G is called a *k-Hamiltonian graph* if $G - U$ contains a Hamiltonian cycle for every subset $U \subseteq V(G)$ with $|U| = k$. An $[a, b]$-factor F of G is called a *Hamiltonian $[a, b]$-factor* if F contains a Hamiltonian cycle. If $G - U$ has a Hamiltonian $[a, b]$-factor for any subset $U \subseteq V(G)$ with $|U| = k$, then we say that G admits a *k-Hamiltonian $[a, b]$-factor*. A *k-Hamiltonian r-factor* is a k-Hamiltonian $[r, r]$-factor. In particular, a 0-Hamiltonian graph is said to be a *Hamiltonian graph*; a 0-Hamiltonian $[a, b]$-factor is a *Hamiltonian $[a, b]$-factor*.

Ore[11] obtained a classic sufficient degree condition for a graph to have a Hamiltonian cycle.

Theorem 1.1[11]**.** *Let G be a graph of order n. If G satisfies*

$$d_G(x) + d_G(y) \ge n$$

for each pair of nonadjacent vertices $x, y \in V(G)$, then G has a Hamiltonian cycle.

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†Corresponding author.

Cai, Li and Kano[2] presented a result on the existence of Hamiltonian $[k, k+1]$-factor in a graph.

Theorem 1.2[2]. *Let $k \geq 2$ be an integer and let G be a graph of order $n \geq 3$ with $n \geq 8k-16$ for even n and $n \geq 6k-13$ for odd n. If G satisfies*

$$d_G(x) + d_G(y) \geq n$$

for each pair of nonadjacent vertices x and y in G, then G contains a Hamiltonian $[k, k+1]$-factor.

Matsuda[10] obtained a sufficient condition for a 2-connected graph to have a Hamiltonian $[k, k+1]$-factor.

Theorem 1.3[10]. *Let $k \geq 2$ be an integer and G a 2-connected graph of order $n \geq 3$ with $n \geq 8k-16$ for even n and with $n \geq 6k-13$ for odd n. If G satisfies*

$$\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$$

for each pair of nonadjacent vertices x and y in G, then G has a Hamiltonian $[k, k+1]$-factor.

Matsuda[9] proved the following result on the existence of Hamiltonian $[a, b]$-factor in a graph, which is an extension of Theorem 1.3.

Theorem 1.4[9]. *Let $2 \leq a < b$ be integers and let G a Hamiltonian graph of order n with $n \geq \frac{(a+b-4)(2a+b-6)}{b-2}$. If $\delta(G) \geq a$ and*

$$\max\{d_G(x), d_G(y)\} \geq \frac{(a-2)n}{a+b-4} + 2$$

for each pair of nonadjacent vertices x and y in G, then G admits a Hamiltonian $[a, b]$-factor.

For the relationships between degree conditions and graph factors, we refer the reader to [1, 3, 5, 8, 12, 14, 19, 22, 23, 25]. Some other results on factors of graphs see [4, 6, 13, 15–18, 20, 21, 24]. We verify the following theorem, which is a generalization of Theorem 1.4.

Theorem 1.5. *Let a, b, k be nonnegative integers with $2 \leq a < b$, and let G a k-Hamiltonian graph of order n with $n \geq \frac{(a+b-4)(2a+b+k-6)}{b-2} + k$. If $\delta(G) \geq a+k$ and*

$$\max\{d_G(x), d_G(y)\} \geq \frac{(a-2)n + (b-2)k}{a+b-4} + 2$$

for each pair of nonadjacent vertices x and y in G, then G admits a k-Hamiltonian $[a, b]$-factor.

If $k = 0$ in Theorem 1.5, then Theorem 1.4 is obtained immediately. Hence, Theorem 1.4 is a special case of Theorem 1.5. Unfortunately, the author does not know whether the result on Theorem 1.5 is sharp or not.

2 The Proof of Theorem 1.5

The Proof of Theorem 1.5 relies on the following theorem, which is a special case of Lovász's (g, f)-factor theorem[7].

Theorem 2.1[7]. *Let $1 \leq a < b$ be integers and let G be a graph. Then G admits an $[a, b]$-factor if and only if*

$$\delta_G(S, T) = b|S| + d_{G-S}(T) - a|T| \geq 0$$

for any disjoint subsets S and T of $V(G)$.

Proof of Theorem 1.5. According to the condition of Theorem 1.5, G admits a k-Hamiltonian cycle C. It is easy to see that C is a k-Hamiltonian $[2, b]$-factor of G, and so Theorem 1.5 holds for $a = 2$. So we may assume that $a \geq 3$.

We write $H = G - U$, where $U \subseteq V(G)$ with $|U| = k$. According to the assumption of Theorem 1.5 and the definition of k-Hamiltonian graph, H has a Hamiltonian cycle C. Set $R = H - E(C)$. Note that $V(R) = V(H) = V(G) \setminus U$ and $\delta(R) = \delta(H) - 2 \geq \delta(G) - k - 2 \geq a - 2$.

Clearly, G has the desired property if and only if R has an $[a-2, b-2]$-factor. Suppose that R has no $[a-2, b-2]$-factor. Then from Theorem 2.1, there exist disjoint subsets S and T of $V(R)$ such that

$$\delta_R(S, T) = (b-2)|S| + d_{R-S}(T) - (a-2)|T| \leq -1. \tag{2.1}$$

We choose subsets S and T such that $|T|$ is minimum.

Claim 1. $|T| \geq b - 1$.

Proof. Suppose $|T| \leq b-2$. Then by $|S| + d_{R-S}(x) \geq d_R(x) \geq \delta(R) \geq a-2$ for $x \in V(G) \setminus S$, we have

$$\begin{aligned}\delta_R(S, T) =& (b-2)|S| + d_{R-S}(T) - (a-2)|T| \\ \geq& |T||S| + d_{R-S}(T) - (a-2)|T| \\ =& \sum_{x \in T} (|S| + d_{R-S}(x) - (a-2)) \geq 0,\end{aligned}$$

which contradicts (2.1). □

Claim 2. $d_{R-S}(x) \leq a-3$ for each $x \in T$.

Proof. Assume that $d_{R-S}(x) \geq a-2$ for some $x \in T$. Then the subsets S and $T \setminus \{x\}$ satisfy (2.1), which contradicts the choice of S and T. □

Claim 3. $S \neq \emptyset$.

Proof. Note that $|S| + d_{R-S}(x) \geq a-2$ for $x \in V(G) \setminus S$. If $S = \emptyset$, then we obtain $d_R(x) \geq a-2$ for $x \in V(G) \setminus S$, and so $d_R(T) \geq (a-2)|T|$. Combining this with (2.1), we have

$$-1 \geq \delta_R(S, T) = (b-2)|S| + d_{R-S}(T) - (a-2)|T| = d_R(T) - (a-2)|T| \geq 0,$$

which is a contradiction. □

Write $X = \{x \in V(G) : d_G(x) \geq \lceil \frac{(a-2)n+(b-2)k}{a+b-4} \rceil + 2\}$ and $Y = V(G) \setminus X$, and set $T_X = T \cap X$ and $T_Y = T \cap Y$.

Claim 4. $G[Y]$ is a complete graph.

Proof. Assume that $G[Y]$ is not a complete graph. Then there exist $x, y \in Y$ satisfying $xy \notin E(G)$. In terms of the condition of Theorem 1.5, we obtain

$$\max\{d_G(x), d_G(y)\} \geq \left\lceil \frac{(a-2)n + (b-2)k}{a+b-4} \right\rceil + 2. \tag{2.2}$$

On the other hand, it follows from the definition of Y that

$$\max\{d_G(x), d_G(y)\} \leq \left\lceil \frac{(a-2)n + (b-2)k}{a+b-4} \right\rceil + 1,$$

which contradicts (2.2). □

Claim 5. $|S| \leq \lceil \frac{(a-2)n-(a-2)k}{a+b-4} \rceil - 2$.

Proof. Assume that $|S| \geq \lceil \frac{(a-2)n-(a-2)k}{a+b-4} \rceil - 1$. In the following, we consider two cases.

Case 1. $|S| \geq \lceil \frac{(a-2)n-(a-2)k}{a+b-4} \rceil$.

In terms of $|S| + |T| + k \leq n$, we have

$$\begin{aligned}\delta_R(S,T) =&(b-2)|S| + d_{R-S}(T) - (a-2)|T| \\ \geq&(b-2)|S| + d_{R-S}(T) - (a-2)(n-k-|S|) \\ =&(a+b-4)|S| + d_{R-S}(T) - (a-2)n + (a-2)k \\ \geq&(a+b-4)\cdot \left\lceil \frac{(a-2)n-(a-2)k}{a+b-4} \right\rceil + d_{R-S}(T) - (a-2)n + (a-2)k \\ \geq&(a+b-4)\cdot \frac{(a-2)n-(a-2)k}{a+b-4} + d_{R-S}(T) - (a-2)n + (a-2)k \\ =&d_{R-S}(T) \geq 0,\end{aligned}$$

which contradicts (2.1).

Case 2. $|S| = \lceil \frac{(a-2)n-(a-2)k}{a+b-4} \rceil - 1$.

In this case, we first verify

$$d_{R-S}(T) \geq |T| - 2. \tag{2.3}$$

For each $x \in T_X$, we have

$$\begin{aligned}d_{R-S}(x) \geq&d_R(x) - |S| = d_H(x) - 2 - |S| \geq d_G(x) - k - 2 - |S| \\ \geq&\left\lceil \frac{(a-2)n+(b-2)k}{a+b-4} \right\rceil + 2 - k - 2 - \left(\left\lceil \frac{(a-2)n-(a-2)k}{a+b-4} \right\rceil - 1 \right) \\ \geq&\frac{(a-2)n+(b-2)k}{a+b-4} - k - \left\lceil \frac{(a-2)n-(a-2)k}{a+b-4} \right\rceil + 1 \\ >&\frac{(a-2)n+(b-2)k}{a+b-4} - k - \left(\frac{(a-2)n-(a-2)k}{a+b-4} + 1 \right) + 1 \\ =&0.\end{aligned}$$

According to the integrity of $d_{R-S}(x)$, we obtain

$$d_{R-S}(x) \geq 1$$

for each $x \in T_X$, and so

$$d_{R-S}(T_X) \geq |T_X|. \tag{2.4}$$

If $T_Y = \emptyset$, i.e., $T = T_X$, then (2.3) holds by (2.4). Thus, we may assume that $T_Y \neq \emptyset$. Then we have $|E(G[T_Y])| \geq \frac{|T_Y|(|T_Y|-1)}{2}$ by Claim 4. Since C is a Hamiltonian cycle of H, $|E(G[T_Y]) \cap E(C)| \leq |T_Y| - 1$ holds. Thus, we obtain

$$\begin{aligned}d_{R-S}(T_Y) = \sum_{x \in T_Y} d_{R-S}(x) \geq 2|E(G[T_Y]) \setminus E(C)| \\ \geq|T_Y|(|T_Y|-1) - 2(|T_Y|-1) = (|T_Y|-1)(|T_Y|-2) \geq |T_Y| - 2.\end{aligned}$$

Combining this with (2.4), we have

$$d_{R-S}(T) = d_{R-S}(T_X) + d_{R-S}(T_Y) \geq |T_X| + |T_Y| - 2 = |T| - 2.$$

According to this inequality, (2.1), $|S| + |T| + k \leq n$ and $n \geq \frac{(a+b-4)(2a+b+k-6)}{b-2} + k$, we obtain

$$-1 \geq \delta_R(S,T) = (b-2)|S| + d_{R-S}(T) - (a-2)|T|$$

$$
\begin{aligned}
&\geq (b-2)|S| + |T| - 2 - (a-2)|T| \\
&= (b-2)|S| - (a-3)|T| - 2 \\
&\geq (b-2)|S| - (a-3)(n-k-|S|) - 2 \\
&= (a+b-5)|S| - (a-3)n + (a-3)k - 2 \\
&= (a+b-5)\Big(\Big\lceil \frac{(a-2)n-(a-2)k}{a+b-4} \Big\rceil - 1\Big) - (a-3)n + (a-3)k - 2 \\
&\geq (a+b-5)\Big(\frac{(a-2)n-(a-2)k}{a+b-4} - 1\Big) - (a-3)n + (a-3)k - 2 \\
&= \frac{(b-2)n-(b-2)k}{a+b-4} - (a+b-3) \geq (2a+b+k-6) - (a+b-3) \\
&= a+k-3 \geq 0,
\end{aligned}
$$

which is a contradiction. □

Claim 6. $|T_X| \geq 1$.

Proof. Assume that $|T_X| = 0$. Then $T = T_Y$. Combining this with Claim 4, we obtain

$$|E(G[T])| = \frac{|T|(|T|-1)}{2}.$$

Since C is a Hamiltonian cycle of H, $|E(G[T]) \cap C| \leq |T| - 1$ holds. Thus, we have

$$d_{R-S}(T) \geq 2|E(G[T]) \setminus E(C)| \geq |T|(|T|-1) - 2(|T|-1) = (|T|-1)(|T|-2). \tag{2.5}$$

Using (2.1), (2.5), Claims 1 and 3, we obtain

$$
\begin{aligned}
-1 \geq &\delta_R(S,T) = (b-2)|S| + d_{R-S}(T) - (a-2)|T| \\
\geq &(b-2)|S| + (|T|-1)(|T|-2) - (a-2)|T| = (b-2)|S| + |T|^2 - (a+1)|T| + 2 \\
\geq &(b-2) + |T|^2 - (a+1)|T| + 2 > |T|^2 - (a+1)|T| + a = (|T|-1)(|T|-a) \geq 0,
\end{aligned}
$$

which is a contradiction. □

Claim 7. $|T_Y| \geq 1$.

Proof. Assume that $|T_Y| = 0$, i.e., $T = T_X$. According to Claim 2 and the definition of T_X, we have

$$
\begin{aligned}
&\frac{(a-2)n+(b-2)k}{a+b-4} + 2 \leq \Big\lceil \frac{(a-2)n+(b-2)k}{a+b-4} \Big\rceil + 2 \leq d_G(x) \\
\leq &d_H(x) + k = d_R(x) + 2 + k \leq d_{R-S}(x) + |S| + 2 + k \leq |S| + k + a - 1
\end{aligned}
$$

for any $x \in T$, which implies

$$d_{R-S}(x) \geq \frac{(a-2)n+(b-2)k}{a+b-4} - |S| - k \tag{2.6}$$

for any $x \in T$, and

$$\frac{(a-2)n+(b-2)k}{a+b-4} - |S| - k - a + 2 \leq -1. \tag{2.7}$$

It follows from (2.6), (2.7) and $|S| + |T| + k \leq n$ that

$$\delta_R(S,T) = (b-2)|S| + d_{R-S}(T) - (a-2)|T|$$

$$
\begin{aligned}
&\geq (b-2)|S| + \Big(\frac{(a-2)n+(b-2)k}{a+b-4} - |S| - k\Big)|T| - (a-2)|T| \\
&= (b-2)|S| + \Big(\frac{(a-2)n+(b-2)k}{a+b-4} - |S| - k - a + 2\Big)|T| \\
&\geq (b-2)|S| + \Big(\frac{(a-2)n+(b-2)k}{a+b-4} - |S| - k - a + 2\Big)(n - |S| - k) \\
&= (b-2)|S| + \Big(\frac{(a-2)n-(a-2)k}{a+b-4} - |S| - a + 2\Big)(n - |S| - k).
\end{aligned}
$$

Let $f(|S|) = (b-2)|S| + (\frac{(a-2)n-(a-2)k}{a+b-4} - |S| - a + 2)(n - |S| - k)$. Using $n \geq \frac{(a+b-4)(2a+b+k-6)}{b-2} + k$ and Claim 5, we obtain

$$
\begin{aligned}
f'(|S|) &= a+b-4+k-n - \frac{(a-2)n-(a-2)k}{a+b-4} + 2|S| \\
&\leq a+b-4+k-n - \frac{(a-2)n-(a-2)k}{a+b-4} + 2\Big(\Big\lceil \frac{(a-2)n-(a-2)k}{a+b-4} \Big\rceil - 2\Big) \\
&< a+b-4+k-n - \frac{(a-2)n-(a-2)k}{a+b-4} + 2\Big(\frac{(a-2)n-(a-2)k}{a+b-4} - 1\Big) \\
&= \frac{-(b-2)n+(b-2)k}{a+b-4} + a + b - 6 \\
&\leq -(2a+b+k-6) + a + b - 6 = -(a+k) < 0,
\end{aligned}
$$

and so

$$
\begin{aligned}
f(|S|) &\geq f\Big(\Big\lceil \frac{(a-2)n-(a-2)k}{a+b-4} \Big\rceil - 2\Big) > f\Big(\frac{(a-2)n-(a-2)k}{a+b-4} - 1\Big) \\
&= (b-2)\Big(\frac{(a-2)n-(a-2)k}{a+b-4} - 1\Big) + (3-a)\Big(\frac{(b-2)n-(b-2)k}{a+b-4} + 1\Big) \\
&= \frac{(b-2)n-(b-2)k}{a+b-4} - a - b + 5 \\
&\geq (2a+b+k-6) - a - b + 5 = a + k - 1 > 0,
\end{aligned}
$$

which contradicts (2.1). □

Claim 8. $|T_Y| \leq a + k$.

Proof. Suppose that $|T_Y| \geq a + k + 1$. Note that $T_Y \neq \emptyset$ by Claim 7 and $G[T_Y]$ is a complete graph by Claim 4. Thus, we obtain

$$
d_{R-S}(x) \geq d_{H-S}(x) - 2 \geq d_{G-S}(x) - k - 2 \geq (|T_Y| - 1) - k - 2 \geq a - 2
$$

for each $x \in T_Y \subseteq T$, which contradicts Claim 2. □

Note that $d_G(x) \geq \lceil \frac{(a-2)n+(b-2)k}{a+b-4} \rceil + 2$ for any $x \in T_X$. Hence, we have

$$
\begin{aligned}
d_{R-S}(x) &\geq d_{H-S}(x) - 2 \geq d_{G-S}(x) - k - 2 \geq d_G(x) - |S| - k - 2 \\
&\geq \Big\lceil \frac{(a-2)n+(b-2)k}{a+b-4} \Big\rceil - |S| - k \qquad (2.8)
\end{aligned}
$$

for any $x \in T_X$. From (2.8) and Claim 2, we obtain

$$
\Big\lceil \frac{(a-2)n+(b-2)k}{a+b-4} \Big\rceil - |S| - k - a + 2 \leq -1. \qquad (2.9)
$$

In terms of (2.9) and Claim 5, we have $a \geq 5$.

It follows from (2.1), (2.8), (2.9), Claims 5 and 8, $|T_X| \leq n-k-|S|-|T_Y|$ and $n \geq \frac{(a+b-4)(2a+b+k-6)}{b-2}+k$ that

$$\begin{aligned}
-1 \geq& \delta_R(S,T) = (b-2)|S| + d_{R-S}(T) - (a-2)|T| \\
=&(b-2)|S| + d_{R-S}(T_X) - (a-2)|T_X| + d_{R-S}(T_Y) - (a-2)|T_Y| \\
\geq&(b-2)|S| + \Big(\Big\lceil \frac{(a-2)n+(b-2)k}{a+b-4} \Big\rceil - |S| - k - a + 2\Big)|T_X| - (a-2)|T_Y| \\
\geq&(b-2)|S| + \Big(\Big\lceil \frac{(a-2)n+(b-2)k}{a+b-4} \Big\rceil - |S| - k - a + 2\Big)(n-k-|S|-|T_Y|) \\
&-(a-2)|T_Y| \\
\geq&(b-2)|S| + \Big(\Big\lceil \frac{(a-2)n+(b-2)k}{a+b-4} \Big\rceil - |S| - k - a + 2\Big)\Big(\Big\lceil \frac{(a-2)n-(a-2)k}{a+b-4} \Big\rceil \\
&+ \frac{(b-2)n-(b-2)k}{a+b-4} - |S| - |T_Y|\Big) - (a-2)|T_Y| \\
\geq&(b-2)|S| + \Big(\Big\lceil \frac{(a-2)n-(a-2)k}{a+b-4} \Big\rceil - |S| - a + 2\Big)\Big(\Big\lceil \frac{(a-2)n-(a-2)k}{a+b-4} \Big\rceil \\
&+ \frac{(b-2)n-(b-2)k}{a+b-4} - |S| - |T_Y|\Big) - (a-2)|T_Y| \\
=&(b-2)|S| + \Big(\Big\lceil \frac{(a-2)n-(a-2)k}{a+b-4} \Big\rceil - |S| - 2\Big)^2 + \Big(\Big\lceil \frac{(a-2)n-(a-2)k}{a+b-4} \Big\rceil \\
&-|S|-2\Big)\Big(\frac{(b-2)n-(b-2)k}{a+b-4} - |T_Y| - a + 6\Big) \\
&-(a-4)\Big(\frac{(b-2)n-(b-2)k}{a+b-4} - |T_Y| + 2\Big) - (a-2)|T_Y| \\
\geq&(b-2)|S| + \Big(\Big\lceil \frac{(a-2)n-(a-2)k}{a+b-4} \Big\rceil - |S| - 2\Big)\Big(\frac{(b-2)n-(b-2)k}{a+b-4} - |T_Y| \\
&-a+6\Big) - (a-4)\Big(\frac{(b-2)n-(b-2)k}{a+b-4} - |T_Y| + 2\Big) - (a-2)|T_Y| \\
=&\Big(\Big\lceil \frac{(a-2)n-(a-2)k}{a+b-4} \Big\rceil - |S| - 2\Big)\Big(\frac{(b-2)n-(b-2)k}{a+b-4} - |T_Y| - a + 6\Big) \\
&-(b-2)\Big(\frac{(a-2)n-(a-2)k}{a+b-4} - |S| - 2\Big) \\
&+2\Big(\frac{(b-2)n-(b-2)k}{a+b-4} - a - b - |T_Y| + 6\Big) \\
\geq&\Big(\Big\lceil \frac{(a-2)n-(a-2)k}{a+b-4} \Big\rceil - |S| - 2\Big)\Big(\frac{(b-2)n-(b-2)k}{a+b-4} - |T_Y| - a + 6\Big) \\
&-(b-2)\Big(\Big\lceil \frac{(a-2)n-(a-2)k}{a+b-4} \Big\rceil - |S| - 2\Big) \\
&+2\Big(\frac{(b-2)n-(b-2)k}{a+b-4} - a - b - |T_Y| + 6\Big) \\
=&\Big(\Big\lceil \frac{(a-2)n-(a-2)k}{a+b-4} \Big\rceil - |S| - 2\Big)\Big(\frac{(b-2)n-(b-2)k}{a+b-4} - |T_Y| - a - b + 8\Big) \\
&+2\Big(\frac{(b-2)n-(b-2)k}{a+b-4} - a - b - |T_Y| + 6\Big) \\
\geq&\Big(\Big\lceil \frac{(a-2)n-(a-2)k}{a+b-4} \Big\rceil - |S| - 2\Big)\Big(\frac{(b-2)n-(b-2)k}{a+b-4} - 2a - b - k + 8\Big)
\end{aligned}$$

$$+2\Big(\frac{(b-2)n-(b-2)k}{a+b-4}-2a-b-k+6\Big)$$
$$\geq 0,$$

which is a contradiction. This completes the proof of Theorem 1.5. □

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