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Isolated Toughness and k-Hamiltonian $[a, b]$-factors

Zhi-ren SUN[1], **Si-zhong ZHOU**[2,†]

[1]School of Mathematical Sciences, Nanjing Normal University, Nanjing 210046, China
(E-mail: 05119@njnu.edu.cn)
[2]School of Science, Jiangsu University of Science and Technology, Mengxi Road 2, Zhenjiang 212003, China
(E-mail: zsz_cumt@163.com)

Abstract Let a, b and k be nonnegative integers with $a \geq 2$ and $b \geq a(k+1)+2$. A graph G is called a k-Hamiltonian graph if after deleting any k vertices of G the remaining graph of G has a Hamiltonian cycle. A graph G is said to have a k-Hamiltonian $[a, b]$-factor if after deleting any k vertices of G the remaining graph of G admits a Hamiltonian $[a, b]$-factor. Let G is a k-Hamiltonian graph of order n with $n \geq a+k+2$. In this paper, it is proved that G contains a k-Hamiltonian $[a, b]$-factor if $\delta(G) \geq a+k$ and $\delta(G) \geq I(G) \geq a-1+\frac{a(k+1)}{b-2}$.

Keywords isolated toughness; k-Hamiltonian graph; k-Hamiltonian $[a, b]$-factor
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1 Introduction

We begin with notations and definitions. In this paper, we consider only finite undirected graphs which do not contain loops and multiple edges. Let G be a graph. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For any $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G, and by $N_G(x)$ the set of vertices adjacent to x in G. For any $X \subseteq V(G)$, $N_G(X) = \bigcup_{x \in X} N_G(x)$, $G[X]$ denotes the subgraph of a graph G induced by X and $G - X$ denotes the subgraph of a graph G induced by $V(G) \setminus X$. The minimum degree and the maximum degree of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. We use $i(G)$ to denote the number of isolated vertices in a graph G. The isolated toughness $I(G)$ of a graph G was first introduced by Ma and Liu[14],

$$I(G) = \min\Big\{\frac{|X|}{i(G-X)} : X \subseteq V(G),\ i(G-X) \geq 2\Big\},$$

if G is not a complete graph; otherwise, $I(G) = +\infty$.

A subset X of $V(G)$ is said to be an independent set (a covering set) of G if each edge of G is incident with at most (at least) one vertex of X. It is easy to deduce that a subset X of $V(G)$ is an independent set of G if and only if $V(G) \setminus X$ is a covering set of G.

Let $a \leq b$ be two positive integers. A spanning subgraph F of a graph G with $a \leq d_F(x) \leq b$ for each $x \in V(G)$ is called an $[a, b]$-factor. Especially, an $[r, r]$-factor is simply called an r-factor. An $[a, b]$-factor including a Hamiltonian cycle is called a Hamiltonian $[a, b]$-factor. A graph G is a k-Hamiltonian graph if $G - U$ contains a Hamiltonian cycle for any $U \subseteq V(G)$ with $|U| = k$. We say that a graph G includes a k-Hamiltonian $[a, b]$-factor if $G - U$ admits

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†Corresponding author.

a Hamiltonian $[a,b]$-factor for all $U \subseteq V(G)$ with $|U| = k$. It is obvious that a 0-Hamiltonian $[a,b]$-factor is simply called a Hamiltonian $[a,b]$-factor.

Many authors investigated factors and fractional factors [4–6, 8, 9, 11, 12, 17–19, 21–27] of graphs, and Hamiltonian factors [1, 3, 15, 16, 20] in graphs. Some results on the relationship between graph factors and isolated toughness see [2, 13, 14]. In this paper, we show a new result on the relationship between graph factors and isolated toughness, which is the following theorem.

Theorem 1.1. *Let a, b and k be three nonnegative integers with $a \geq 2$ and $b \geq a(k+1)+2$, and let G be a k-Hamiltonian graph of order n with $n \geq a+k+2$. Then G has a k-Hamiltonian $[a,b]$-factor if $\delta(G) \geq a+k$ and $\delta(G) \geq I(G) \geq a-1+\frac{a(k+1)}{b-2}$.*

If $k = 0$ in Theorem 1.1, then we obtain the following corollary.

Corollary 1.2. *Let a and b be two nonnegative integers with $b-1 > a \geq 2$, and let G be a Hamiltonian graph of order n with $n \geq a+2$. Then G has a Hamiltonian $[a,b]$-factor if $\delta(G) \geq I(G) \geq a-1+\frac{a}{b-2}$.*

2 The Proof of Theorem 1.1

We use the following lemmas to prove Theorem 1.1.

Lemma 2.1 ([10])**.** *Let G be a graph, and let a and b be two nonnegative integers with $a < b$. Then G contains an $[a,b]$-factor if and only if for each subset S of $V(G)$,*

$$a|T| - d_{G-S}(T) \leq b|S|,$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq a-1\}$ and $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$.

Lemma 2.2 ([7])**.** *Let H be a graph, and let a be an integer with $a \geq 1$. Let $T_1, T_2, \cdots, T_{a-1}$ be a partition of $V(H)$ satisfying $d_H(x) \leq j$ for $\forall\, x \in T_j$ (where T_j may be empty sets), $j = 1, 2, \cdots, a-1$. Then there exist an independent set I and a covering set C of H satisfying*

$$\sum_{j=1}^{a-1}(a-j)c_j \leq \sum_{j=1}^{a-1}(a-1)(a-j)i_j,$$

where $i_j = |I \cap T_j|$, $c_j = |C \cap T_j|$, $j = 1, 2, \cdots, a-1$.

Lemma 2.3 ([20])**.** *Let a and b be two integers with $2 \leq a < b$, and let G be a graph of order n with $n \geq a+2$. If G is complete, then G includes a Hamiltonian $[a,b]$-factor.*

Proof of Theorem 1.1. For any $U \subseteq V(G)$ with $|U| = k$, $G' = G - U$. Obviously, G' includes a Hamiltonian cycle C. Set $H = G' - E(C)$. It is easy to see that $V(H) = V(G') = V(G) \setminus U$ and $\delta(H) = \delta(G') - 2 \geq \delta(G) - k - 2$.

Assume that G is a complete graph. Then G' also is a complete graph. It follows from Lemma 2.3 that G' has a Hamiltonian $[a,b]$-factor, and so G has a k-Hamiltonian $[a,b]$-factor. In the following, we assume that G is not a complete graph. Clearly, G includes the desired factor if and only if H has an $[a-2, b-2]$-factor. By way of contradiction, suppose that H has no $[a-2, b-2]$-factor. Then from Lemma 2.1, there exists some vertex subset S' of H satisfying

$$(a-2)|T| - d_{H-S'}(T) > (b-2)|S'|, \tag{2.1}$$

where $T = \{x : x \in V(H) \setminus S',\ d_{H-S'}(x) \leq a-3\}$. According to $H = G' - E(C) = G - U - E(C)$, we have

$$d_{H-S'}(x) \geq d_{G'-S'}(x) - 2 = d_{G-U-S'}(x) - 2$$

for each $x \in T$. We write $S = S' \cup U$. Then we obtain

$$d_{G-S}(x) \leq d_{H-S'}(x) + 2 \leq (a-3) + 2 = a - 1, \tag{2.2}$$

for each $x \in T$. It follows from (2.1), (2.2), $|U| = k$ and $S = S' \cup U$ that

$$a|T| - d_{G-S}(T) > (b-2)|S| - (b-2)k. \tag{2.3}$$

Claim 1. $|S| \geq k+1$.
Proof. Note that $S = S' \cup U$ and $|U| = k$. Hence, we have $|S| \geq k$. Assume that $|S| = k$. In terms of $\delta(G) \geq a + k$, we obtain

$$d_{G-S}(x) \geq d_G(x) - |S| \geq \delta(G) - |S| \geq a$$

for any $x \in T$, which contradicts (2.2). Thus, we have $|S| \geq k+1$. Claim 1 is proved. □
Claim 2. $(b-2)|S| - (b-2)k \geq \frac{(b-2)|S|}{k+1}$.
Proof. It follows from Claim 1 that

$$\begin{aligned} &(b-2)(k+1)|S| - (b-2)k(k+1) - (b-2)|S| = (b-2)k|S| - (b-2)k(k+1) \\ =&(b-2)k(|S| - (k+1)) \geq 0, \end{aligned}$$

that is,

$$(b-2)|S| - (b-2)k \geq \frac{(b-2)|S|}{k+1}$$

The proof of Claim 2 is complete. □

According to (2.3) and Claim 2, we obtain

$$a|T| - d_{G-S}(T) > \frac{(b-2)|S|}{k+1}. \tag{2.4}$$

We write $T_j = \{x : x \in T,\ d_{G-S}(x) = j\}$, and $|T_j| = t_j$, $j = 0, 1, \cdots, a-1$. Let $H = G[T_1 \cup T_2 \cup \cdots \cup T_{a-1}]$. Apparently, $d_H(x) \leq j$ for any $x \in T_j$. In terms of Lemma 2.2, there exist an independent set I of H and a covering set C satisfying

$$\sum_{j=1}^{a-1}(a-j)c_j \leq \sum_{j=1}^{a-1}(a-1)(a-j)i_j, \tag{2.5}$$

where $i_j = |I \cap T_j|$, $c_j = |C \cap T_j|$, $j = 1, 2, \cdots, a-1$. We may assume that I is a maximum independent set of H. Then $C = V(H) - I$, and so $t_j = i_j + c_j$. Set $W = G - (S \cup T)$ and $Q = S \cup C \cup (N_G(I) \cap V(W))$. Note that $|C| + |N_G(I) \cap V(W)| \leq \sum_{j=1}^{a-1} j i_j$. Thus, we obtain

$$|Q| \leq |S| + \sum_{j=1}^{a-1} j i_j \tag{2.6}$$

and

$$i(G-Q) \geq t_0 + \sum_{j=1}^{a-1} i_j, \tag{2.7}$$

where $t_0 = |T_0|$. In the following, we consider two cases.
Case 1. $i(G-Q) \geq 2$ or $i(G-Q) = 0$.

In this case, the following inequality obviously holds

$$|Q| \geq I(G)i(G-Q). \tag{2.8}$$

Note that $a|T| - d_{G-S}(T) = at_0 + \sum_{j=1}^{a-1}(a-j)t_j$ and $t_j = i_j + c_j$. It follows from (2.4) that

$$at_0 + \sum_{j=1}^{a-1}(a-j)i_j + \sum_{j=1}^{a-1}(a-j)c_j > \frac{(b-2)|S|}{k+1}. \tag{2.9}$$

According to (2.6), (2.7) and (2.8), we have

$$|S| \geq I(G)\Big(t_0 + \sum_{j=1}^{a-1} i_j\Big) - \sum_{j=1}^{a-1} j i_j.$$

Combining this with (2.9), we obtain

$$at_0 + \sum_{j=1}^{a-1}(a-j)i_j + \sum_{j=1}^{a-1}(a-j)c_j > \frac{b-2}{k+1}\Big(I(G)\Big(t_0 + \sum_{j=1}^{a-1} i_j\Big) - \sum_{j=1}^{a-1} j i_j\Big).$$

In view of $I(G) \geq a - 1 + \frac{a(k+1)}{b-2}$, $a \geq 2$ and $b \geq a(k+1)+2$, we have

$$\begin{aligned}\frac{b-2}{k+1}I(G) \geq& \frac{b-2}{k+1}\Big(a-1+\frac{a(k+1)}{b-2}\Big) = \frac{(a-1)(b-2)}{k+1} + a\\ \geq& \frac{a(a-1)(k+1)}{k+1} + a = a^2 > a.\end{aligned}$$

Thus, we obtain

$$\sum_{j=1}^{a-1}(a-j)i_j + \sum_{j=1}^{a-1}(a-j)c_j > \frac{b-2}{k+1}\Big(I(G)\sum_{j=1}^{a-1} i_j - \sum_{j=1}^{a-1} j i_j\Big).$$

Combining this with (2.5), we have

$$\sum_{j=1}^{a-1}(a-1)(a-j)i_j > \frac{b-2}{k+1}\Big(I(G)\sum_{j=1}^{a-1} i_j - \sum_{j=1}^{a-1} j i_j\Big) - \sum_{j=1}^{a-1}(a-j)i_j,$$

that is,

$$\sum_{j=1}^{a-1}\Big(\frac{(b-2)I(G)}{k+1} - \frac{(b-2)j}{k+1} - a(a-j)\Big)i_j < 0. \tag{2.10}$$

Using (2.10), $b \geq a(k+1)+2$, $0 \leq j \leq a-1$ and $I(G) \geq a - 1 + \frac{a(k+1)}{b-2}$, we obtain

$$\begin{aligned}0 >& \sum_{j=1}^{a-1}\Big(\frac{(b-2)I(G)}{k+1} - \frac{(b-2)j}{k+1} - a(a-j)\Big)i_j\\ =& \sum_{j=1}^{a-1}\Big(\frac{(b-2)I(G)}{k+1} - a^2 + \frac{a(k+1)-b+2}{k+1}j\Big)i_j\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{j=1}^{a-1}\Big(\frac{(b-2)I(G)}{k+1}-a^2+\frac{a(k+1)-b+2}{k+1}(a-1)\Big)i_j\\
&=\sum_{j=1}^{a-1}\Big(\frac{(b-2)I(G)}{k+1}-\frac{(a-1)(b-2)}{k+1}-a\Big)i_j\\
&\geq \sum_{j=1}^{a-1}\Big(\frac{(b-2)(a-1+\frac{a(k+1)}{b-2})}{k+1}-\frac{(a-1)(b-2)}{k+1}-a\Big)i_j\\
&=0,
\end{aligned}
$$

which is a contradiction.

Case 2. $i(G-Q)=1$.

In terms of (2.7), we obtain

$$1=i(G-Q)\geq t_0+\sum_{j=1}^{a-1} i_j.$$

Subcase 2.1. $t_0=i_j=0$ for all $j=1,2,\cdots,a-1$.

In this case, it is obvious that $T=\emptyset$. Combining this with (2.4), Claim 1 and $b\geq a(k+1)+2\geq a+2$, we have

$$0=a|T|-d_{G-S}(T)>\frac{(b-2)|S|}{k+1}\geq b-2\geq a>0,$$

which is a contradiction.

Subcase 2.2. $t_0=1$ and $i_j=0$ for all $j=1,2,\cdots,a-1$.

Clearly, T is an isolated vertex, and so $d_{G-S}(T)=0$. It follows from (2.4), $b\geq a(k+1)+2\geq a+2$ and Claim 1 that

$$a=a|T|-d_{G-S}(T)>\frac{(b-2)|S|}{k+1}\geq b-2\geq a,$$

which is a contradiction.

Subcase 2.3. There exists some $j_0\in\{1,2,\cdots,a-1\}$ satisfying $i_{j_0}=1$, and $t_0=0$.

Obviously, $T_0=\emptyset$ and H is a complete graph. Thus, we may write $I=\{v\}$. Note that $C=V(H)-I$ is a covering set of H. Then we obtain

$$|Q|=|S\cup C\cup(N_G(v)\cap V(W))|\geq|S|+d_{G-S}(v)\geq d_G(v)\geq\delta(G)\geq I(G). \tag{2.11}$$

According to (2.11) and $i(G-Q)=1$, (2.8) holds. Then we may obtain some contradictions by using the same method as Case 1. This completes the proof of Theorem 1.1. □

Finally, we present the following problem.

Problem. Let a,b,k be three nonnegative integers with $a\geq 2$ and $b\geq a(k+1)+2$, and let G a k-Hamiltonian graph of order n with $n\geq a+k+2$ and $\delta(G)\geq a+k$. For any little real $\epsilon>0$, $\delta(G)\geq I(G)\geq a-1+\frac{a(k+1)}{b-2}-\epsilon$. Does G include a k-Hamiltonian $[a,b]$-factor?

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