Acta Mathematicae Applicatae Sinica, English Series
Vol. 36, No. 3 (2020) 581–589
https://doi.org/10.1007/s10255-020-0950-3
http://www.ApplMath.com.cn & www.SpringerLink.com

Acta Mathematicae Applicatae Sinica, English Series

Acyclic Edge Coloring of IC-planar Graphs

Wen-yao SONG[1,*], Yuan-yuan DUAN[1,†], Juan WANG[2], Lian-ying MIAO[2]

[1]School of Mathematics and Statistics, Zaozhuang University, Zaozhuang 277160, China
(E-mail: *songwenyao@cumt.edu.cn, †dyy19860827@163.com)
[2]School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China

Abstract A proper edge coloring of a graph G is acyclic if there is no 2-colored cycle in G. The *acyclic chromatic index* of G is the least number of colors such that G has an acyclic edge coloring and denoted by $\chi'_a(G)$. An IC-plane graph is a topological graph where every edge is crossed at most once and no two crossed edges share a vertex. In this paper, it is proved that $\chi'_a(G) \le \Delta(G)+10$, if G is an IC-planar graph without adjacent triangles and $\chi'_a(G) \le \Delta(G)+8$, if G is a triangle-free IC-planar graph.

Keywords Acyclic chromatic index; acyclic edge coloring; IC-planar graph
2000 MR Subject Classification 05C15

1 Introduction

All graphs considered are finite, simple and undirected. Let G be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ to denote its vertex set, edge set, maximum degree and minimum degree, respectively. For a planar graph G, $F(G)$ denotes its face set, $d(v)$ denotes the degree of a vertex v in G. The length or degree of a face f, denoted by $d(f)$, is the length of the boundary walk of f in G. We call v a k-vertex, or a k^+-vertex, or a k^--vertex if $d(v)=k$, or $d(v)\ge k$, or $d(v)\le k$, respectively and call f a k-face, or a k^+-face, or a k^--face if $d(f)=k$, or $d(f)\ge k$, or $d(f)\le k$, respectively. Any undefined notation follows that of Bondy and Murty[6].

A proper edge k-coloring of a graph G is a mapping $\phi: E(G)\to\{1,2,\cdots,k\}$ such that no pair of adjacent edges are colored with the same color. A proper edge coloring of a graph G is acyclic if there is no 2-colored cycle in G. The acyclic chromatic index of G is the least number of colors such that G has an acyclic edge coloring and denoted by $\chi'_a(G)$. Fiamčik[9] and later Alon et al.[3] proposed the following conjecture:

Conjecture 1. For any graph G, $\chi'_a(G)\le\Delta(G)+2$.

Alon et al.[2] proved that $\chi'_a(G)\le 64\Delta(G)$ for any graph G. Molloy and Reed[18] improved this bound to that $\chi'_a(G)\le 16\Delta(G)$. Něsetřil and Wormald[20] proved that $\chi'_a(G)\le \Delta(G)+1$ for a random $\Delta(G)$-regular graph G. The acyclic edge coloring of some special classes of graphs has been studied widely, including graphs with maximum degree 4 (Basavaraju and Chandran[4]), graphs with large girths (Lin et al.[17]), subcubic graphs (Basavaraju and Chandran[5]; Fiamčik[9]; Skulrattanakulchai[23]), series-parallel graphs (Hou et al.[13]; Wang and Shu[26]), outerplanar graphs (Hou et al.[14]; Muthu et al.[19]), planar graphs (Cohen et al.[7]; Dong and Xu[8]; Fiedorowicz et al.[10]; Guan et al.[11]; Hou et al.[12]; Shu and Wang[21, 22]; Wang et al.[27]; Yu et al.[29]) and 1-planar graphs (Chen et al.[32]; Song and Miao[24]; Zhang et

Manuscript received December 7, 2017. Accepted on December 6, 2018.
This paper is supported by the National Natural Science Foundation of China (No. 11771443), Natural Science Foundation of Shandong Province (No. ZR2019BA016) and by the foundation of innovative Science and technology for youth in universities of Shandong Province (No. 2019KJI001). The work of the second author is also under the financial support from the Zaozhaung University Research Fund Project in 2019.
†Corresponding author.

al.[30]). In this paper, we prove that $\chi'_a(G) \le \Delta(G) + 10$, if G is an IC-planar graph without adjacent triangles and $\chi'_a(G) \le \Delta(G) + 8$, if G is a triangle-free IC-planar graph.

An IC-plane graph is a topological graph where every edge is crossed at most once and no two crossed edges share a vertex, i.e., two distinct crossings are independent that is the end-vertices of the crossed pair of edges are mutually different. If a graph G has a drawing on the plane in which every two crossings are independent, then we call G a plane graph with independent crossings or *IC*-planar graph for short. The definition of IC-planar graph was introduced by Alberson[1] in 2008. Making a conjecture of Alberson[1], Král and Stacho[16] proved that every IC-planar graph is 5-colorable. Obviously, every IC-planar graph also is a 1-planar graphs. A graph is 1-*planar* if it can be drawn on the plane so that each edge is crossed by at most one other edges.

Every IC-planar graph G in this paper has been embedded on a plane such that every edge is crossed by at most one other edge and the number of crossings is as small as possible. In other words, we call G an IC-plane graph. The associated plane graph $G^\times$ of G is obtained by turning all crossings of G into new 4-vertices on a plane. For convenience, a vertex in $G^\times$ is called false if it is not a vertex of G and real otherwise. A false face means a face f in $G^\times$ that is incident with one false vertex; otherwise, f is a normal face. For a vertex $v \in V(G)$, we call $f_i(v)$ the number of i-faces which are incident with v. We use $n_i(v)$ to denote the number of i-vertices which are adjacent to v in G. One can see that every real vertex in $G^\times$ is adjacent to at most one false vertex and incident with at most two false 3-faces in $G^\times$.

In this paper, we prove the following results.

Theorem 1. *Let G be an IC-planar graph without adjacent triangles. Then* $\chi'_a(G) \le \Delta(G)+10$.

Theorem 2. *Let G be a triangle-free IC-planar graph. Then* $\chi'_a(G) \le \Delta(G) + 8$.

2 Notations and Lemmas

Before proving our main results, we introduce some notations on acyclic edge coloring and structural properties on IC-planar graphs.

In this paper, we use C to denote the set of colors under an acyclic edge coloring c. For $e \in E(G)$, the color α of C is said to be *candidate* for e with respect to a partial acyclic edge coloring c of G if none of the adjacent edges of e is colored with α.

An (α, β)-*maximal bichromatic path* with respect to a partial coloring c in G is a maximal path whose edges are colored by the colors α and β alternatingly. An (α, β, u, v)-*maximal bichromatic path* is an (α, β)-maximal bichromatic path which starts out from the vertex u and ends at the vertex v. An (α, β, uv)-critical path for an edge uv is an (α, β, u, v)-maximal bichromatic path which starts at the vertex u with an edge colored α and ends at the vertex v with an edge colored α.

A graph G with $\Delta(G) \le k$ is *k-deletion-minimal* if $\chi'_a(G) > k$ and $\chi'_a(H) \le k$ for every proper subgraph H of G. Under an acyclic edge coloring c of G, we denote $C(v)$ by the set of colors which are assigned to the edges incident to v. We use $c(uv)$ to denote the color of edge uv with respect to the coloring c. Let $C_{uv} = C(v) - c(uv)$.

A multiset is a generalized set where each member can appear multiple in the set. If an element x appears t times in the multiset S, then we say the multiplicity of x in S is t, denoted by $D_S(x)$. We use $\parallel S \parallel = \sum_{x \in S} D_S(x)$ to denote the cardinality of finite multiset. Let S and S' be two multisets. A multiset, denoted by $S \uplus S'$, is said to be the union of S and S' if the multiset $S \uplus S'$ has all the members of S and S' and $D_{S \uplus S'}(x) = D_S(x) + D_{S'}(x)$ for any member $x \in S \uplus S'$.

Lemma 1 [5]. *Given a pair of colors α and β of a proper coloring c of G, there can be at most one (α, β)-maximal bichromatic path containing a particular vertex v, with respect to c.*

Lemma 2 [28]. *If G be a k-deletion-minimal graph, then G is 2-connected.*

Lemma 3 [28]. *Let G be a k-deletion-minimal graph. If v is adjacent to a 2-vertex v_0 and $N_G(v_0) = \{w, v\}$, then v is adjacent to at least $k-d(w)+1$ vertices of degree at least $k-d(v)+2$. Moreover, if $k \geq \Delta(G)+2$ and v is adjacent to precisely $k-\Delta(G)+1$ vertices of degree at least $k-\Delta(G)+2$, then v is adjacent to at most $d(v)+\Delta(G)-k-3$ vertices of degree two and $d(v) \geq k-\Delta(G)+4$.*

Lemma 4 [31]. *Let G be a 1-plane graph and $G^{\times}$ be its associated plane graph. If $d_G(u) = 3$ and v is a false vertex in $G^{\times}$, then either $uv \notin E(G^{\times})$ or uv is not incident with two 3-faces.*

Lemma 5 [25]. *Let G be an IC-plane graph without adjacent triangles and $G^{\times}$ be its associated plane graph. If $d_G(v) = 3$ and v is incident with two 3-faces in $G^{\times}$, then v must be incident with a 5^+-face.*

Lemma 6. *Let G be an IC-plane graph without adjacent triangles and $G^{\times}$ be its associated plane graph. If $d_G(v) \geq 4$, then v is incident with at most $\lceil \frac{d_G(v)}{2} \rceil$ 3-faces in $G^{\times}$.*

Proof. Let v be a d-vertex in G, where $d \geq 4$, and let $v_1, v_2, \cdots, v_d$ be its d neighbors in $G^{\times}$ that occur around v in a clockwise order. By f_i denote the face incident with vv_i and vv_{i+1} in $G^{\times}$, where the addition on subscripts are taken modulo d.

If v is not adjacent to any false vertex in $G^{\times}$, then we claim that v is incident with at most $\lfloor \frac{d(v)}{2} \rfloor$ 3-faces. Otherwise, we may easily find adjacent triangles. If v is adjacent to a false vertex in $G^{\times}$, say v_2, then we consider two cases. If v is incident with two false 3-faces in $G^{\times}$, without loss of generality, we assume that f_1 and f_2 are false 3-faces. Then neither f_3 nor f_d is a 3-face in $G^{\times}$ since vv_1v_3v is a 3-face in G. Thus, v is incident with at most $\lfloor \frac{d(v)-3}{2} \rfloor + 2 = \lceil \frac{d(v)}{2} \rceil$ 3-faces in $G^{\times}$. Otherwise, v is incident with exactly one false 3-faces in $G^{\times}$, without loss of generality, we assume that f_1 is a false 3-faces. Then f_2 must be a 4^+-face. Moreover, f_3 and f_d may be 3-faces in $G^{\times}$. Therefore, v is incident with at most $\lfloor \frac{d(v)-1}{2} \rfloor + 1 = \lceil \frac{d(v)}{2} \rceil$ 3-faces in $G^{\times}$. □

Lemma 7. Let G be an IC-planar graph without adjacent triangles and $\delta(G) \geq 2$, then there is a vertex $v \in V(G)$ with d neighbors $v_1, \cdots, v_d$, where $d(v_1) \leq \cdots \leq d(v_d)$ such that one of the followings holds:

- $(A1)$ $d = 3,\ d(v_1) \leq 9$;
- $(A2)$ $d = 4,\ d(v_1) \leq 6,\ d(v_2) \leq 8$;
- $(A3)$ $d = 5,\ d(v_1) \leq d(v_2) \leq d(v_3) \leq 5$;
- $(A4)$ $d = 2$.

Proof. We apply the discharging method on associated plane graph $G^{\times}$ of G and complete the proof by contradiction. Since $G^{\times}$ is a plane graph, we have

$$\sum_{v \in V(G^{\times})} (d\,(v) - 4) + \sum_{f \in F(G^{\times})} (d\,(f) - 4) = -8.$$

Now we define the initial charge function $\mathrm{ch}\,(x)$ of $x \in V\,(G^{\times}) \cup F\,(G^{\times})$. Let $\mathrm{ch}\,(v) = d(v) - 4$ if $v \in V\,(G^{\times})$ and $\mathrm{ch}\,(f) = d(f) - 4$ if $f \in F\,(G^{\times})$. Note that any discharging procedure retains the total charge of G. If we can define suitable discharging rules to shift the initial charge function $\mathrm{ch}\,(x)$ such that the final charge function $\mathrm{ch}'(x) \geq 0$ for all $x \in V\,(G^{\times}) \cup F\,(G^{\times})$, then $0 \leq \sum_{x \in V(G^{\times}) \cup F(G^{\times})} \mathrm{ch}'(x) = \sum_{x \in V(G^{\times}) \cup F(G^{\times})} \mathrm{ch}\,(x) = -8$, a contradiction.

For $v \in V(G^{\times})$ and $f \in F(G^{\times})$, we define the discharging rules as follows. Note that $d_G(v) = d_{G^{\times}}(v)$ for each real vertex, so we denote $d(v)$ by the degree of each real vertex in the following.

(R1) Let f be a false 3-face in $G^\times$. Then f receives $\frac{1}{2}$ from each incident real vertex.

(R2) Let f be a normal 3-face in $G^\times$. Then f receives $\frac{1}{3}$ from each incident vertex.

(R3) Let f be a 5^+-face in $G^\times$. Then f sends $\frac{d(f)-4}{t(f)}$ to each 3-vertex incident with it, where $t(f)$ denotes the number of 3-vertices incident with f.

(R4) Let v be a 3-vertex. Then v receives $\frac{1}{2}$ from each of its real neighbors in G.

(R5) Let v be a 6^+-vertex. Then v sends $\frac{1}{9}$ to each 5-vertex adjacent to it in G.

(R6) Let v be a d-vertex, where $7 \le d \le 8$. Then v sends $\frac{1}{4}$ to each 4-vertex adjacent to it in G.

(R7) Let v be a 9^+-vertex. Then v sends $\frac{1}{3}$ to each 4-vertex adjacent to it in G.

Now we prove that $\mathrm{ch}'(x) \ge 0$ for each $x \in V(G^\times) \cup F(G^\times)$.

Let f be a face of $G^\times$. Clearly, if $d(f) = 4$, then $\mathrm{ch}'(f) = \mathrm{ch}(f) = d(f) - 4 = 0$ and if $d(f) \ge 5$, then $\mathrm{ch}'(f) \ge \mathrm{ch}(f) - \frac{d(f)-4}{t(f)} \times t(f) = 0$ by (R2). Now we check the final charge of 3-faces in $G^\times$. If f is a false 3-face, then f receives $\frac{1}{2}$ from each real vertex incident with it by (R1). Thus, we have $\mathrm{ch}'(f) = \mathrm{ch}(f) + \frac{1}{2} \times 2 = 0$. If f is a normal 3-face, then f receives $\frac{1}{3}$ from each real vertex incident with it by (R2). Thus, we have $\mathrm{ch}'(f) = \mathrm{ch}(f) + \frac{1}{3} \times 3 = 0$.

We next check the final charge of the vertex $v \in V(G^\times)$. Since G has no $(A4)$, it follows $d(v) \ge 3$.

Suppose $d(v) = 3$. Since G has no $(A1)$, each neighbor of v is a 10^+-vertex. By (R4), v receives $\frac{1}{2} \times 3 = \frac{3}{2}$ from its neighbors. Since G does not contain adjacent triangles, v is incident with at most two 3-faces in $G^\times$. If v is incident with at most one 3-faces, then v sends at most $\frac{1}{2}$ to the false 3-face incident with it in $G^\times$ by (R1). So we have $\mathrm{ch}'(f) = \mathrm{ch}(f) + \frac{3}{2} - \frac{1}{2} = 0$. If v is incident with exactly two 3-faces, then v is incident with a 5^+-face by Lemma 5. By (R3), v receives at least $\lfloor \frac{d(f)-4}{t(f)} \rfloor \ge \frac{1}{3}$ from the 5^+-face incident with it. And v sends at most $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ to the 3-faces incident with it in $G^\times$ by (R1), (R2) and Lemma 2.4. Thus, $\mathrm{ch}'(v) \ge \mathrm{ch}(v) + \frac{3}{2} + \frac{1}{3} - \frac{5}{6} = 0$.

Suppose $d(v) = 4$. If v is false, then $\mathrm{ch}'(v) = \mathrm{ch}(v) = 0$. Otherwise, v is real in G. Since G has no $(A2)$, we have $d(v_1) \ge 7$, or $d(v_1) \le 6$ and $d(v_2) \ge 9$. So v receives totally at least $\min\{\frac{1}{4} \times 4, \frac{1}{3} \times 3\} = 1$ by (R6) and (R7). And by Lemma 6, v is incident with at most two 3-faces. So v sends at most $\frac{1}{2} \times 2 = 1$ to the 3-faces incident with it in $G^\times$ by (R1) and (R2). Hence, $\mathrm{ch}'(v) \ge \mathrm{ch}(v) + 1 - 2 \times \frac{1}{2} = 0$.

Suppose $d(v) = 5$. Since G has no $(A3)$, v is adjacent to at least three 6^+-vertices. Hence v receives at least $\frac{1}{9} \times 3 = \frac{1}{3}$ by (R5). By Lemma 6, v is incident with at most three 3-faces. So v sends at most $\frac{1}{2} \times 2 + \frac{1}{3} = \frac{4}{3}$ to the 3-faces incident with it in $G^\times$ by (R1) and (R2). Hence, we have $\mathrm{ch}'(v) \ge \mathrm{ch}(v) + \frac{1}{3} - \frac{4}{3} = 0$.

Suppose $d(v) = 6$. By Lemma 6, v is incident with at most three 3-faces. By (R1) and (R2), v sends at most $\frac{1}{2} \times 2 + \frac{1}{3} = \frac{4}{3}$ to the 3-faces incident with it in $G^\times$. v also sends at most $\frac{1}{9} \times 6 = \frac{2}{3}$ to the 5-vertices adjacent to it in G by (R5). Hence, $\mathrm{ch}'(v) \ge \mathrm{ch}(v) - \frac{4}{3} - \frac{2}{3} = 0$.

Suppose $d(v) = 7$. By Lemma 6, v is incident with at most four 3-faces.

If $f_3(v) = 4$, then $n_4(v) \le 4$ and $n_{7^+}(v) \ge n_4(v) - 1$ while $n_4(v) \ge 2$ in G since G contains neither adjacent triangles nor $(A2)$. Furthermore, $n_5(v) \le 8 - 2n_4(v)$ in G by the same argument. By (R1) and (R2), v sends at most $\frac{1}{2} \times 2 + \frac{1}{3} \times 2 = \frac{5}{3}$ to the 3-faces incident with it in $G^\times$. Thus, we have $\mathrm{ch}'(v) \ge \mathrm{ch}(v) - \frac{5}{3} - \frac{1}{4} - 6 \times \frac{1}{9} > 0$ by (R5) and (R6) if $n_4(v) = 1$, and $\mathrm{ch}'(v) \ge \mathrm{ch}(v) - \frac{5}{3} - \frac{n_4(v)}{4} - \frac{8-2n_4(v)}{9} = \frac{16-n_4(v)}{36} > 0$ by (R5) and (R6) if $n_4(v) \ge 2$.

If $f_3(v) = 3$, then $n_4(v) \le 5$ since G contains neither adjacent triangles nor $(A2)$. By (R1) and (R2), v sends at most $\frac{1}{2} \times 2 + \frac{1}{3} = \frac{4}{3}$ to the 3-faces incident with it in $G^\times$. Thus, we have $\mathrm{ch}'(v) \ge \mathrm{ch}(v) - \frac{4}{3} - \frac{n_4(v)}{4} - \frac{7-n_4(v)}{9} = \frac{32-5n_4(v)}{36} > 0$ by (R5) and (R6).

If $f_3(v) \le 2$, by (R1) and (R2), v sends at most $\frac{1}{2} \times 2 = 1$ to the 3-faces incident with it in $G^\times$. v also sends at most $\frac{1}{4} \times 7 = \frac{7}{4}$ to the 4-vertices adjacent to it in G by (R6). Hence, $\mathrm{ch}'(v) \ge \mathrm{ch}(v) - \frac{7}{4} - 1 > 0$.

Suppose $d(v) = 8$. By Lemma 6, v is incident with at most four 3-faces. By (R1) and (R2), v sends at most $\frac{1}{2} \times 2 + \frac{1}{3} \times 2 = \frac{5}{3}$ to the 3-faces incident with it in $G^\times$. v also sends at most $\frac{1}{4} \times 8 = 2$ to the 4-vertices adjacent to it in G by (R6). Hence, $\mathrm{ch}'(v) \geq \mathrm{ch}\,(v) - \frac{5}{3} - 2 > 0$.

Suppose $d(v) = 9$. By Lemma 6, v is incident with at most five 3-faces. By (R1) and (R2), v sends at most $\frac{1}{2}\times 2+\frac{1}{3}\times 3 = 2$ to the 3-faces incident with it in $G^\times$. v also sends at most $\frac{1}{3}\times 9 = 3$ to the 4-vertices adjacent to it in G by (R5) and (R7). Hence, $\mathrm{ch}'(v) \geq \mathrm{ch}\,(v) - 3 - 2 = 0$.

Suppose $d(v) \geq 10$. If $f_3(v) = 0$, then v gives at most $\frac{1}{2} \times d(v)$ away by (R4)–(R7). Thus, $\mathrm{ch}'(v) \geq \mathrm{ch}\,(v) - \frac{d(v)}{2} = \frac{d(v)-8}{2} > 0$. If $f_3(v) = 1$, then v sends at most $\frac{1}{2}$ to the 3-faces incident with it in $G^\times$ by (R1) and (R2) and at most $\frac{1}{2} \times d(v)$ to the vertices adjacent to it in G by (R4)–(R7). Thus, $\mathrm{ch}'(v) \geq \mathrm{ch}\,(v) - \frac{1}{2} - \frac{d(v)}{2} = \frac{d(v)-9}{2} > 0$. If $f_3(v) \geq 2$, then we may assume that the number of false 3-faces is t_1 and the number of normal 3-faces is t_2. Obviously, we have $t_1 + t_2 \leq \lceil \frac{d(v)}{2} \rceil$ and $t_1 \leq 2$. Then v sends at most $\frac{t_1}{2} + \frac{t_2}{3}$ to the 3-faces incident with it in $G^\times$ by (R1) and (R2). Moreover, we assume that the number of normal 3-faces each of which is incident with a 3-vertex is t_2'. If $t_2' = 0$, v is adjacent to at least $2t_2$ 4^+-vertices since G has no (A1). Then v sends at most $\frac{d(v)-2t_2}{2} + \frac{2t_2}{3}$ to the vertices adjacent to it in G by (R4)-(R7). Thus, $\mathrm{ch}'(v) \geq \mathrm{ch}\,(v) - \frac{t_1}{2} - \frac{t_2}{3} - \frac{d(v)-2t_2}{2} - \frac{2t_2}{3} = \frac{d(v)-8-t_1}{2} \geq 0$. Otherwise, $t_2' \geq 1$, then v is adjacent to at least $(t_1 + t_2 - 1)$ 4^+-vertices since G has no (A1). Thus, v sends at most $\frac{d(v)+1-t_1-t_2}{2} + \frac{t_1+(t_2-t_2')-1}{3}$ to the vertices adjacent to it in G by (R4)–(R7). Thus, $\mathrm{ch}'(v) \geq \mathrm{ch}\,(v) - \frac{t_1}{2} - \frac{t_2}{3} - \frac{d(v)+1-t_1-t_2}{2} - \frac{t_1+t_2-1}{3} + \frac{t_2'}{3} = \frac{3d(v)-25-\lceil \frac{d(v)}{2} \rceil - t_1 + 2t_2'}{6} \geq \frac{3d(v)-25-\lceil \frac{d(v)}{2} \rceil}{6} \geq 0$.

This completes the proof of Lemma 7. □

Lemma 8. Let G be a triangle-free IC-planar graph and $\delta(G) \geq 2$, then there is a vertex $v \in V(G)$ with d neighbors $v_1, \cdots, v_d$, where $d(v_1) \leq \cdots \leq d(v_d)$ such that one of the followings holds:

(*A*1) $d = 3$, $d(v_1) \leq 8$;
(*A*2) $d = 4$, $d(v_1) \leq d(v_2) \leq 5$;
(*A*3) $d = 2$.

Proof. We apply the discharging method on associated plane graph $G^\times$ of G and complete the proof by contradiction. Since $G^\times$ is a plane graph, we have

$$\sum_{v\in V(G^\times)} (d\,(v) - 6) + \sum_{f\in F(G^\times)} (2d\,(f) - 6) = -12.$$

Now we define the initial charge function $\mathrm{ch}\,(x)$ of $x \in V\,(G^\times) \cup F\,(G^\times)$. Let $\mathrm{ch}\,(v) = d(v) - 6$ if $v \in V\,(G^\times)$ and $\mathrm{ch}\,(f) = 2d(f) - 6$ if $f \in F\,(G^\times)$. Note that any discharging procedure retains the total charge of G. If we can define suitable discharging rules to shift the initial charge function $\mathrm{ch}\,(x)$ such that the final charge function $\mathrm{ch}'(x) \geq 0$ for all $x \in V\,(G^\times) \cup F\,(G^\times)$, then $0 \leq \sum_{x\in V(G^\times)\cup F(G^\times)} \mathrm{ch}'(x) = \sum_{x\in V(G^\times)\cup F(G^\times)} \mathrm{ch}\,(x) = -12$, a contradiction.

For $v \in V(G^\times)$ and $f \in F(G^\times)$, we define the discharging rules as follows. Note that $d_G(v) = d_{G^\times}(v)$ for each real vertex, so we denote $d(v)$ by the degree of each real vertex in the following.

(R1) Let f be a 4^+-face in $G^\times$. Then f sends $\frac{2d(f)-6}{d(f)}$ to each of vertex incident with it in $G^\times$.

(R2) Let v be a false vertex in $G^\times$. Then v receives $\frac{1}{4}$ from each of its neighbors in $G^\times$.

(R3) Every 3-vertex receives $\frac{3}{4}$ from each of its neighbors in G.

(R4) Every 6^+-vertex sends $\frac{1}{4}$ to each 4-vertex adjacent to it in G.

Now we prove that $\mathrm{ch}'(x) \geq 0$ for each $x \in V\,(G^\times) \cup F\,(G^\times)$.

Let f be a face of $G^{\times}$. Clearly, if $d(f)=3$, then $\mathrm{ch}'(f)=\mathrm{ch}\,(f)=2d(f)-6=0$ and if $d(f)\geq 4$, then $\mathrm{ch}'(f)\geq \mathrm{ch}\,(f)-\frac{2d(f)-6}{d(f)}\times d(f)=0$ by (R1).

We next check the final charge of the vertex $v\in V(G^{\times})$. Since G has no (A3), it follows $d(v)\geq 3$.

Suppose $d(v)=3$. Since G has no $(A1)$, each neighbor of v is a 9^{+}-vertex. By (R3), v receives $\frac{3}{4}\times 3=\frac{9}{4}$ from its neighbors in G. And v is incident with at most one 3-faces in $G^{\times}$ since G has no triangles. So v is incident with at least two 4^{+}-faces. By (R1), v receives at least $\frac{1}{2}$ from each of 4^{+}-face incident with it in $G^{\times}$. And v sends at most $\frac{1}{4}$ to the false vertex adjacent to it in $G^{\times}$ by (R2). So $\mathrm{ch}'(v)\geq \mathrm{ch}\,(v)+\frac{9}{4}+2\times\frac{1}{2}-\frac{1}{4}=0$.

Suppose $d(v)=4$. If v is false, then v is incident with at least two 4^{+}-faces in $G^{\times}$ since G has no triangles. Hence, $\mathrm{ch}'(v)\geq \mathrm{ch}\,(v)+2\times\frac{1}{2}+\frac{1}{4}\times 4=0$ by (R1) and (R2).

If v is real in G. Since G has no $(A2)$, then v is adjacent to at least three 6^{+}-vertices. So v receives totally at least $\frac{1}{4}\times 3=\frac{3}{4}$ by (R4). v is also incident with at least three 4^{+}-faces in $G^{\times}$ since G has no triangles. So v receives at least $\frac{1}{2}\times 3=\frac{3}{2}$ by (R1). And v sends at most $\frac{1}{4}$ to the false vertex adjacent to it in $G^{\times}$ by (R2). Hence, $\mathrm{ch}'(v)\geq \mathrm{ch}\,(v)+\frac{3}{4}+\frac{3}{2}-\frac{1}{4}=0$.

Suppose $d(v)=5$. v is incident with at least four 4^{+}-faces in $G^{\times}$ since G has no triangles. Hence, $\mathrm{ch}'(v)\geq \mathrm{ch}\,(v)+4\times\frac{1}{2}-\frac{1}{4}>0$ by (R1)and (R2).

Suppose $6\leq d(v)\leq 8$. v is incident with at least $(d(v)-1)$ 4^{+}-faces in $G^{\times}$ since G has no triangles. By (R1), v receives at least $\frac{1}{2}$ from each of 4^{+}-face incident with it. v sends at most $\frac{1}{4}$ to the false vertex adjacent to it and $\frac{1}{4}$ to each 4-vertex adjacent to it in $G^{\times}$ by (R2) and (R4). Hence, $\mathrm{ch}'(v)\geq \mathrm{ch}\,(v)+(d(v)-1)\times\frac{1}{2}-d(v)\times\frac{1}{4}-\frac{1}{4}\geq\frac{5d(v)-27}{4}>0$.

Suppose $d(v)\geq 9$. v is incident with at least $(d(v)-1)$ 4^{+}-faces since G has no triangles. By (R1), v receives at least $\frac{1}{2}$ from each of 4^{+}-face incident with it. v sends at most $\frac{1}{4}$ to the false vertex adjacent to it by (R2). By (R3) and (R4), v sends $\frac{3}{4}$ to each 3-vertex adjacent to it and $\frac{1}{4}$ to each real 4-vertex adjacent to it in $G^{\times}$. Hence, $\mathrm{ch}'(v)\geq \mathrm{ch}\,(v)+(d(v)-1)\times\frac{1}{2}-n_3(v)\times\frac{3}{4}-n_4(v)\times\frac{1}{4}-\frac{1}{4}\geq\frac{3d(v)-27}{4}\geq 0$ since $d(v)\geq 9$.

This completes the proof of Lemma 8. □

3 Proof of Theorem 1 and Theorem 2

Proof of Theorem 1. Let G be a k-deletion-minimal graph with $k=\Delta(G)+10$. Then G is 2-connected by Lemma 2 . By Lemma 7, there exists a vertex $v\in V(G)$ and $d(v)=d$ such that v admits one of configurations (A1)–(A4). Let $H=G-vv_1$. By the minimality of G, H has an acyclic edge k-coloring c with the color set $C=\{1,2,\cdots,k\}$. Moreover, among all the acyclic edge coloring of H, we choose the coloring c such that the value of $|C(v)\cap C(v_1)|=m$ is minimum.

Suppose $m=0$. Since

$$|C(v)\cup C(v_1)|\leq 4+\Delta(G)-1=\Delta(G)+3<\Delta(G)+10=|C|,$$

we have at least one available colors for the edge uv_1 such that no bichromatic cycles are created. So we consider that $m\geq 1$.

Case 1. $d(v)=3$. The proof of this case is similar to that of Lemma 4 in[15], we omit it here.

Case 2. $d(v)=4$. Let $N_G(v)=\{v_1,v_2,v_3,v_4\}$, where $d(v_1)\leq d(v_2)\leq d(v_3)\leq d(v_4)$ and $d(v_1)\leq 6, d(v_2)\leq 8$. Without loss of generality, assume that $d(v_1)=6, d(v_2)=8$. Let x_1,x_2,x_3,x_4,x_5 be the five neighbors of v_1 other than v. Let $S_v=C_{vv_2}\uplus C_{vv_3}\uplus C_{vv_4}$.

Suppose $m=1$, without loss of generality, assume that $c(vv_4)=c(v_1x_1)$ =1. Let $\mathcal{U}=C(v)\cup C(v_1)$ in H. Therefore, $|C\backslash\mathcal{U}|=\Delta(G)+3$ and $|C\setminus(\mathcal{U}\cup C(v_4))|\neq\emptyset$. Thus we get an acyclic edge k-coloring of G by coloring vv_1 with a color in $C\setminus(\mathcal{U}\cup C(v_4))$, a contradiction.

When $m \geq 2$, without loss of generality, assume that $c(vv_{i+1}) = c(v_1x_i) = i$, for $i \in \{1, \cdots, m\}$. Let $\mathcal{U} = C(v) \cup C(v_1)$ in H. Therefore, $|C \backslash \mathcal{U}| = \Delta(G) + 2 + m$. If there exists a color $\alpha \in C\backslash\mathcal{U}$ such that the edge vv_1 is colored by α and no bichromatic cycles are created in G, then we get an acyclic edge k-coloring of G, a contradiction. Otherwise, for any color $\gamma \in C\backslash\mathcal{U}$, there exists an (i, γ, vv_1)-critical path for $i \in \{1, \cdots, m\}$ under c. Since

$$\|S_v\| = d(v_3) - 1 + d(v_4) - 1 + d(v_2) - 1 \leq 2\Delta(G) + 5 < 2(\Delta(G) + 2 + m),$$

then there must be a color $\beta \in C\backslash\mathcal{U}$ such that $D_{S_v}(\beta) \leq 1$ in S_v. Without loss of generality, assume that $\beta \in C(v_2)$. There is no $(1, \beta, vv_3)$-critical path through v_2 under coloring c since there exist a $(1, \beta, vv_1)$-critical path through v_2 under coloring c by Lemma 1. We recolor the edge vv_3 with β to obtain an acyclic edge coloring c' of H, but the number of common colors on the edges which are incident with v and v_1 becomes smaller, a contradiction.

Case 3. $d(v) = 5$. Let $N_G(v) = \{v_1, v_2, v_3, v_4, v_5\}$, where $d(v_1) \leq d(v_2) \leq d(v_3) \leq d(v_4) \leq d(v_5)$ and $d(v_1) \leq d(v_2) \leq d(v_3) \leq 5$. Without loss of generality, assume that $d(v_1) = 5, d(v_2) = 5, d(v_3) = 5$. Let x_1, x_2, x_3, x_4 be the four neighbors of v_1 other than v. Let $S_v = C_{vv_2} \uplus C_{vv_3} \uplus C_{vv_4} \uplus C_{vv_5}$.

Suppose $m = 1$. Without loss of generality, assume that $c(vv_5) = c(v_1x_1) = 1$. Let $\mathcal{U} = C(v) \cup C(v_1)$ in H. Therefore, $|C\backslash\mathcal{U}| = \Delta(G) + 3$ and $|C \setminus (\mathcal{U} \cup C(v_5))| \neq \emptyset$. Thus, we can get an acyclic edge k-coloring of G by coloring vv_1 with a color in $C \setminus (\mathcal{U} \cup C(v_5))$, a contradiction.

When $m \geq 2$, without loss of generality, assume that $c(vv_{i+1}) = c(v_1x_i) = i$, for $i \in \{1, \cdots, m\}$. Let $\mathcal{U} = C(v) \cup C(v_1)$ in H. Therefore, $|C\backslash\mathcal{U}| = \Delta(G) + 2 + m$. If there exists a color $\alpha \in C\backslash\mathcal{U}$ such that the edge vv_1 is colored by α and no bichromatic cycles are created in G, then we can get an acyclic edge k-coloring of G, a contradiction. Otherwise, for any color $\gamma \in C\backslash\mathcal{U}$, there exists an (i, γ, vv_1)-critical path for $i \in \{1, \cdots, m\}$ under c. Since

$$\|S_v\| = d(v_5) - 1 + d(v_4) - 1 + d(v_3) - 1 + d(v_2) - 1 \leq 2\Delta(G) + 6 < 2(\Delta(G) + 2 + m),$$

then there must be a color $\beta \in C\backslash\mathcal{U}$ such that $D_{S_v}(\beta) \leq 1$ in S_v. Without loss of generality, we assume that $\beta \in C(v_2)$. There is no $(1, \beta, vv_3)$-critical path through v_2 under coloring c since there exists a $(1, \beta, vv_1)$-critical path through v_2 under coloring c by Lemma 1. So we recolor the edge vv_3 with β to obtain an acyclic edge coloring c' of H, but the number of common colors on the edges which are incident with v and v_1 becomes smaller, a contradiction.

Now we consider the situation that there is no vertex v that belongs to configurations (A1), (A2) and (A3).

Case 4. G contains a 2-vertex. We remove all the 2-vertices in G to get a graph G'. By Lemma 3, if $d_{G'}(x) < d_G(x)$, then $d_{G'}(x) \geq 11$. So G' has no 2-vertices. Now we consider G'. By Lemma 7, there exists a vertex in G' such that at least one of (A1), (A2) and (A3) holds, say the vertex is v, then $3 \leq d_{G'}(v) = d_G(v) \leq 5$ and $d_{G'}(v_1) \leq 9$. But by Lemma 3, we have $d_{G'}(v_1) \geq 11$, a contradiction. □

Proof of Theorem 2. Let G be a k-deletion-minimal graph with $k = \Delta(G) + 8$. Then G is 2-connected by Lemma 2 . By Lemma 8, there exists a vertex $v \in V(G)$ and $d(v) = d$ such that v admits one of configurations (A1)–(A3). Let $H = G - vv_1$. By the minimality of G, H has an acyclic edge k-coloring c with the color set C=$\{1, 2, \cdots, k\}$. Moreover, among all the acyclic edge coloring of H, we choose the coloring c such that the value of $|C(v) \cap C(v_1)| = m$ is minimum.

Suppose $m = 0$. Since

$$|C(v) \cup C(v_1)| \leq 3 + \Delta(G) - 1 = \Delta(G) + 2 < \Delta(G) + 8 = |C|,$$

we have at least one available colors for the edge uv_1 such that no bichromatic cycles are created. So we only consider that $m \geq 1$.

Case 1. $d(v) = 3$. The proof of this case is similar to that of Lemma 4 in [15], we omit it here.
Case 2. $d(v) = 4$. Let $N_G(v) = \{v_1, v_2, v_3, v_4\}$, where $d(v_1) \leq d(v_2) \leq d(v_3) \leq d(v_4)$ and $d(v_1) \leq d(v_2) \leq 5$. Without loss of generality, assume that $d(v_1) = 5, d(v_2) = 5$. Let x_1, x_2, x_3, x_4 be the four neighbors of v_1 other than v and $S_v = C_{vv_2} \uplus C_{vv_3} \uplus C_{vv_4}$.

Suppose $m = 1$, without loss of generality, assume that $c(vv_4) = c(v_1x_1)$ =1. Let $\mathcal{U} = C(v) \cup C(v_1)$ in H. Therefore, $|C \backslash \mathcal{U}| = \Delta(G) + 2$ and $|C \setminus (\mathcal{U} \cup C(v_4))| \neq \emptyset$. Thus, we can get an acyclic edge k-coloring of G by coloring vv_1 with a color in $C \setminus (\mathcal{U} \cup C(v_4))$, a contradiction.

When $m \geq 2$, without loss of generality, assume that $c(vv_{i+1}) = c(v_1x_i) = i$, for $i \in \{1, \cdots, m\}$. Let $\mathcal{U} = C(v) \cup C(v_1)$ in H. Therefore, $|C \backslash \mathcal{U}| = \Delta(G) + 1 + m$. If there exists a color $\alpha \in C \backslash \mathcal{U}$ such that the edge vv_1 is colored by α and no bichromatic cycles are created in G, then we get an acyclic edge k-coloring of G, a contradiction. Otherwise, for any color $\gamma \in C \backslash \mathcal{U}$, there exists an (i, γ, vv_1)-critical path for $i \in \{1, \cdots, m\}$ under c. Since

$$\|S_v\| = d(v_3) - 1 + d(v_4) - 1 + d(v_2) - 1 \leq 2\Delta(G) + 2 < 2(\Delta(G) + 1 + m),$$

then there must be a color $\beta \in C \backslash \mathcal{U}$ such that $D_{S_v}(\beta) \leq 1$ in S_v. Without loss of generality, assume that $\beta \in C(v_2)$. There is no $(1, \beta, vv_3)$-critical path through v_2 under coloring c since there exists a $(1, \beta, vv_1)$-critical path through v_2 under coloring c by Lemma 1. We recolor the edge vv_3 with β to obtain an acyclic edge coloring c' of H, but the number of common colors on the edges which are incident with v and v_1 becomes smaller, a contradiction.

Now we consider the situation that there is no vertex v that belongs to configurations (A1) or (A2).
Case 3. G contains a 2-vertex. We remove all the 2-vertices in G to get a graph G'. By Lemma 3, if $d_{G'}(x) < d_G(x)$, then $d_{G'}(x) \geq 9$. So G' has no 2-vertices. Now we consider G'. By Lemma 8, there exists a vertex in G' such that at least one of (A1) and (A2) holds, say the vertex is v, then $3 \leq d_{G'}(v) = d_G(v) \leq 4$ and $d_{G'}(v_1) \leq 8$. But by Lemma 3, we have $d_{G'}(v_1) \geq 9$, a contradiction. □

Acknowledgments. The authors would like to express their thanks to the referees for their valuable corrections and suggestions of the manuscript that greatly improve the format and correctness of it.

References

[1] Alberson, M.O. Chromatic number, independent ratio, and crossing number. *Ars Math. Contemp*, 1: 1–6 (2008)
[2] Alon, N., McDiarmid, C., Reed, B. Acyclic coloring of graphs. *Random Structures Algorithms*, 2: 277–288 (1991)
[3] Alon, N., Sudakov, B., Zaks, A. Acyclic edge colorings of graphs. *J. Graph Theory*, 37: 157–167 (2001)
[4] Basavaraju, M., Chandran, L.S. Acyclic edge coloring of graphs with maximum degree 4. *J. Graph Theory*, 61: 192–209 (2009)
[5] Basavaraju, M., Chandran, L.S. Acyclic edge coloring of subcubic graphs. *Discrete Math.*, 308: 6650–6653 (2008)
[6] Bondy, J.A., Murty, U.S.R. Graph Theory with Applications. North-Holland, New York, 1976
[7] Cohen, N., Havet, F., Müller, T. Acyclic edge-colouring of planar graphs. Extended abstract. *Electron. Notes Discrete Math.*, 34: 417–421 (2009)
[8] Dong, W., Xu, B. Some results on acyclic edge coloring of plane graphs. *Inform. Process. Lett.*, 110: 887–892 (2010)
[9] Fiamčik, J. The acyclic chromatic class of a graph (in Russian). *Math. Slovaca*, 28: 139–145 (1978)
[10] Fiedorowicz, A., Haluszczak, M., Narayanan, N. About acyclic edge colourings of planar graphs. *Inform. Process. Lett.*, 108: 412–417 (2008)
[11] Guan, Y., Hou, J., Yang, Y. An improved bound on acyclic chromatic index of planar graphs. *Discrete Math.*, 313: 1098–1103 (2013)
[12] Hou, J., Wang, W., Zhang, X. Acyclic edge coloring of planar graphs with girth at least 5. *Discrete Appl. Math.*, 161: 2958–2967 (2013)

[13] Hou, J., Wu, J., Liu, G., Liu, B. Acyclic edge colorings of planar graphs and series-parallel graphs. *Sci. China Ser. A*, 52: 605–616 (2009)
[14] Hou, J., Wu, J., Liu, G., Liu, B. Acyclic edge chromatic number of outerplanar graphs. *J. Graph Theory*, 64, 22–36 (2010)
[15] Hou, J., Roussel, N., Wu, J. Acyclic chromatic index of planar graphs with triangles. *Inform. Process. Lett.*, 111 (17): 836–840 (2011)
[16] Král, D., Stacho, L. Coloring plane graphs with independent crossings. *J. Graph Theory*, 64(3): 184–205 (2010)
[17] Lin, Q., Hou, J., Liu, Y. Acyclic edge coloring of graphs with large girths. *Science China Mathematics*, 12: 2593–2600 (2012)
[18] Molloy, M., Reed, B. Further algorithmic aspects of the local lemma. In: Proceedings of the 30th annual ACM symposium on theory of computing, ACM, New York, 1988, 524–529
[19] Muthu, R., Narayanan, N., Subramanian, C. Acyclic edge colouring of outerplanar graphs. *Lecture Notes in Comput. Sci.*, 4508: 144–152 (2007)
[20] Němetřil, J., Wormald, N.C. The acyclic edge chromatic number of a randomd-regular graph is $d+1$. *J. Graph Theory*, 49: 69–74 (2005)
[21] Shu, Q., Wang, W., Wang, Y. Acyclic chromatic indices of planar graphs with girth at least 4. *J. Graph Theory*, 73: 386–399 (2013)
[22] Shu, Q., Wang, W. Acyclic chromatic indices of planar graphs with girth at least five. *J. Comb. Optim.*, 23: 140–157 (2012)
[23] Skulrattanakulchai, S. Acyclic colorings of subcubic graphs. *Inform. Process. Lett.*, 92: 161–167 (2004)
[24] Song, W., Miao, L. Acyclic edge coloring of triangle-free 1-planar graphs. *Acta Mathematica Sinica, English Series*, 31(10): 1563–570 (2015)
[25] Song, W., Miao, L., Li, J., Zhao, Y., Pang, J. Neighbor sum distinguishing total coloring of sparse IC-planar graphs. *Discrete Applied Mathematics*, 239: 183–192 (2018)
[26] Wang, W., Shu, Q. Acyclic chromatic indices of K_4-minor free graphs. *Sci. China Ser. A*, 41: 733–744 (2011)
[27] Wang, W., Shu, Q., Kan, W., Wang, P. Acyclic chromatic indices of planar graphs with large girth. *Discrete Appl. Math.*, 159: 1239–1253 (2011)
[28] Wang, T., Zhang, Y. Acyclic edge coloring of graphs. *Discrete Appl. Math.*, 167: 290–303 (2014)
[29] Yu, D., Hou, J., Liu, G., Liu, B., Xu, L. Acyclic edge coloring of planar graphs with large girth. *Theoret. Comput. Sci.*, 410: 5196–5200 (2010)
[30] Zhang, X., Liu, G, Wu, J. Structural properties of 1-planar graphs and an application to acyclic edge coloring. *Scientia Sinica Mathematica*, 40: 1025–1032 (2010)
[31] Zhang, X., Wu, J. On edge colorings of 1-planar graphs. *Inform. Process. Lett.*, 111(3): 124–128 (2011)
[32] Chen, J., Wang, T., Zhang, H. Acyclic Chromatic Index of Triangle-free 1-Planar Graphs, *Graph and Combinatorics*, 33(4): 859–868 (2017)