Acta Mathematicae Applicatae Sinica, English Series
Vol. 36, No. 2 (2020) 390–400
https://doi.org/10.1007/s10255-020-0946-z
http://www.ApplMath.com.cn & www.SpringerLink.com

Acta Mathematicae Applicatae Sinica, English Series

Multiple Periodic Solutions of Differential Delay Equations with $2k-1$ Lags

Lin LI, Hua-fei SUN†, Wei-gao GE

School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China
(†E-mail: huafeisun@bit.edu.cn)

Abstract In this paper, we study the periodic solutions to a type of differential delay equations with $2k-1$ lags. The $4k$-periodic solutions are obtained by using the variational method and the method of Kaplan-Yorke coupling system. This is a new type of differential delay equations compared with all the previous researches. And this paper provides a theoretical basis for the study of differential delay equations. An example is given to demonstrate our main results.

Keywords differential delay equation; periodic solutions; critical point theory; variational method
2000 MR Subject Classification 34K13; 58E50

1 Introduction

The differential delay equations have useful applications in various fields such as age-structured population growth, control theory, and the models involving responses with nonzero delays.

Given $f \in C^0(R, R)$ with $f(-x) = -f(x),\ xf(x) > 0,\ x \neq 0$. Kaplan and Yorke[17] studied the existence of 4-periodic and 6-periodic solutions to the differential delay equations

$$x'(t) = -f(x(t-1)) \tag{1.1}$$

and

$$x'(t) = -f(x(t-1)) - f(x(t-2)) \tag{1.2}$$

respectively. The method they applied is transforming the two equations into corresponding ordinary differential equations by regarding the retarded functions $x(t-1)$ and $x(t-2)$ as independent variables. They guessed that the existence of $2(n+1)$-periodic solutions to the equation

$$x'(t) = -\sum_{i=1}^{n} f(x(t-i)) \tag{1.3}$$

could be studied under the restriction

$$x(t-(n+1)) = -x(t), \tag{1.4}$$

which was proved by Nussbaum[19] in 1978 by use of a fixed point theorem on cones.

Manuscript received March 13, 2018. Accepted on September 24, 2019.
This subject is supported by the National Natural Science Foundations of China (No. 61179031).
†Corresponding author.

After then a lot of papers[3–17] discussed the existence and multiplicity of $2(n+1)$-periodic solutions to equation (1.3) and its extension

$$x'(t) = -\sum_{i=1}^{n} \nabla F(x(t-i)) \tag{1.5}$$

where $F \in C^1(R^N, R),\ \nabla F(-x) = -\nabla F(x),\ F(0) = 0.$

In this paper, we study the periodic orbits to a type of differential delay equations with $2k-1$ lags in the form

$$x'(t) = -\sum_{i=1}^{2k-1} (-1)^{i+1} f(x(t-i)), \tag{1.6}$$

which is different from (1.3) and can be regarded as a new extension of (1.3). The method applied in this paper is the variational approach in the critical point theory[1, 2, 18].

We suppose that

$$f \in C^0(R, R), f(-x) = -f(x) \tag{1.7}$$

and there are $\alpha, \beta \in R$ such that

$$\lim_{x\to 0} \frac{f(x)}{x} = \alpha, \qquad \lim_{x\to\infty} \frac{f(x)}{x} = \beta. \tag{1.8}$$

Let $F(x) = \int_0^x f(s)ds$. Then $F(-x) = F(x)$ and $F(0) = 0$. For convenience, we make the following assumptions.

(S_1) f satisfies (1.7) and (1.8),

(S_2) $|F(x) - \frac{1}{2}\beta x^2| \to \infty$ as $|x| \to \infty$,

$(S_3^{\pm})$ $\pm[F(x) - \frac{1}{2}\beta x^2] > 0,\ |x| \to \infty$,

$(S_4^{\pm})$ $\pm[F(x) - \frac{1}{2}\alpha x^2] > 0,\ 0 < |x| \ll 1$.

In this paper, we need the following lemma as the basis of our discussion.

Let X be a Hilbert space, $L : X \to X$ be a linear operator, and $\Phi : X \to R$ be a differentiable functional. Besides, $P^{-1} : X \to X$ is a linear operator determined by the definition of the inner product and A_0, A_∞ are defined by

$$\nabla F(x) = A_0 x + \circ(|x|), \qquad |x| \to 0, \tag{1.9}$$

$$\nabla F(x) = A_\infty x + \circ(|x|), \qquad |x| \to \infty. \tag{1.10}$$

Lemma 1.1 ([2], Theorem 2.4; [3], Lemma 2.4). *Assume that there are two closed S^1-invariant linear subspaces, X^+ and X^-, and $r > 0$ such that*

(a) $X^+ \cup X^-$ is closed and of finite codimensions in X,

(b) $\widehat{L}(X^-) \subset X^-, \widehat{L} = L + P^{-1}A_0$ or $\widehat{L} = L + P^{-1}A_\infty$,

(c) there exists $c_0 \in R$ such that

$$\inf_{x\in X^+} \Phi(x) \geq c_0,$$

(d) there is $c_\infty \in R$ such that

$$\Phi(x) \leq c_\infty < \Phi(0) = 0,\ \forall\, x \in X^- \cap S_r = \{x \in X^- : \|x\| = r\},$$

(e) Φ satisfies $(P.S)_c$-condition, $c_0 < c < c_\infty$. Then Φ has at least $\frac{1}{2}[\dim(X^+ \cap X^-) - \mathrm{codim_X}(X^+ \cup X^-)]$ geometrically different critical orbits in $\Phi^{-1}([c_0, c_\infty])$ if

$$[\dim(X^+ \cap X^-) - \mathrm{codim_X}(X^+ \cup X^-)] > 0.$$

Remark 1.1. We may use (P.S)-condition (Palais-Smale condition) to replace condition (e) in Lemma 1.1 since (P.S)-condition implies that (P.S)$_c$-condition holds for each $c \in R$.

2 Space X, Functional Φ and Its Differential Φ'

We are concerned with the $4k$-periodic solutions to (1.6) and suppose

$$x(t-2k)=-x(t), \qquad k\geq 1. \tag{2.1}$$

Let

$$\begin{aligned}\widehat{X}=&\{x\in C_T: x(t-2k)=-x(t)\}\\ =&\Big\{\sum_{i=0}^{\infty}(a_i\cos\frac{(2i+1)\pi t}{2k}+b_i\sin\frac{(2i+1)\pi t}{2k}): a_i,b_i\in R\Big\},\\ X=&\mathrm{cl}\Big\{\sum_{i=0}^{\infty}\Big(a_i\cos\frac{(2i+1)\pi t}{2k}+b_i\sin\frac{(2i+1)\pi t}{2k}\Big): a_i,b_i\in R,\ \sum_{i=0}^{\infty}(2i+1)(a_i^2+b_i^2)<\infty\Big\},\end{aligned}$$

and define $P: X\to L^2$ by

$$\begin{aligned}Px(t)=&P\Big(\sum_{i=0}^{\infty}(a_i\cos\frac{(2i+1)\pi t}{2k}+b_i\sin\frac{(2i+1)\pi t}{2k})\Big)\\ =&\sum_{i=0}^{\infty}(2i+1)\Big(a_i\cos\frac{(2i+1)\pi t}{2k}+b_i\sin\frac{(2i+1)\pi t}{2k}\Big).\end{aligned} \tag{2.2}$$

Then the inverse P^{-1} of P exists and

$$P^{-1}x(t)=\sum_{i=0}^{\infty}\frac{1}{2i+1}\Big(a_i\cos\frac{(2i+1)\pi t}{2k}+b_i\sin\frac{(2i+1)\pi t}{2k}\Big).$$

For $x\in X$, define

$$\langle x,y\rangle=\int_0^{4k}(Px(t),y(t))dt, \qquad \|x\|=\sqrt{\langle x,x\rangle},$$
$$\langle x,y\rangle_2=\int_0^{4k}(x(t),y(t))dt, \qquad \|x\|_2=\sqrt{\langle x,x\rangle_2}.$$

Therefore $(X,\|\cdot\|)$ is an $H^{\frac{1}{2}}$ space.

Define functional $\Phi: X\to R$ by

$$\Phi(x)=\frac{1}{2}\langle Lx,x\rangle+\int_0^{4k}F(x(t))dt \tag{2.3}$$

where

$$Lx=-P^{-1}\sum_{i=1}^{2k-1}x'(t-i). \tag{2.4}$$

Let

$$X(i)=\Big\{x(t)=a_i\cos\frac{(2i+1)\pi t}{2k}+b_i\sin\frac{(2i+1)\pi t}{2k}: a_i,b_i\in R\Big\}.$$

Then we have

$$X=\sum_{l=0}^{\infty}\Big[\sum_{i=0}^{k-1}\Big(X(2lk+i)+X(2lk+2k-i-1)\Big)\Big]. \tag{2.5}$$

If $x_i(t) = a_i \cos\frac{(2i+1)\pi t}{2k} + b_i \sin\frac{(2i+1)\pi t}{2k} \in X(i),\ i \in N$, we have

$$Lx = -\frac{\pi}{2k}\Big(\sum_{i=0}^{\infty} x_i \cot\frac{(2i+1)\pi}{4k}\Big). \tag{2.6}$$

Obviously $L|_{X(i)} : X(i) \to X(i)$ is invertible.

Based on the theorem given by Mawhen and Willem (see [18], Theorem 1.4) the functional Φ is differentiable, and its differential is

$$\Phi'(x) = Lx + K(x) \tag{2.7}$$

where $K(x) = P^{-1}f(x)$. It is easy to prove that $K : (X, \|\bullet\|^2) \to (X, \|\bullet\|_2^2)$ is compact.

Therefore, from (2.6) we have that if

$$x(t) = \sum_{i=0}^{\infty}\Big(a_i \cos\frac{(2i+1)\pi t}{2k} + b_i \sin\frac{(2i+1)\pi t}{2k}\Big),$$

then

$$\begin{aligned}\langle Lx, x\rangle =& -\sum_{i=0}^{\infty}(2i+1)\pi(a_i^2+b_i^2)\cot\frac{(2i+1)\pi}{4k}\\ =&\sum_{l=0}^{\infty}\Big[-\sum_{i=0}^{k-1}(4lk+2i+1)\pi(a_{2lk+i}^2+b_{2lk+i}^2)\cot\frac{(2i+1)\pi}{4k}\\ &+\sum_{i=0}^{k-1}(4lk+4k-2i-1)\pi(a_{2lk+2k-i-1}^2+b_{2lk+2k-i-1}^2)\cot\frac{(2i+1)\pi}{4k}\Big].\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle P^{-1}\beta x, x\rangle =& \sum_{i=0}^{\infty}2k\beta(a_i^2+b_i^2)\\ =&\sum_{l=0}^{\infty}\Big[\sum_{i=0}^{k-1}2k\beta(a_{2lk+i}^2+b_{2lk+i}^2)+\sum_{i=0}^{k-1}2k\beta(a_{2lk+2k-i-1}^2+b_{2lk+2k-i-1}^2)\Big].\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\langle (L+P^{-1}\beta)x, x\rangle\\ =&2k\sum_{l=0}^{\infty}\Big[\sum_{i=0}^{k-1}\Big(-\frac{(4lk+2i+1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k}+\beta\Big)(a_{2lk+i}^2+b_{2lk+i}^2)\\ &+\sum_{i=0}^{k-1}\Big(\frac{(4lk+4k-2i-1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k}+\beta\Big)(a_{2lk+2k-i-1}^2+b_{2lk+2k-i-1}^2)\Big].\end{aligned} \tag{2.8}$$

Lemma 2.1. *Each critical point of the functional Φ is a $4k$-periodic solution of equation (1.6) satisfying (2.1).*

Proof. Let x be a critical point of the functional Φ. Then $x(t)$ satisfies

$$-\sum_{i=1}^{2k-1}x'(t-i)+f(x(t))=0. \tag{2.9}$$

Consequently,

$$-\sum_{i=1}^{2k-1} x'(t-i-1) + f(x(t-1)) = 0, \tag{2.9.1}$$

$$-\sum_{i=1}^{2k-1} x'(t-i-2) + f(x(t-2)) = 0, \tag{2.9.2}$$

$$-\sum_{i=1}^{2k-1} x'(t-i-3) + f(x(t-3)) = 0, \tag{2.9.3}$$

$$\vdots$$

$$-\sum_{i=0}^{2k-1} x'(t-i-(2k-1)) + f(x(t-(2k-1))) = 0. \tag{2.9.(2k-1)}$$

Calculating (2.9.1) − (2.9.2) + (2.9.3) − ⋯ + (2.9.$(2k-1)$), we can get

$$x'(t) + \sum_{i=1}^{2k-1} (-1)^{i+1} f(x(t-i)) = 0,$$

namely,

$$x'(t) = -\sum_{i=1}^{2k-1} (-1)^{i+1} f(x(t-i)),$$

which implies that x is a solution to (1.6). □

3 Partition of Space X and Symbols

Let

$$\begin{aligned}
X_\infty^+ =&\Big\{X(2lk+i) : l \geq 0,\ 0 \leq i \leq k-1, -\frac{(4lk+2i+1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k} + \beta > 0\Big\} \\
&\cup \Big\{X(2lk+2k-i-1) : l \geq 0,\ 0 \leq i \leq k-1, \frac{(4lk+4k-2i-1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k} + \beta > 0\Big\}, \\
X_\infty^- =&\Big\{X(2lk+i) : l \geq 0,\ 0 \leq i \leq k-1, -\frac{(4lk+2i+1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k} + \beta < 0\Big\} \\
&\cup \Big\{X(2lk+2k-i-1) : l \geq 0,\ 0 \leq i \leq k-1, \frac{(4lk+4k-2i-1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k} + \beta < 0\Big\}, \\
X_0^+ =&\Big\{X(2lk+i) : l \geq 0, 0 \leq i \leq k-1, -\frac{(4lk+2i+1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k} + \alpha > 0\Big\} \\
&\cup \Big\{X(2lk+2k-i-1) : l \geq 0,\ 0 \leq i \leq k-1, \frac{(4lk+4k-2i-1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k} + \alpha > 0\Big\}, \\
X_0^- =&\Big\{X(2lk+i) : l \geq 0, 0 \leq i \leq k-1, -\frac{(4lk+2i+1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k} + \alpha < 0\Big\} \\
&\cup \Big\{X(2lk+2k-i-1) : l \geq 0,\ 0 \leq i \leq k-1, \frac{(4lk+4k-2i-1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k} + \alpha < 0\Big\}.
\end{aligned}$$

On the other hand,

$$X_\infty^0 =\Big\{X(2lk+i) : l \geq 0, 0 \leq i \leq k-1, -\frac{(4lk+2i+1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k} + \beta = 0\Big\}$$

$$\cup\Big\{X(2lk+2k-i-1): l\geq 0,\ 0\leq i\leq k-1, \frac{(4lk+4k-2i-1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k}+\beta=0\Big\},$$

$$X_0^0=\Big\{X(2lk+i): l\geq 0, 0\leq i\leq k-1, -\frac{(4lk+2i+1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k}+\alpha=0\Big\}$$

$$\cup\Big\{X(2lk+2k-i-1): l\geq 0,\ 0\leq i\leq k-1, \frac{(4lk+4k-2i-1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k}+\alpha=0\Big\},$$

Obviously, $\dim X_\infty^0<\infty$ and $\dim X_0^0<\infty$.

Lemma 3.1. *Under assumptions (S_1) and (S_2), there is $\sigma>0$ such that*

$$\big\langle (L+P^{-1}\beta)x,x\big\rangle>\sigma\|x\|^2,\quad x\in X_\infty^+ \text{ and } \big\langle (L+P^{-1}\beta)x,x\big\rangle<-\sigma\|x\|^2,\quad x\in X_\infty^- . \tag{3.1}$$

Proof. First, we have that, for $\beta\geq 0, i\in\{0,1,\cdots,k-1\}$,

$$-\frac{(4lk+2i+1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k}+\beta>-\frac{(4l^+(i)k+2i+1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k}+\beta>0$$

where $l^+(i)=\max\big\{l\in N: -\frac{(4lk+2i+1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k}+\beta>0\big\}$ and

$$-\frac{(4lk+2i+1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k}+\beta<-\frac{(4l^-(i)k+2i+1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k}+\beta<0,$$

where $l^-(i)=\min\big\{l\in N: -\frac{(4lk+2i+1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k}+\beta<0\big\}$.

In this case, we may choose

$$\sigma_i=\min\Big\{-\frac{\pi}{2k}\cot\frac{(2i+1)\pi}{4k}+\frac{\beta}{4l^+(i)k+2i+1},\ \frac{\pi}{2k}\cot\frac{(2i+1)\pi}{4k}-\frac{\beta}{4l^-(i)k+2i+1}\Big\}>0,$$

and let $\sigma=\min\{\sigma_0,\sigma_1,\cdots,\sigma_{k-1}\}>0$. The proof for the case $\beta<0$ is similar. We omit it. The inequalities in (3.1) are proved. □

Lemma 3.2. *Under conditions (S_1) and (S_2), the functional Φ defined by (2.3) satisfies (P.S)-condition.*

Proof. Let Π, N, Z be the orthogonal projections from X onto $X_\infty^+, X_\infty^-, X_\infty^0$, respectively. From the second condition in (1.8) it follows that

$$\big|\big\langle P^{-1}(f(x)-\beta x),x\big\rangle\big|<\frac{\sigma}{2}\|x\|^2+M,\qquad x\in X \tag{3.2}$$

for some $M>0$.

Assume that $\{x_n\}\subset X$ is a subsequence such that $\Phi'(x_n)\to 0$ and $\Phi(x_n)$ is bounded. Let $w_n=\Pi x_n, y_n=Nx_n, z_n=Zx_n$. Then we have

$$\Pi(L+P^{-1}\beta)=(L+P^{-1}\beta)\Pi, N(L+P^{-1}\beta)=(L+P^{-1}\beta)N. \tag{3.3}$$

From

$$\big\langle\Phi'(x_n),x_n\big\rangle=\big\langle Lx_n+P^{-1}f(x_n),x_n\big\rangle=\big\langle(L+P^{-1}\beta)x_n,x_n\big\rangle+\big\langle P^{-1}(f(x_n)-\beta x_n),x_n\big\rangle,$$

and (3.3), we have

$$\big\langle\Pi\Phi'(x_n),x_n\big\rangle=\big\langle\Pi(L+P^{-1}\beta)x_n,x_n\big\rangle+\big\langle\Pi P^{-1}(f(x_n)-\beta x_n),x_n\big\rangle$$

$$= \big\langle (L+P^{-1}\beta)w_n, w_n\big\rangle + \big\langle \Pi P^{-1}(f(x_n)-\beta x_n), w_n\big\rangle,$$

and then, by (3.1), we have

$$\big\langle (L+P^{-1}\beta)w_n, w_n\big\rangle + \big\langle \Pi P^{-1}(f(x_n)-\beta x_n), w_n\big\rangle > \frac{\sigma}{2}\|w_n\|^2 - M\|w_n\|,$$

which, together with $\Pi\Phi'(x_n) \to 0$, implies the boundedness of w_n. Similarly we have the boundedness of y_n. At the same time, (S_2) yields

$$\begin{aligned}\Phi(x_n) &= \frac{1}{2}\big\langle (L+P^{-1}\beta)x_n, x_n\big\rangle + \int_0^{4k} F(x_n)dt - \frac{\beta}{2}\|x_n\|_2^2 \\ &= \frac{1}{2}\big\langle (L+P^{-1}\beta)w_n, w_n\big\rangle + \frac{1}{2}\big\langle (L+P^{-1}\beta)y_n, y_n\big\rangle \\ &\quad + \int_0^{4k} F(x_n)dt - \frac{\beta}{2}\big(\|w_n\|_2^2 + \|y_n\|_2^2 + \|z_n\|_2^2\big).\end{aligned}$$

Then the boundedness of $\Phi(x)$ implies that $\|z_n\|_2$ is bounded. Consequently, $\|z_n\|$ is bounded since X_∞^0 is finite-dimensional. Therefore, $\|x_n\|$ is bounded.

It follows from (2.7) that

$$\begin{aligned}(\Pi+N)\Phi'(x_n) &= (\Pi+N)Lx_n + (\Pi+N)Kx_n \\ &= L(w_n+y_n) + (\Pi+N)Kx_n.\end{aligned}$$

From the compactness of operator K and the boundedness of x_n we have that $K(x_n) \to u$. Then

$$L|_{x_\infty^+ + x_\infty^-}(w_n+y_n) \to -(\Pi+N)u. \tag{3.4}$$

The finite-dimensionality of X_∞^0 and the boundedness of $z_n = Zx_n$ imply $z_n \to \varphi \in X_\infty^0$. Therefore,

$$x_n = z_n + w_n + y_n \to \varphi - (L|_{x_\infty^+ + x_\infty^-})^{-1}(\Pi+N)u,$$

which implies (P.S)-condition. □

4 Notations and Main Results of This Paper

We first give some notation.

Denote

$$N(\alpha) = \begin{cases} -\sum\limits_{i=0}^{k-1} \text{card}\Big\{ l \geq 0 : 0 < \dfrac{(4lk+4k-2i-1)\pi}{2k} \cot\dfrac{(2i+1)\pi}{4k} < -\alpha \Big\}, & \alpha < 0, \\ \sum\limits_{i=0}^{k-1} \text{card}\Big\{ l \geq 0 : 0 < \dfrac{(4lk+2i+1)\pi}{2k} \cot\dfrac{(2i+1)\pi}{4k} < \alpha \Big\}, & \alpha \geq 0. \end{cases}$$

$$N(\beta) = \begin{cases} -\sum\limits_{i=0}^{k-1} \text{card}\Big\{ l \geq 0 : 0 < \dfrac{(4lk+4k-2i-1)\pi}{2k} \cot\dfrac{(2i+1)\pi}{4k} < -\beta \Big\}, & \beta < 0, \\ \sum\limits_{i=0}^{k-1} \text{card}\Big\{ l \geq 0 : 0 < \dfrac{(4lk+2i+1)\pi}{2k} \cot\dfrac{(2i+1)\pi}{4k} < \beta \Big\}, & \beta \geq 0, \end{cases}$$

and

$$N^0(\alpha_-) = \sum_{i=0}^{k-1} \text{card}\Big\{ l \geq 0 : 0 < \frac{(4lk+4k-2i-1)\pi}{2k} \cot\frac{(2i+1)\pi}{4k} = -\alpha \Big\}, \qquad \alpha < 0,$$

$$N^0(\alpha_+)=\sum_{i=0}^{k-1}\mathrm{card}\Big\{l\geq 0: 0<\frac{(4lk+2i+1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k}=\alpha\Big\},\qquad \alpha\geq 0,$$
$$N^0(\beta_-)=\sum_{i=0}^{k-1}\mathrm{card}\Big\{l\geq 0: 0<\frac{(4lk+4k-2i-1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k}=-\beta\Big\},\qquad \beta<0,$$
$$N^0(\beta_+)=\sum_{i=0}^{k-1}\mathrm{card}\Big\{l\geq 0: 0<\frac{(4lk+2i+1)\pi}{2k}\cot\frac{(2i+1)\pi}{4k}=\beta\Big\},\qquad \beta\geq 0.$$

Now we give the main results of this paper.

Theorem 4.1. *Suppose that (S_1) and (S_2) hold. Then equation (1.6) possesses at least*

$$n=\max\big\{N(\beta)-N(\alpha)-N^0(\beta_-)-N^0(\alpha_+),N(\alpha)-N(\beta)-N^0(\alpha_-)-N^0(\beta_+)\big\},$$

$4k$-periodic solutions satisfying $x(t-2k)=-x(t)$ provided that $n>0$.

Theorem 4.2. *Suppose that $(S_1),(S_2),(S_3^+)$, and (S_4^-) hold. Then equation (1.6) possesses at least*

$$n=N(\beta)-N(\alpha)+N^0(\beta_+)+N^0(\alpha_-)$$

$4k$-periodic solutions satisfying $x(t-2k)=-x(t)$ provided that $n>0$.

Theorem 4.3. *Suppose that $(S_1),(S_2),(S_3^-)$, and (S_4^+) hold. Then equation (1.6) possesses at least*

$$n=N(\alpha)-N(\beta)+N^0(\alpha_+)+N^0(\beta_-)$$

$4k$-periodic solutions satisfying $x(t-2k)=-x(t)$ provided that $n>0$.

5 Proof of Main Results of This Paper

Proof of Theorem 4.1. Suppose without loss of generality that

$$n=N(\beta)-N(\alpha)-N^0(\beta_-)-N^0(\alpha_+).$$

Let $X^+=X_\infty^+$ and $X^-=X_0^-$. Then

$$X\setminus(X^+\cup X^-)=X\setminus(X_\infty^+\cup X_0^-)\subseteq X_\infty^0\cup X_0^0\cup(X_\infty^+\cap X_0^-).$$

Obviously

$$\mathrm{codim_X(X^++X^-)\leq\dim X_\infty^0+\dim X_0^0+\dim(X_\infty^+\cap X_0^-)<\infty},$$

which implies that condition (a) in Lemma 1.1 holds. Let $A_\infty=\beta$. Then condition (b) in Lemma 1.1 holds since for each $j\in N$, we have that $x\in X(j)$ yields $(L+P^{-1}\beta)x\in X(j)$.

At the same time, Lemma 3.2 gives the (P.S)-condition.

Now it suffices to show that conditions (c) and (d) in Lemma 1.1 hold under assumptions (S_1) and (S_2).

In fact, condition (S_1) implies that on X^- we have $\Phi(x)<0$ if $0<\|x\|\ll 1$, that is, there are $r>0$ and $c_\infty<0$ such that

$$\Phi(x)\leq c_\infty<0=\Phi(0),\forall x\in X^-\cap S_r=\{x\in X:\|x\|=r\}.$$

On the other hand, we have shown in Lemma 3.1 that there is $\sigma>0$ such that $\big\langle(L+P^{-1}\beta)x,x\big\rangle>\sigma\|x\|^2,\ x\in X_\infty^+$. On the other hand $|F(x)-\frac{1}{2}\beta x^2|<\frac{1}{4}\sigma|x|^2+M_1,\ x\in R$ for some $M_1>0$.

Then

$$\begin{aligned}\Phi(x) &= \frac{1}{2}\big\langle (L+P^{-1}\beta)x, x\big\rangle + \int_0^{4k}\Big[F(x(t)) - \frac{1}{2}\beta|x(t)|^2\Big]dt \\ &\geq \frac{1}{2}\sigma\|x\|^2 - \frac{1}{4}\sigma\|x\|^2 - 4kM_1 \geq \frac{1}{4}\sigma\|x\|^2 - 4kM_1\end{aligned}$$

if $x \in X^+$. Clearly, there is $c_0 < c_\infty$ such that $\Phi(x) \geq c_0,\ x \in X^+$.

Our last task is to compute the value of

$$\begin{aligned} n &= \frac{1}{2}\big[\dim(X^+\cap X^-) - \mathrm{codim_X(X^+ + X^-)}\big] \\ &= \frac{1}{2}\big[\dim(X^+_\infty \cap X^-_0) - \mathrm{codim_X(X^+_\infty + X^-_0)}\big] \\ &= \frac{1}{2}\sum_{j=0}^{\infty}\big[\dim(X^+_\infty(j)\cap X^-_0(j)) - \mathrm{codim_{X(j)}(X^+_\infty(j) + X^-_0(j))}\big].\end{aligned}$$

By computation we get that, for each $i \in \{0, 1, \cdots, k-1\}$,

$$\big\langle (L+P^{-1}\beta)x, x\big\rangle = \Big(-\frac{\pi}{2k}\cot\frac{(2i+1)\pi}{4k} + \frac{\beta}{4lk+2i+1}\Big)\|x\|^2, \quad x \in X(2lk+i), \tag{5.1}$$

$$\big\langle (L+P^{-1}\beta)x, x\big\rangle = \Big(\frac{\pi}{2k}\cot\frac{(2i+1)\pi}{4k} + \frac{\beta}{4lk+4k-2i-1}\Big)\|x\|^2, \quad x \in X(2lk+2k-i-1), \tag{5.2}$$

$$\big\langle (L+P^{-1}\alpha)x, x\big\rangle = \Big(-\frac{\pi}{2k}\cot\frac{(2i+1)\pi}{4k} + \frac{\alpha}{4lk+2i+1}\Big)\|x\|^2, \quad x \in X(2lk+i), \tag{5.3}$$

$$\big\langle (L+P^{-1}\alpha)x, x\big\rangle = \Big(\frac{\pi}{2k}\cot\frac{(2i+1)\pi}{4k} + \frac{\alpha}{4lk+4k-2i-1}\Big)\|x\|^2, \quad x \in X(2lk+2k-i-1). \tag{5.4}$$

Therefore,

$X^+_\infty(2lk+i) = X^+_\infty \cap X(2lk+i) = \emptyset$,

$X^+_\infty(2lk+2k-i-1) = X^+_\infty \cap X(2lk+2k-i-1) = X(2lk+2k-i-1)$,

$X^-_0(2lk+i) = X^-_0 \cap X(2lk+i) = X(2lk+i)$,

$X^-_0(2lk+2k-i-1) = X^-_0 \cap X(2lk+2k-i-1) = \emptyset$

if $i \in \{0, 1, \cdots, k-1\}$ and $l \geq 0$ is large enough, which means that there is $M > 0$ such that $\dim(X^+_\infty(j)\cap X^-_0(j)) - \mathrm{codim_X(X^+_\infty(j) + X^-_0(j))} = 0,\ \mathrm{j > M}$, from which it follows that

$$\begin{aligned} n &= \frac{1}{2}\sum_{j=0}^{M}\big[\dim(X^+_\infty(j)\cap X^-_0(j)) - \mathrm{codim_{X(j)}(X^+_\infty(j) + X^-_0(j))}\big] \\ &= \frac{1}{2}\sum_{j=0}^{M}\big[\dim X^+_\infty(j) + \dim X^-_0(j) - 2\big] \\ &= \frac{1}{2}\sum_{j=0}^{M}\big[\dim X^+_\infty(j) + \dim X^-_0(j)\big] - (M+1).\end{aligned}$$

Then we have

$$\sum_{j=0}^{M}\dim(X_\infty^+(j))$$
$$=2\begin{cases} N(\beta)+\mathrm{card}\{2lk+2k-i-1:0\le 2lk+2k-i-1\le M\}, & \beta\ge 0,\\ N(\beta)-N^0(\beta_-)+\mathrm{card}\{2lk+2k-i-1:0\le 2lk+2k-i-1\le M\}, & \beta<0,\end{cases} \tag{5.5}$$

$$\sum_{j=0}^{M}\dim(X_0^-(j))$$
$$=2\begin{cases} -N(\alpha)-N^0(\alpha_+)+\mathrm{card}\{2lk+i:0\le 2lk+i\le M\}, & \alpha\ge 0,\\ -N(\alpha)+\mathrm{card}\{2lk+i:0\le 2lk+i\le M\}, & \alpha<0,\end{cases} \tag{5.6}$$

and

$$\sum_{j=0}^{M}\big[\dim X_\infty^+(j)+\dim X_0^-(j)\big]=2\big[N(\beta)-N(\alpha)-N^0(\beta_-)-N^0(\alpha_+)\big]+2(M+1). \tag{5.7}$$

Therefore

$$n=N(\beta)-N(\alpha)-N^0(\beta_-)-N^0(\alpha_+).$$

Theorem 4.1 is proved. □

Proof of Theorem 4.2 and Theorem 4.3. Since the proof for the two theorems is similar, we prove only Theorem 4.2.

Let $X^+=X_\infty^+ + X_\infty^0$, $X^-=X_-^0+X_0^0$. Then as in the proof of Theorem 4.1, we check the conditions (a), (b), (c), (d), and (e). In the present case, we may suppose that (5.7) still holds for some $M>0$. Let $X_\infty^0(i)=X_\infty^0\cap X(i)$, $X_0^0(i)=X_0^0\cap X(i)$. Then

$$\begin{aligned} n&=\frac12\sum_{i=0}^{M}\big[\dim(X_\infty^+(i)\cap X_0^-(i))-\mathrm{codim}_{\mathrm{X(i)}}(\mathrm{X}_\infty^+(\mathrm{i})+\mathrm{X}_0^-(\mathrm{i}))\big]+\big(\dim \mathrm{X}_\infty^0+\dim \mathrm{X}_0^0\big)\\ &=\frac12\sum_{i=0}^{M}\big[\dim X_\infty^+(i)+\dim X_0^-(i)-2\big]+\big(\dim X_\infty^0+\dim X_0^0\big)\\ &=\frac12\sum_{i=0}^{M}\big[\dim X_\infty^+(i)+\dim X_0^-(i)\big]-(M+1)+\big(\dim X_\infty^0+\dim X_0^0\big)\\ &=N(\beta)-N(\alpha)-N^0(\beta_-)-N^0(\alpha_+)+\big(N^0(\beta_+)+N^0(\beta_-)+N^0(\alpha_+)+N^0(\alpha_-)\big)\\ &=N(\beta)-N(\alpha)+N^0(\beta_+)+N^0(\alpha_-).\end{aligned}$$

Our proof is completed. □

6 Example

Suppose that $f\in C^0(R,R)$ satisfies

$$f(x)=\begin{cases} 3\pi x+x^{\frac13}, & |x|\gg 1,\\ \pi x-x^3, & |x|\ll 1.\end{cases}$$

We are to discuss the multiplicity of 12-periodic solutions of the equation

$$x'(t)=-f(x(t-1))+f(x(t-2))-f(x(t-3))+f(x(t-4))-f(x(t-5)). \tag{6.1}$$

In this case, $k=3, \alpha=\pi, \beta=3\pi$. This yields that

$$\begin{aligned} N(\alpha) =&\mathrm{card}\Big\{l\geq 0: 0<\frac{(12l+1)\pi}{6}\cot\frac{\pi}{12}<\pi\Big\}\\ &+\mathrm{card}\Big\{l\geq 0: 0<\frac{(12l+3)\pi}{6}\cot\frac{3\pi}{12}<\pi\Big\}\\ &+\mathrm{card}\Big\{l\geq 0: 0<\frac{(12l+5)\pi}{6}\cot\frac{5\pi}{12}<\pi\Big\}=4,\\ N(\beta) =&\mathrm{card}\Big\{l\geq 0: 0<\frac{(12l+1)\pi}{6}\cot\frac{\pi}{12}<3\pi\Big\}\\ &+\mathrm{card}\Big\{l\geq 0: 0<\frac{(12l+3)\pi}{6}\cot\frac{3\pi}{12}<3\pi\Big\}\\ &+\mathrm{card}\Big\{l\geq 0: 0<\frac{(12l+5)\pi}{6}\cot\frac{5\pi}{12}<3\pi\Big\}=9, \end{aligned}$$

$$N^0(\alpha_+)=N^0(\beta_-)=N^0(\alpha_-)=N^0(\beta_+)=0.$$

Applying Theorem 4.2, we conclude that equation (6.1) possesses at least 5 different 12-periodic orbits satisfying $x(t-6)=-x(t)$.

References

[1] Benci, V. On critical point theory for indefinite functionals in the presence of symmetries. *Trans. Amer. Math. Soc.*, 274: 533–572 (1982)
[2] Fannio, L. Multiple periodic solutions of Hamiltonian systems with strong resonance at infinity. *Discrete and Cont. Dynamical Sys.*, 3: 251–264 (1997)
[3] Fei, G. Multiple periodic solutions of differential delay equations via Hamiltonian systems (I). *Nonlinear Analysis*, 65: 25–39 (2006)
[4] Fei, G. Multiple periodic solutions of differential delay equations via Hamiltonian systems (II). *Nonlinear Analysis*, 65: 40–58 (2006)
[5] Ge, W. Periodic solutions of the differential delay equation $x'(t)=-f(x(t-1))$. *Acta Math. Sinica (New Series)*, 12: 113–121 (1996)
[6] Ge, W. On the existence of periodic solutions of differential delay equations with multiple lags. *Acta Appl. Math. Sinica (in Chinese)*, 17: 173–181 (1994)
[7] Ge, W. Two existence theorems of periodic solutions for differential delay equations. *Chin. Ann. Math.*, 15: 217–224 (1994)
[8] Ge, W. Oscillatory periodic solutions of differential delay equations with multiple lags. *Chin. Sci. Bulltin.*, 42 444–447 (1997)
[9] Ge, W., Zhang, L. Multiple periodic solutions of delay differential systems with $2k-1$ lags via variational approach. *Discrete and Continuous Dynamical Systems*, 36: 4925–4943 (2016)
[10] Guo, Z., Yu, J. Multiple results for periodic solutions to delay differential equations via critical point theory. *JDE*, 218: 15–35 (2005)
[11] Guo, Z., Yu, J. Multiple results on periodic solutions to higher dimensional differential equations with multiple delays. *J. Dyn. Diff. Eqns.*, 23: 1029–1052 (2011)
[12] Li, L., Xue, C., Ge, W. Periodic orbits to Kaplan-Yorke like differential delay equations with two lags of ratio $(2k-1)/2$. *Adv. Difference Equ.*, 247 (2016)
[13] Li, L., Sun, H., Ge, W. On the Number of Periodic Solutions to KaplanCYorke-like High Order Differential Delay Equations with 2k Lags. *International Journal of Bifurcation and Chaos*, 29 (2019)
[14] Li, J., He, X. Proof and generalization of Kaplan-Yorke conjecture under the condition $f'(0)>0$ on periodic solution of differential delay equations. *Sci. China Series A.*, 42: 957–964 (1999)
[15] Li, J., He, X., Liu, Z. Hamiltonian symmetric group and multiple periodic solutions of differential delay equations. *Nonlinear Analysis TMA.*, 35: 957–964 (1999)
[16] Li, S., Liu, J. Morse theory and asymptotically linear Hamiltonian systems. *JDE*, 78: 53–73 (1989)
[17] Kaplan, J., Yorke, J. Ordinary differential equations which yield periodic solution of delay equations. *J. Math. Anal. Appl.*, 48: 317–324 (1974)
[18] Mawhen, J., Willem, M. Critical Point Theory and Hamiltonian Systems. Springer-Verlag. New Yorke, 1989.
[19] Nussbaum, R. Periodic solutions of special differential delay equations: an example in nonlinear functional analysis. *Proc. Royal Soc. Edingburgh.*, 81: 131–151 (1978)