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Dispersion Analysis of Multi-symplectic Scheme for the Nonlinear Schrödinger Equations

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Abstract In this paper, we study the dispersive properties of multi-symplectic discretizations for the nonlinear Schrödinger equations. The numerical dispersion relation and group velocity are investigated. It is found that the numerical dispersion relation is relevant when resolving the nonlinear Schrödinger equations.

Keywords the nonlinear Schrödinger equation;multi-symplectic scheme; dispersion analysis; group velocity
2000 MR Subject Classification 65D17

1 Introduction

A large class of PDEs, such as the KdV equation, the Klein-Gordon equation, the linear and nonlinear Schrödinger equation, etc, can be written as the following PDEs form

$$Mz_t + Kz_x = \nabla_z S(z), \qquad z \in R^n, \quad (x,t) \in R^2, \tag{1.1}$$

where M and K are skew-symmetric matrices on R^n, $n \geq 3$ and $S : R^n \longrightarrow R$ is a smooth function. Eq.(1.1) is called as multi-symplectic Hamiltonian system, since it satisfies a multi-symplectic conservation law [1, 2, 5–11, 13–18]

$$\frac{\partial}{\partial t}\omega + \frac{\partial}{\partial x}\kappa = 0, \tag{1.2}$$

where $\omega = \frac{1}{2} dz \wedge M dz$, $\kappa = \frac{1}{2} dz \wedge K dz$ are two form, $\wedge$ is the standard product of the differential form.

Recently, the multi-symplectic integrators, such as the multi-symplectic Preissman box scheme (MSBS), the multi-symplectic Runge-Kutta method, the multi-symplectic leapfrog scheme, etc, which can preserve the multi-symplectic geometric structure under appropriate discretizations, have been proposed [5, 10, 11, 15, 17, 19, 20]. The MSBS for the nonlinear Schrödinger (NLS) equations was presented [10]. The multi-symplectic integrators have displayed much better numerical behaviors for long time computation. As it is well known, dispersion and group velocity analysis are essential tools in understanding the behavior of discretization of linear and nonlinear wave equation. McLachlan, Frank, Schober, etc analyzed dispersion and group velocity of these multi-symplectic schemes. Theories and numerical results showed the MSBS can well preserve the sign of the group velocity [2–4, 21–23]. We find that the numerical group velocities of the schemes are related to the choice of $\triangle x$ and $\triangle t$ for the linear wave and sine-Gordon equations [12]. In this paper, we investigate the dispersive properties of multi-symplectic discretizations for the NLS equations.

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The rest of the paper is arranged as follows: In Section 2, the numerical dispersion relation and group velocity of the MSBS for the NLS equation is obtained. In section 3, the numerical dispersion relation and group velocity of the MSBS for the coupled nonlinear Schrödinger (C-NLS) equations is obtained. In section 4,we investigate the numerical dispersion effect of the MSBS for the NLS equations.

2 Dispersion of Multi-symplectic Scheme for the NLS Equation

We consider the NLS equation

$$iu_t + u_{xx} + \lambda|u|^2u = 0 \tag{2.1}$$

with the initial condition $u(x,0) = u_0(x)$, $x \in R$ and $\lambda > 0$ is a constant parameter. This equation is one of the most important completely integrable models in the theory of solitons. Its application can be found in many areas of physics , including nonlinear optics and plasma physics. Eq.(2.1) can be expressed in the Hamiltonian system

$$Mz_t + Kz_x = \nabla_z S(z) \tag{2.2}$$

where $z = (p,q,v,w)^T$, $S(z) = \frac{1}{2}(v^2 + w^2 + \frac{\lambda}{2}(p^2+q^2)^2)$, and

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Eq.(2.2) satisfies the multi-symplectic conservation law

$$\partial_t(-dp \wedge dq) + \partial_x(dp \wedge dv + dq \wedge dw) = 0. \tag{2.3}$$

Letting

$$\delta_t^+ z_i^j = \frac{z_i^{j+1} - z_i^j}{\triangle t}, \qquad \delta_x^+ z_i^j = \frac{z_{i+1}^j - z_i^j}{\triangle x}, \quad z_i^{j+\frac{1}{2}} = \frac{1}{2}(z_i^j + z_i^{j+1}),$$
$$z_{i+\frac{1}{2}}^j = \frac{1}{2}(z_i^j + z_{i+1}^j), \qquad z_{i+\frac{1}{2}}^{j+\frac{1}{2}} = \frac{1}{4}(z_i^j + z_i^{j+1} + z_{i+1}^j + z_{i+1}^{j+1}),$$

the MSBS for Eq.(1.1) is

$$\mathbf{K}\delta_t^+ z_{i+\frac{1}{2}}^j + \mathbf{L}\delta_x^+ z_i^{j+\frac{1}{2}} = \nabla_z S(z_{i+\frac{1}{2}}^{j+\frac{1}{2}}). \tag{2.4}$$

The MSBS (2.4) has the discrete multi-symplectic conservation law

$$\delta_t^+ \omega_{i+\frac{1}{2}}^j + \delta_x^+ \kappa_i^{j+\frac{1}{2}} = 0, \tag{2.5}$$

where $\omega_{i+\frac{1}{2}}^j = dz_{i+\frac{1}{2}}^j \wedge \mathbf{K} dz_{i+\frac{1}{2}}^j$, $\kappa_i^{j+\frac{1}{2}} = dz_i^{j+\frac{1}{2}} \wedge \mathbf{L} dz_i^{j+\frac{1}{2}}$.

We obtain a MSBS for the NLS equation

$$i\frac{(u_{l-1}^{n+1} + 2u_l^{n+1} + u_{l+1}^{n+1}) - (u_{l-1}^n + 2u_l^n + u_{l+1}^n)}{2\triangle t}$$
$$+ \frac{u_{l+1}^n + u_{l+1}^{n+1} - 2(u_l^n + u_l^{n+1}) + u_{l-1}^n + u_{l-1}^{n+1}}{\triangle x^2}$$

$$+\lambda\Big(\Big|\frac{u_{l-1}^n+u_l^n+u_{l-1}^{n+1}+u_l^{n+1}}{4}\Big|^2\Big)\frac{u_{l-1}^n+u_l^n+u_{l-1}^{n+1}+u_l^{n+1}}{4} \tag{2.6}$$
$$+\lambda\Big(\Big|\frac{u_l^n+u_{l+1}^n+u_l^{n+1}+u_{l+1}^{n+1}}{4}\Big|^2\Big)\frac{u_l^n+u_{l+1}^n+u_l^{n+1}+u_{l+1}^{n+1}}{4}=0.$$

The NLS equation supports plane wave solutions of the form

$$u(x,t)=ae^{i(\kappa x-\omega t)}, \tag{2.7}$$

where k denotes the wave number and ω denotes the wave frequency. We obtain

$$\omega ae^{i(\kappa x-\omega t)}-k^2ae^{i(\kappa x-\omega t)}+\lambda|a|^2ae^{i(\kappa x-\omega t)}=0. \tag{2.8}$$

We can get the dispersion relation of the NLS equation

$$\omega-\kappa^2+\lambda|a|^2=0. \tag{2.9}$$

We take the numerical solution of Eq.(2.1) to be

$$u_l^n=ae^{i(Kx_l-\Omega t_n)}, \tag{2.10}$$

where $x_l=l\triangle x$, $t_n=n\triangle t$ and K is the numerical wave number, Ω is the numerical frequency such that

$$-\pi\le\triangle xK\le\pi,\qquad -\pi\le\triangle t\Omega\le\pi,\qquad x_l=l\triangle x, t_n=n\triangle t.$$

So we have

$$u_{l-1}^n=ae^{i(Kx_l-\Omega t_n)}e^{-iK\triangle x}=u_l^ne^{-iK\triangle x}, \tag{2.11}$$
$$u_{l-1}^{n+1}=ae^{i(Kx_l-\Omega t_n)}e^{-i(K\triangle x+\Omega\triangle t)}=u_l^ne^{-i(K\triangle x+\Omega\triangle t)}, \tag{2.12}$$
$$u_l^{n+1}=ae^{i(Kx_l-\Omega t_n)}e^{-i\Omega\triangle t}=u_l^ne^{-i\Omega\triangle t}, \tag{2.13}$$
$$u_{l+1}^n=ae^{i(Kx_l-\Omega t_n)}e^{iK\triangle x}=u_l^ne^{iK\triangle x}, \tag{2.14}$$
$$u_{l+1}^{n+1}=ae^{i(Kx_l-\Omega t_n)}e^{i(K\triangle x-\Omega\triangle t)}=u_l^ne^{i(K\triangle x-\Omega\triangle t)}. \tag{2.15}$$

We can get

$$\frac{iu_l^n}{2\triangle t}((e^{-i(K\triangle x+\Omega\triangle t)}+2e^{-i\Omega\triangle t}+e^{i(K\triangle x-\Omega\triangle t)})-(e^{-iK\triangle x}+2+e^{iK\triangle x}))$$
$$+\frac{u_l^n}{\triangle x^2}(e^{iK\triangle x}+e^{i(K\triangle x-\Omega\triangle t)}-2(1+e^{-i\Omega\triangle t})+e^{-iK\triangle x}+e^{-i(K\triangle x+\Omega\triangle t)}) \tag{2.16}$$
$$+\lambda|u_l^n|^2\Big|\frac{e^{-iK\triangle x}+1+e^{-i(K\triangle x+\Omega\triangle t)}+e^{-i\Omega\triangle t}}{4}\Big|^2$$
$$\cdot u_l^n\Big(\frac{e^{-iK\triangle x}+1+e^{-i(K\triangle x+\Omega\triangle t)}+e^{-i\Omega\triangle t}}{4}\Big) \tag{2.17}$$
$$+\lambda|u_l^n|^2\Big|\frac{1+e^{iK\triangle x}+e^{-i\Omega\triangle t}+e^{i(K\triangle x-\Omega\triangle t)}}{4}\Big|^2$$
$$\cdot u_l^n\Big(\frac{1+e^{iK\triangle x}+e^{-i\Omega\triangle t}+e^{i(K\triangle x-\Omega\triangle t)}}{4}\Big)=0. \tag{2.18}$$

We can get the numerical dispersion relation of the MSBS for the NLS equation

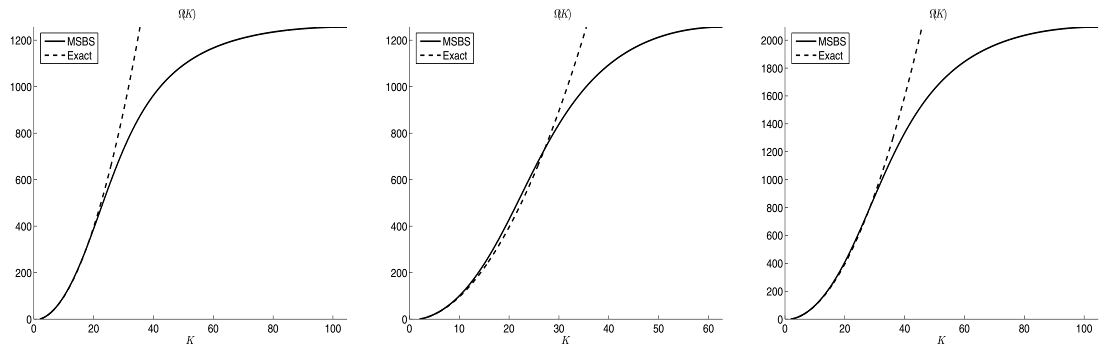
$$\frac{i}{2\triangle t}(e^{-i\Omega\triangle t}-1)(e^{\frac{iK\triangle x}{2}}+e^{-\frac{iK\triangle x}{2}})^2+\frac{1}{\triangle x^2}(e^{-i\Omega\triangle t}+1)(e^{\frac{iK\triangle x}{2}}-e^{-\frac{iK\triangle x}{2}})^2$$

$$+ \lambda|a|^2\Big|\frac{1}{4}(e^{-iK\triangle x}+1)(e^{-i\Omega\triangle t}+1)\Big|^2\frac{1}{4}(e^{-iK\triangle x}+1)(e^{-i\Omega\triangle t}+1)$$
$$+ \lambda|a|^2\Big|\frac{1}{4}(e^{iK\triangle x}+1)(e^{-i\Omega\triangle t}+1)\Big|^2\frac{1}{4}(e^{iK\triangle x}+1)(e^{-i\Omega\triangle t}+1)=0. \tag{2.19}$$

So we can conclude that Ω is a function of K

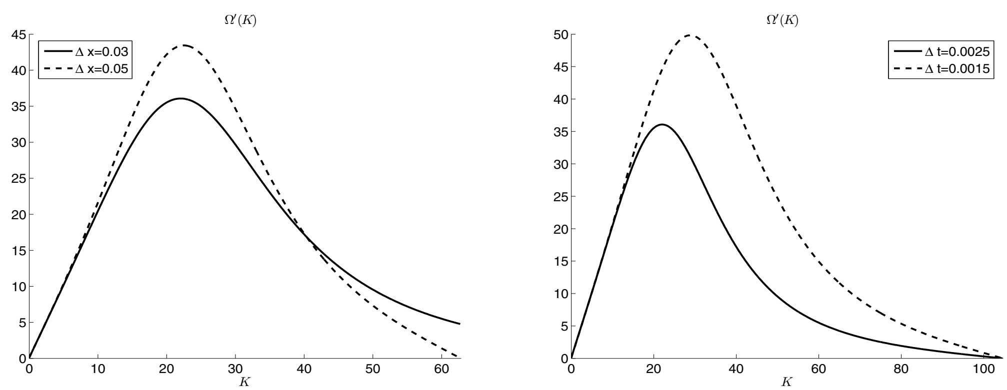
$$\Omega = \Omega(K). \tag{2.20}$$

We give the dispersion properties of the MSBS for the NLS equation by numerical simulations. Figs.(1,2) show the dispersion curves $\Omega(K)$ and the group velocity $\Omega'(K)$ for $\lambda = 2$ with three different values of $\triangle t$ and $\triangle x$. The exact relation is given by $\Omega = K^2 - \lambda|a|^2$. Each plot is shown only for $0 \leq \triangle xK \leq \pi$.



(a) (b) (c)

Fig.1. The dispersion relation for the MSBS discretizations of the NLS equation with (a)$\triangle x = 0.03$, $\triangle t = 0.0025$ (b)$\triangle x = 0.05$, $\triangle t = 0.0025$ (c)$\triangle x = 0.03$, $\triangle t = 0.0015$.



(a) (b)

Fig.2. Group velocities for the MSBS of the NLS equation with (a) $\triangle x = 0.03, 0.05$ and $\triangle t = 0.0025$ (b) $\triangle x = 0.03$ and $\triangle t = 0.015, 0.0025$.

From Fig.1, we can see that the dispersion curves for the MSBS of the NLS equation appear very close for small wave number K with different values of $\triangle t$ and $\triangle x$. And the dispersion curve for the MSBS is monotonically increasing of K given by its numerical group velocities (see Fig.2) So we can conclude that for the NLS equation, higher frequency indicates higher wave number for the MSBS and the exact solution, and the numerical results and the analytical ones will be the same for small wave number.

Fig.2 shows the group velocity for the MSBS for different $\triangle x$ and $\triangle t$ respectively. We can get the relationship between the propagation speed for the MSBS and $\triangle x$ or $\triangle t$. From Fig.2, we can see that with the increasing of the $\triangle x$ ($\triangle t$), the max group velocity increases (decreases). So we can conclude that the max numerical propagation speed for the MSBS is a increasing (decreasing) function of $\triangle x$ ($\triangle t$). Further more, Fig.2 also shows that the group velocities of the MSBS is positive, which shows that the direction of energy transport is preserved.

3 Dispersion of Multi-symplectic Scheme for the CNLS Equations

The following CNLS equations

$$iu_t + u_{xx} + (|u|^2 + \beta|v|^2)u = 0, \tag{3.1}$$

$$iv_t + v_{xx} + (|v|^2 + \beta|u|^2)v = 0, \tag{3.2}$$

is equal to the following forms by $u(x,t) = p(x,t) + q(x,t)i$ and $v(x,t) = \mu(x,t) + \zeta(x,t)i$:

$$i(p_t + q_t i) + p_{xx} + q_{xx} i + ((p^2 + q^2) + \beta(\mu^2 + \zeta^2))(p + qi) = 0, \tag{3.3}$$

$$i(\mu_t + \zeta_t i) + \mu_{xx} + \zeta_{xx} i + ((\mu^2 + \zeta^2) + \beta(p^2 + q^2))(\mu + \zeta i) = 0. \tag{3.4}$$

Eqs.(3.1,3.2) can be rewritten as

$$p_t + q_{xx} + (p^2 + q^2 + \beta(\mu^2 + \zeta^2))q = 0, \qquad q_t - p_{xx} - (p^2 + q^2 + \beta(\mu^2 + \zeta^2))p = 0,$$
$$\mu_t + \zeta_{xx} + (\mu^2 + \zeta^2 + \beta(p^2 + q^2))\zeta = 0, \qquad \zeta_t - \mu_{xx} - (\mu^2 + \zeta^2 + \beta(p^2 + q^2))\mu = 0.$$

The CNLS equations can be expressed in the Hamiltonian system

$$Mz_t + Kz_x = \nabla_z S(z) \tag{3.5}$$

where $z = (p, q, b, a, \mu, \xi, d, c)^T$, $S(z) = \frac{1}{2}(v^2 + w^2 + \frac{\alpha}{2}(p^2 + q^2)^2)$, and

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Eq.(3.5) satisfies the multi-symplectic conservation law

$$\partial_t(-dp \wedge dq - d\mu \wedge d\xi) + \partial_x(dp \wedge dv + dq \wedge dw + d\mu \wedge dd + d\xi \wedge dc) = 0, \tag{3.6}$$

Then we can obtain a MSBS for the CNLS equations

$$i\frac{(u_{l-1}^{n+1} + 2u_l^{n+1} + u_{l+1}^{n+1}) - (u_{l-1}^n + 2u_l^n + u_{l+1}^n)}{2\triangle t} + \frac{u_{l+1}^n + u_{l+1}^{n+1} - 2(u_l^n + u_l^{n+1}) + u_{l-1}^n + u_{l-1}^{n+1}}{\triangle x^2}$$
$$+ \Big(\Big|\frac{u_{l-1}^n + u_l^n + u_{l-1}^{n+1} + u_l^{n+1}}{4}\Big|^2 + \beta\Big|\frac{v_{l-1}^n + v_l^n + v_{l-1}^{n+1} + v_l^{n+1}}{4}\Big|^2\Big)\frac{u_{l-1}^n + u_l^n + u_{l-1}^{n+1} + u_l^{n+1}}{4}$$
$$+ \Big(\Big|\frac{u_l^n + u_{l+1}^n + u_l^{n+1} + u_{l+1}^{n+1}}{4}\Big|^2 + \beta\Big|\frac{v_l^n + v_{l+1}^n + v_l^{n+1} + v_{l+1}^{n+1}}{4}\Big|^2\Big)\frac{u_l^n + u_{l+1}^n + u_l^{n+1} + u_{l+1}^{n+1}}{4}$$

$$= 0, \tag{3.7}$$

$$i\frac{(v_{l-1}^{n+1}+2v_l^{n+1}+v_{l+1}^{n+1})-(v_{l-1}^{n}+2v_l^{n}+v_{l+1}^{n})}{2\triangle t}+\frac{v_{l+1}^{n}+v_{l+1}^{n+1}-2(v_l^n+v_l^{n+1})+v_{l-1}^{n}+v_{l-1}^{n+1}}{\triangle x^2}$$
$$+\Big(\Big|\frac{v_{l-1}^{n}+v_l^{n}+v_{l-1}^{n+1}+v_l^{n+1}}{4}\Big|^2+\beta\Big|\frac{u_{l-1}^{n}+u_l^{n}+u_{l-1}^{n+1}+u_l^{n+1}}{4}\Big|^2\Big)\frac{v_{l-1}^{n}+v_l^{n}+v_{l-1}^{n+1}+v_l^{n+1}}{4}$$
$$+\Big(\Big|\frac{v_l^{n}+v_{l+1}^{n}+v_l^{n+1}+v_{l+1}^{n+1}}{4}\Big|^2+\beta\Big|\frac{u_l^{n}+u_{l+1}^{n}+u_l^{n+1}+u_{l+1}^{n+1}}{4}\Big|^2\Big)\frac{v_l^{n}+v_{l+1}^{n}+v_l^{n+1}+v_{l+1}^{n+1}}{4}$$
$$= 0. \tag{3.8}$$

The CNLS equations supports plane wave solutions of the form

$$u(x,t)=a_1e^{i(\kappa_1x-\omega_1t)},\qquad v(x,t)=a_2e^{i(\kappa_2x-\omega_2t)}. \tag{3.9}$$

where κ_1, κ_2 denote the wave number and ω_1, ω_2 denote the wave frequency. We can obtain

$$\omega_1a_1e^{i(\kappa_1x-\omega_1t)}-k_1^2a_1e^{i(\kappa_1x-\omega_1t)}+(|a_1|^2+\beta|a_2|^2)a_1e^{i(\kappa_1x-\omega_1t)}=0, \tag{3.10}$$
$$\omega_2a_2e^{i(\kappa_2x-\omega_2t)}-k_2^2a_2e^{i(\kappa_2x-\omega_2t)}+(|a_2|^2+\beta|a_1|^2)a_2e^{i(\kappa_2x-\omega_2t)}=0. \tag{3.11}$$

So we can get the dispersion relation of the CNLS equations

$$\omega_1-\kappa_1^2+(|a_1|^2+\beta|a_2|^2)=0, \tag{3.12}$$
$$\omega_2-\kappa_2^2+(|a_2|^2+\beta|a_1|^2)=0. \tag{3.13}$$

We take the numerical solutions of Eqs.(3.1,3.2) to be

$$u_l^n=a_1e^{i(K_1x_l-\Omega_1t_n)}, \tag{3.14}$$
$$v_l^n=a_2e^{i(K_2x_l-\Omega_2t_n)}, \tag{3.15}$$

where $x_l=l\triangle x,\ \ t_n=n\triangle t$ and K_1, K_2 are the numerical wave number and Ω_1, Ω_2 is the numerical frequency such that

$$-\pi\le\triangle xK_1,\quad \triangle xK_2\le\pi,\quad -\pi\le\triangle t\Omega_1,\quad \triangle t\Omega_2\le\pi,\quad x_l=l\triangle x,\quad t_n=n\triangle t.$$

From Eqs.(3.14,3.15), we can get

$$u_{l-1}^n=a_1e^{i(K_1x_l-\Omega_1t_n)}e^{-iK_1\triangle x}=u_l^ne^{-iK_1\triangle x}, \tag{3.16}$$
$$u_{l-1}^{n+1}=a_1e^{i(K_1x_l-\Omega_1t_n)}e^{-i(K_1\triangle x+\Omega_1\triangle t)}=u_l^ne^{-i(K_1\triangle x+\Omega_1\triangle t)}, \tag{3.17}$$
$$u_l^{n+1}=a_1e^{i(K_1x_l-\Omega_1t_n)}e^{-i\Omega_1\triangle t}=u_l^ne^{-i\Omega_1\triangle t}, \tag{3.18}$$
$$u_{l+1}^n=a_1e^{i(K_1x_l-\Omega_1t_n)}e^{iK_1\triangle x}=u_l^ne^{iK_1\triangle x}, \tag{3.19}$$
$$u_{l+1}^{n+1}=a_1e^{i(K_1x_l-\Omega_1t_n)}e^{i(K_1\triangle x-\Omega_1\triangle t)}=u_l^ne^{i(K_1\triangle x-\Omega_1\triangle t)}, \tag{3.20}$$
$$v_{l-1}^n=a_2e^{i(K_2x_l-\Omega_2t_n)}e^{-iK_2\triangle x}=u_l^ne^{-iK_2\triangle x}, \tag{3.21}$$
$$v_{l-1}^{n+1}=a_2e^{i(K_2x_l-\Omega_2t_n)}e^{-i(K_2\triangle x+\Omega_2\triangle t)}=u_l^ne^{-i(K_2\triangle x+\Omega_2\triangle t)}, \tag{3.22}$$
$$v_l^{n+1}=a_2e^{i(K_2x_l-\Omega_2t_n)}e^{-i\Omega_2\triangle t}=u_l^ne^{-i\Omega_2\triangle t}, \tag{3.23}$$
$$v_{l+1}^n=a_2e^{i(K_2x_l-\Omega_2t_n)}e^{iK_2\triangle x}=u_l^ne^{iK_2\triangle x}, \tag{3.24}$$
$$v_{l+1}^{n+1}=a_2e^{i(K_2x_l-\Omega_2t_n)}e^{i(K_2\triangle x-\Omega_2\triangle t)}=u_l^ne^{i(K_2\triangle x-\Omega_2\triangle t)}. \tag{3.25}$$

From Eqs.(3.7,3.8), we can get

$$\frac{iu_l^n}{2\triangle t}((e^{-i(K_1\triangle x+\Omega_1\triangle t)}+2e^{-i\Omega_1\triangle t}+e^{i(K_1\triangle x-\Omega_1\triangle t)})-(e^{-iK_1\triangle x}+2+e^{iK_1\triangle x}))$$

$$
\begin{aligned}
&+\frac{u_l^n}{\triangle x^2}(e^{iK_1\triangle x}+e^{i(K_1\triangle x-\Omega_1\triangle t)}-2(1+e^{-i\Omega_1\triangle t})+e^{-iK_1\triangle x}+e^{-i(K_1\triangle x+\Omega_1\triangle t)})\\
&+\Big(|u_l^n|^2\Big|\frac{e^{-iK_1\triangle x}+1+e^{-i(K_1\triangle x+\Omega_1\triangle t)}+e^{-i\Omega_1\triangle t}}{4}\Big|\\
&+\beta|v_l^n|^2\Big|\frac{e^{-iK_2\triangle x}+1+e^{-i(K_2\triangle x+\Omega_2\triangle t)}+e^{-i\Omega_2\triangle t}}{4}\Big|^2\Big)\\
&\times u_l^n\Big(\frac{e^{-iK_1\triangle x}+1+e^{-i(K_1\triangle x+\Omega_1\triangle t)}+e^{-i\Omega_1\triangle t}}{4}\Big)\\
&+\Big(|u_l^n|^2\Big|\frac{1+e^{iK_1\triangle x}+e^{-i\Omega_1\triangle t}+e^{i(K_1\triangle x-\Omega_1\triangle t)}}{4}\Big|^2\\
&+\beta|v_l^n|^2\Big|\frac{1+e^{iK_2\triangle x}+e^{-i\Omega_2\triangle t}+e^{i(K_2\triangle x-\Omega_2\triangle t)}}{4}\Big|^2\Big)\\
&\times u_l^n\Big(\frac{1+e^{iK_1\triangle x}+e^{-i\Omega_1\triangle t}+e^{i(K_1\triangle x-\Omega_1\triangle t)}}{4}\Big)=0, \qquad (3.26)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{iv_l^n}{2\triangle t}((e^{-i(K_2\triangle x+\Omega_2\triangle t)}+2e^{-i\Omega_2\triangle t}+e^{i(K_2\triangle x-\Omega_2\triangle t)})-(e^{-iK_2\triangle x}+2+e^{iK_2\triangle x}))\\
&+\frac{v_l^n}{\triangle x^2}(e^{iK_2\triangle x}+e^{i(K_2\triangle x-\Omega_2\triangle t)}-2(1+e^{-i\Omega_2\triangle t})+e^{-iK_2\triangle x}+e^{-i(K_2\triangle x+\Omega_2\triangle t)}) \qquad (3.27)\\
&+\Big(|v_l^n|^2\Big|\frac{e^{-iK_2\triangle x}+1+e^{-i(K_2\triangle x+\Omega_2\triangle t)}+e^{-i\Omega_2\triangle t}}{4}\Big|\\
&+\beta|u_l^n|^2\Big|\frac{e^{-iK_1\triangle x}+1+e^{-i(K_1\triangle x+\Omega_1\triangle t)}+e^{-i\Omega_1\triangle t}}{4}\Big|^2\Big)\\
&\times v_l^n\Big(\frac{e^{-iK_2\triangle x}+1+e^{-i(K_2\triangle x+\Omega_2\triangle t)}+e^{-i\Omega_2\triangle t}}{4}\Big)\\
&+\Big(|v_l^n|^2\Big|\frac{1+e^{iK_2\triangle x}+e^{-i\Omega_2\triangle t}+e^{i(K_2\triangle x-\Omega_2\triangle t)}}{4}\Big|^2\\
&+\beta|u_l^n|^2\Big|\frac{1+e^{iK_1\triangle x}+e^{-i\Omega_1\triangle t}+e^{i(K_1\triangle x-\Omega_1\triangle t)}}{4}\Big|^2\Big)\\
&\times v_l^n\Big(\frac{1+e^{iK_2\triangle x}+e^{-i\Omega_2\triangle t}+e^{i(K_2\triangle x-\Omega_2\triangle t)}}{4}\Big)\\
&=0.
\end{aligned}
$$

So we can get

$$
\begin{aligned}
&\frac{i}{2\triangle t}(e^{-i\Omega_1\triangle t}-1)(e^{\frac{iK_1\triangle x}{2}}+e^{-\frac{iK_1\triangle x}{2}})^2+\frac{1}{\triangle x^2}(e^{-i\Omega_1\triangle t}+1)(e^{\frac{iK_1\triangle x}{2}}-e^{-\frac{iK_1\triangle x}{2}})^2\\
&+\Big(|a_1|^2\Big|\frac{1}{4}(e^{-iK_1\triangle x}+1)(e^{-i\Omega_1\triangle t}+1)\Big|^2\\
&+\beta|a_2|^2\Big|\frac{1}{4}(e^{-iK_2\triangle x}+1)(e^{-i\Omega_2\triangle t}+1)|^2\Big)\frac{1}{4}(e^{-iK_1\triangle x}+1)(e^{-i\Omega_1\triangle t}+1)\\
&+\Big(|a_1|^2\Big|\frac{1}{4}(e^{iK_1\triangle x}+1)(e^{-i\Omega_1\triangle t}+1)\Big|^2\\
&+\beta|a_2|^2\Big|\frac{1}{4}(e^{iK_2\triangle x}+1)(e^{-i\Omega_2\triangle t}+1)\Big|^2\Big)\frac{1}{4}(e^{iK_1\triangle x}+1)(e^{-i\Omega_1\triangle t}+1)=0, \qquad (3.28)\\
&\frac{i}{2\triangle t}(e^{-i\Omega_2\triangle t}-1)(e^{\frac{iK_2\triangle x}{2}}+e^{-\frac{iK_2\triangle x}{2}})^2+\frac{1}{\triangle x^2}(e^{-i\Omega_2\triangle t}+1)(e^{\frac{iK_2\triangle x}{2}}-e^{-\frac{iK_2\triangle x}{2}})^2\\
&+\Big(|a_2|^2\Big|\frac{1}{4}(e^{-iK_2\triangle x}+1)(e^{-i\Omega_2\triangle t}+1)\Big|^2
\end{aligned}
$$

$$+ \beta|a_1|^2\Big|\frac{1}{4}(e^{-iK_1\triangle x}+1)(e^{-i\Omega_1\triangle t}+1)\Big|^2\Big)\frac{1}{4}(e^{-iK_2\triangle x}+1)(e^{-i\Omega_2\triangle t}+1)$$
$$+\Big(|a_2|^2\Big|\frac{1}{4}(e^{iK_2\triangle x}+1)(e^{-i\Omega_2\triangle t}+1)\Big|^2$$
$$+ \beta|a_1|^2\Big|\frac{1}{4}(e^{iK_1\triangle x}+1)(e^{-i\Omega_1\triangle t}+1)\Big|^2\Big)\frac{1}{4}(e^{iK_2\triangle x}+1)(e^{-i\Omega_2\triangle t}+1)=0. \quad (3.29)$$

Considering $u(-x,t)=v(x,t)$, we have

$$K_1=-K_2, \qquad \Omega_1=\Omega_2, \qquad |a_1|=|a_2|=|a|.$$

So we can get the numerical dispersion relation of the MSBS for the CNLS equations

$$\frac{i}{2\triangle t}(e^{-i\Omega\triangle t}-1)(e^{\frac{iK\triangle x}{2}}+e^{-\frac{iK\triangle x}{2}})^2+\frac{1}{\triangle x^2}(e^{-i\Omega\triangle t}+1)(e^{\frac{iK\triangle x}{2}}-e^{-\frac{iK\triangle x}{2}})^2$$
$$+|a|^2\Big(\Big|\frac{1}{4}(e^{-iK\triangle x}+1)(e^{-i\Omega\triangle t}+1)|^2$$
$$+\beta\Big|\frac{1}{4}(e^{iK\triangle x}+1)(e^{-i\Omega\triangle t}+1)\Big|^2\Big)\frac{1}{4}(e^{-iK\triangle x}+1)(e^{-i\Omega\triangle t}+1)$$
$$+|a|^2\Big(\Big|\frac{1}{4}(e^{iK\triangle x}+1)(e^{-i\Omega\triangle t}+1)\Big|^2$$
$$+\beta\Big|\frac{1}{4}(e^{-iK\triangle x}+1)(e^{-i\Omega\triangle t}+1)\Big|^2\Big)\frac{1}{4}(e^{iK\triangle x}+1)(e^{-i\Omega\triangle t}+1)=0. \quad (3.30)$$

In the same way as section 2, we can also conclude that Ω is a function of K

$$\Omega=\Omega(K). \quad (3.31)$$

We give the dispersion properties of the MSBS for the CNLS equations by numerical simulations. Figs.(3,4) show the dispersion curves $\Omega(K)$ and the group velocity $\Omega'(K)$ for $\beta=1$ and three different values of $\triangle t$ and $\triangle x$. The exact relation is given by $\Omega=K^2-(1+\beta)|a|^2$. Each plot is shown only for $0\le\triangle xK\le\pi$.

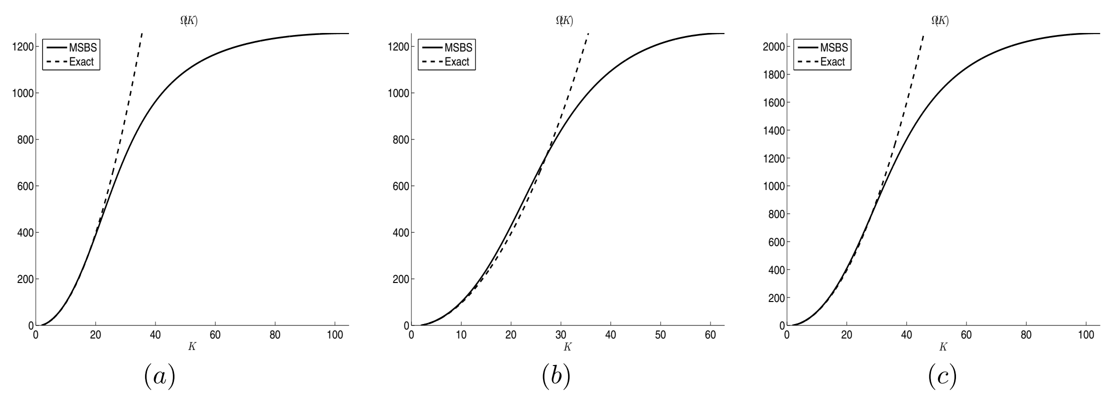
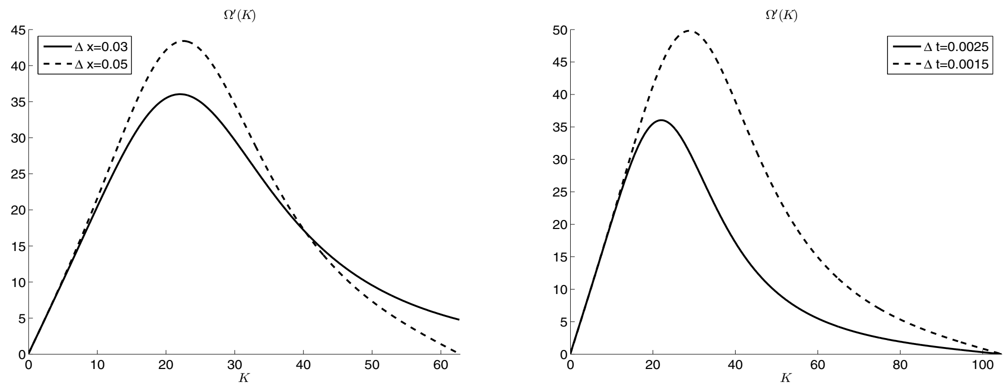


Fig.3. The dispersion relation for the MSBS discretizations of the CNLS equations with (a)$\triangle x=0.03$, $\triangle t=0.0025$ (b)$\triangle x=0.05$, $\triangle t=0.0025$ (c)$\triangle x=0.03$, $\triangle t=0.0015$.

From Fig.3, we can see that the dispersion curves for the MSBS of the CNLS equations appear very close for small wave number K with different values of $\triangle t$ and $\triangle x$. And the dispersion curve for the MSBS is monotonically increasing of K given by its numerical group velocities (see Fig.2) So we can conclude that for the CNLS equations, higher frequency indicates higher wave number for the MSBS and the exact solution, and the numerical results and the analytical ones will be the same for small wave number.



(a) (b)

Fig.4. Group velocities for the MSBS of the CNLS equations with (a) $\triangle x = 0.03, 0.05$ and $\triangle t = 0.0025$ (b) $\triangle x = 0.03$ and $\triangle t = 0.015, 0.0025$.

Fig.4 shows the group velocity for the MSBS for different $\triangle x$ and $\triangle t$ respectively. We can get the relationship between the propagation speed for the MSBS and $\triangle x$ or $\triangle t$. From Fig.4, we can see that with the increasing of the $\triangle x$ ($\triangle t$), the max group velocity increases (decreases). So we can conclude that the max numerical propagation speed for the MSBS is a increasing (decreasing) function of $\triangle x$ ($\triangle t$). Further more, Fig.4 also shows that the group velocities of the MSBS is positive, which shows that the direction of energy transport is preserved.

4 Numerical Examples

4.1 The NLS Equation

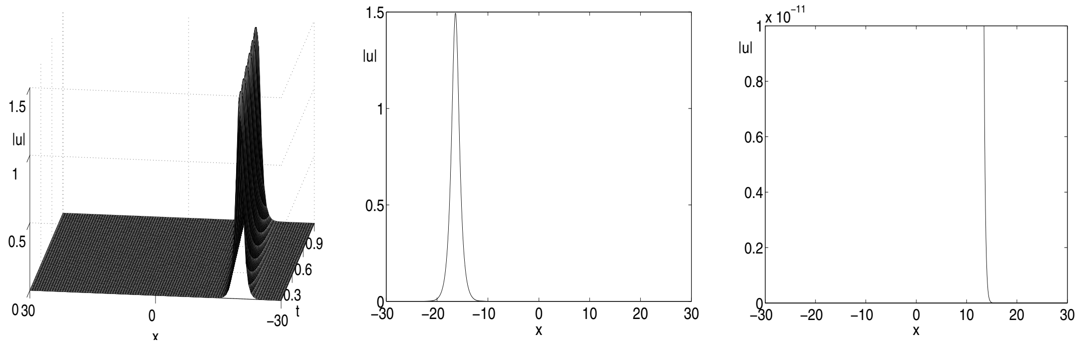
First we consider the NLS equation for initial condition with $\lambda = 2$

$$u(x, 0) = 1.5\text{sech}(1.5x + 30)\exp(2ix). \tag{4.1}$$

The computation is done for $0 \leq t \leq 0.9$, $-30 \leq x \leq 30$, with different $\triangle x$ and $\triangle t$. Fig.5 shows the numerical result with $\triangle x = 0.03$ and $\triangle t = 0.0025$. Fig.5.c shows the propagation of the numerical solution in $0 \leq |u| \leq 1 \times 10^{-11}$. Fig.6 shows the propagation of the numerical solution with $\triangle x = 0.05$ and $\triangle t = 0.0025$. Fig.7 shows the propagation of the numerical solution with $\triangle x = 0.03$ and $\triangle t = 0.0015$. Fig.8 shows the comparison of the dispersion effects with above three different $\triangle x$ and $\triangle t$. From Figs.(5-7), we can see that with $\triangle x(\triangle t)$ increases, the propagation speed of the fast mode increases(decreases), because the max numerical propagation speed for the MSBS is a increasing (decreasing) function of $\triangle x$ ($\triangle t$). From Fig.8, we can see that though the propagation of the small wave number modes appear very close, there is a apparent difference within higher ones as predicted by the analysis (see Fig.2). And we can see

$$V(0.03, 0.0025) < V(0.05, 0.0025) < V(0.03, 0.0015), \tag{4.2}$$

where $V(\triangle x, \triangle t)$ is the propagation speed of the fast mode. Eq.(4.2) can also be get from Fig.2.



(a) (b) (c)

Fig.5. Dispersion effects in the numerical solutions of the MSBS for the NLS equation with $\triangle x = 0.03$, $\triangle t = 0.0025$, $0 \leq t \leq 0.9$, (a) $0 \leq |u| \leq 1.5$, and $t = 0.9$, (b) $0 \leq |u| \leq 1.5$,(c) $0 \leq |u| \leq 1 \times 10^{-11}$.

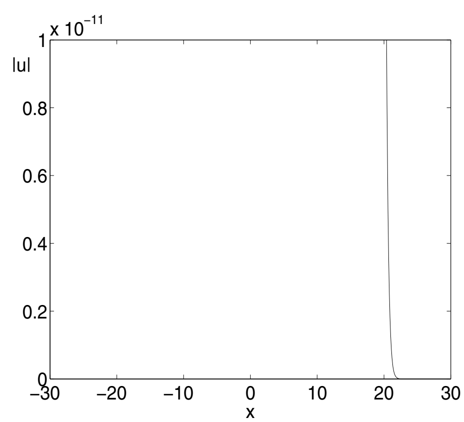


Fig.6. Dispersion effects in the numerical solutions of the MSBS for the NLS equation with $\triangle x = 0.05$, $\triangle t = 0.0025$, $t = 0.9$, $0 \leq |u| \leq 1 \times 10^{-11}$.

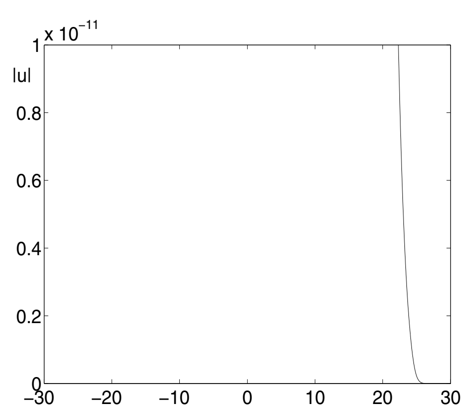
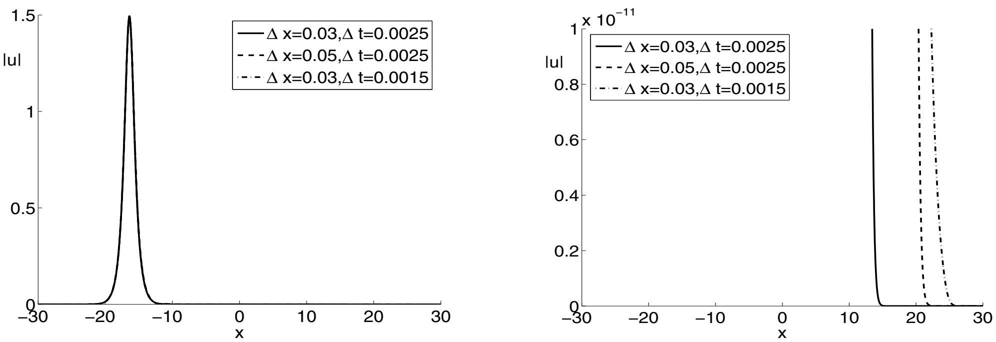


Fig.7. Dispersion effects in the numerical solutions of the MSBS for the NLS equation with $\triangle x = 0.03$, $\triangle t = 0.0015$, $t = 0.9$, $0 \leq |u| \leq 1 \times 10^{-11}$.



(*a*) (*b*)

Fig.8. Comparison of the dispersion effects with three different $\triangle x$ and $\triangle t$,(a) $0 \leq |u| \leq 1.5$, (b) $0 \leq |u| \leq 1 \times 10^{-11}$.

4.2 The CNLS Equations

Then we consider the CNLS equations for the initial condition with $\beta = 1$

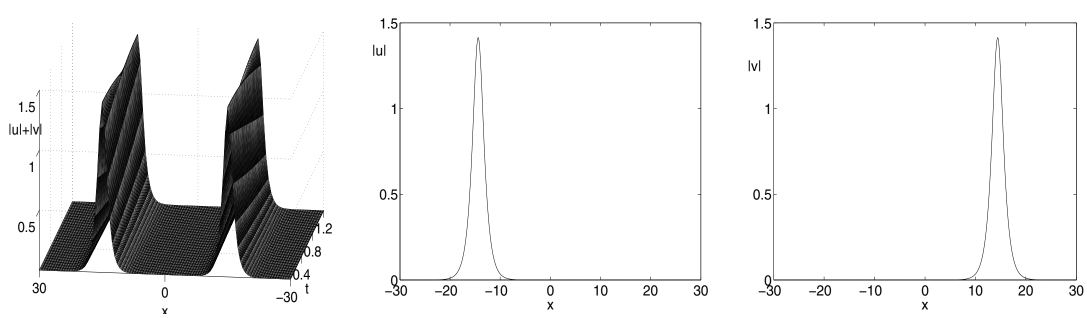
$$u(x, 0) = \sqrt{2}\text{sech}(x + 15) \exp(ix/4), \tag{4.3}$$

$$v(x, 0) = \sqrt{2}\text{sech}(x - 15) \exp(-ix/4). \tag{4.4}$$

The computation is done for $0 \leq t \leq 1.2$, $-30 \leq x \leq 30$, with different $\triangle x$ and $\triangle t$. Fig.9 shows the numerical result with $\triangle x = 0.03$ and $\triangle t = 0.0025$. Fig.10 shows the propagation of the numerical solution in $0 \leq |u| \leq 1 \times 10^{-7}$. Fig.11 shows the propagation of the numerical solution with $\triangle x = 0.05$ and $\triangle t = 0.0025$. Fig.12 shows the propagation of the numerical solution with $\triangle x = 0.03$ and $\triangle t = 0.0015$. From Figs. 10–12, we can see that as $\triangle x(\triangle t)$ increases, the propagation speed of the fast mode increases(decreases), because the max numerical propagation speed for the MSBS scheme is a increasing (decreasing) function of $\triangle x$ ($\triangle t$). Though the propagation of the small wave number modes appear very close, there is a apparent difference within higher ones as predicted by the analysis (see Fig.2). And we can see

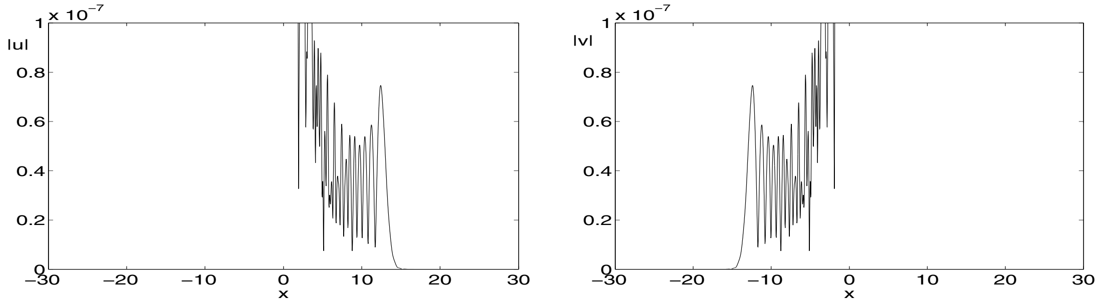
$$V(0.03, 0.0025) < V(0.05, 0.0025) < V(0.03, 0.0015), \tag{4.5}$$

where $V(\triangle x, \triangle t)$ is the propagation speed of the fast mode. Eq.(4.5) can also be get from Fig.4. And the modes with higher or lower wave number all travel slower than the fast one, because the group velocity of the MSBS is not a monotonic function of K (see Fig.4).



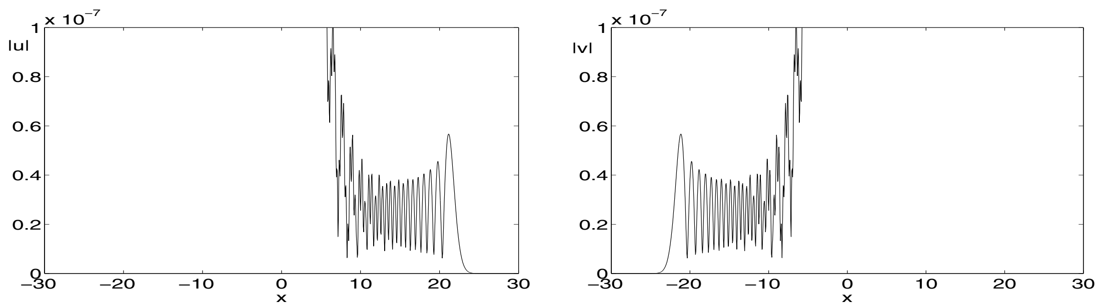
(*a*) (*b*) (*c*)

Fig.9. Numerical result of the MSBS for the CNLS equations with $\triangle x = 0.03$ and $\triangle t = 0.0025$, $0 \leq t \leq 1.2$, (a) $|u| + |v|$, and $t = 1.2$, (b) $|u|$, (c) $|v|$.



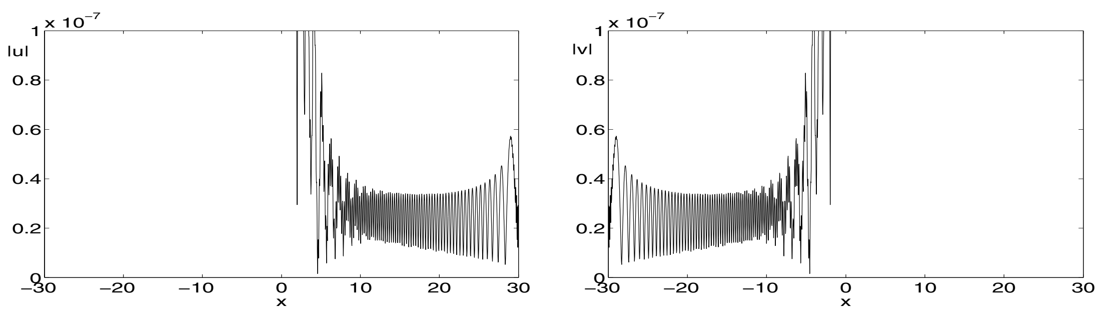
(a) (b)

Fig.10. Dispersion effects in the numerical solutions of the MSBS for the CNLS equations with $\triangle x = 0.03$, $\triangle t = 0.0025$, $t = 1.2$,(a) $0 \leq |u| \leq 1 \times 10^{-7}$, (b) $0 \leq |v| \leq 1 \times 10^{-7}$.



(a) (b)

Fig.11. Dispersion effects in the numerical solutions of the MSBS for the CNLS equations with $\triangle x = 0.05$, $\triangle t = 0.0025$, $t = 1.2$, (a) $0 \leq |u| \leq 1 \times 10^{-7}$, (b) $0 \leq |v| \leq 1 \times 10^{-7}$.



(a) (b)

Fig.12. Dispersion effects in the numerical solutions of the MSBS for the CNLS equations with $\triangle x = 0.03$, $\triangle t = 0.0015$, $t = 1.2$, (a) $0 \leq |u| \leq 1 \times 10^{-7}$,(b) $0 \leq |v| \leq 1 \times 10^{-7}$.

5 Conclusions

In this paper, we study the dispersive properties of multi-symplectic discretizations for the NLS equations. The numerical dispersion relation and group velocity are investigated. We find that the numerical group velocities of the schemes are related to the choice of $\triangle x$ and $\triangle t$ for the NLS equations, and the numerical results confirm it. Numerical results also show that the propagation of the numerical solutions of the MSBS for the NLS equations are dependent on the choice of $\triangle x$ and $\triangle t$. The numerical dispersion relation is relevant when resolving the NLS equations.

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