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Nonconforming Finite Element Methods for the Constrained Optimal Control Problems Governed by Nonsmooth Elliptic Equations

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Abstract In this paper, nonconforming finite element methods (FEMs) are proposed for the constrained optimal control problems (OCPs) governed by the nonsmooth elliptic equations, in which the popular EQ_1^{rot} element is employed to approximate the state and adjoint state, and the piecewise constant element is used to approximate the control. Firstly, the convergence and superconvergence properties for the nonsmooth elliptic equation are obtained by introducing an auxiliary problem. Secondly, the goal-oriented error estimates are obtained for the objective function through establishing the negative norm error estimate. Lastly, the methods are extended to some other well-known nonconforming elements.

Keywords nonconforming finite element; supercloseness and superconvergence; optimal control problems; nonsmooth elliptic equations; goal-oriented error estimate

2000 MR Subject Classification 65N30; 65N15

1 Introduction

The constrained OCPs play a crucial role in many science and engineering applications (cf. [2, 3]). But the exact solution is always difficult to be obtained, so researching the corresponding numerical algorithms becomes very meaningful. For instance, [5] firstly provided the optimality conditions and an important theoretical analysis for OCPs. Based on this research, [6] proved convergence of finite element approximations to OCPs for semi-linear elliptic equations with finitely many state constraints. Later, [7] extended these results to a less regular setting for the states, and gave the convergence analysis of FEMs for semi-linear distributed and boundary control problems. On the other hand, [17] proposed a discretization concept which utilizes for the discretization of the control variable the relation between adjoint state and control. The key feature is not to discretize the space of admissible control, but to implicitly utilize the first order optimality conditions and the discretization of the state and adjoint equations for the discretization of the control. Moreover, the linear element was used to discretize the state equation and the error estimate of L^2-norm was obtained in [12]. Recently, there appeared a lot of studies focusing on the conforming FEMs for OCPs governed by elliptic equations, Stokes equations, convection-dominated diffusion equations, and so on (cf. [22, 23, 37]).

In this paper, we consider the following OCPs with state constrained: find $(y,u)\in Y\times K^*$, such that

$$J(y,u)=\min_{u\in U_{ad}}\Big\{\frac{1}{2}\int_\Omega (y-y_0)^2dx+\frac{\alpha}{2}\int_\Omega u^2dx\Big\}, \tag{1.1}$$

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subject to

$$\begin{cases} -\Delta y + \beta \max(0, y) = u, & \text{in } \Omega, \\ y = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

and

$$U_{ad} = \{v \in L^2(\Omega) : v \le \varphi, \ \text{a.e. in } \Omega\}, \tag{1.3}$$

where Ω is a bounded convex polygon in R^2, $x = (x_1, x_2) \in \Omega$, $\varphi \in L^\infty(\Omega)$, $Y = H_0^1(\Omega) \cap H^3(\Omega)$. α and β are positive constant parameters, $\varphi|_{\partial\Omega} > 0$ and y_0 are given functions. Throughout this paper, the Sobolev spaces and norms are both standard (see [4]).

As we know, the above state equations have been applied in many practical problems, such as equilibrium analysis of confined magnetohydrodynamics plasmas[11], thin stretched membranes partially covered with water[19], reaction-diffusion problems[1], and so on. For the nonsmooth elliptic equations (1.2), [8] investigated a conforming linear FEM and derived the optimal order error estimates both in L^2-norm and H^1-norm by introducing an auxiliary problem. [10] studied the first order conditions, which can be used to verify that a point satisfying the so-called first order conditions is a global or local optimal solution of the OCP(1.1) governed by (1.2). But since the problem is nonsmooth and globally nonconvex, the solution may not be unique in many cases and the first order condition is neither sufficient nor necessary for the original problem. Therefore the H^1-norm and L^2-norm finite element error estimates are not proper for it is not clear that which specific solution is approximated by the numerical solution. Fortunately, [38] gave a conforming FEM motivated by the analysis of [8], and obtained the goal-oriented error estimates, which also indicated that the finite element solution tended to an exact solution in the sense of goal function.

However, the works mentioned above are mainly restricted to the conforming FEMs. In fact, nonconforming elements have attracted much more attentions of the engineers and scholars. Roughly speaking, nonconforming elements have at least two remarkable advantages comparing with the conforming ones. The first aspect is that they are usually easier to be constructed to satisfy the celebrated discrete Babuska-Brezzi, or inf-sup stability condition, which is usually required in the mixed FEMs[4]. The second aspect comes from the domain decomposition point of view. For some Crouzeix-Raviart type elements with the degrees of freedom defined on the edges (or faces) of element, since the unknowns are associated with the element edges or faces, each degree of freedom belongs to at most two elements, the use of the nonconforming elements facilitates the exchange of information across each subdomain and provides spectral radius estimates for the iterative domain decomposition operator[13]. Especially, the nonconforming EQ_1^{rot} element has been applied to many problems. For example, [21] studied its superconvergence properties for the second order elliptic problems, [25] applied this element to solve diffusion-convection-reaction equation, [29] considered its superconvergence behaviors on anisotropic meshes, and [36] used it to deal with the Signorini problem and obtained the global superconvergence results. Furthermore, this element was also employed to solve the Maxwell's equations[30], nonlinear Sobolev equation[33] and some other different problems[26, 27]. Recently, we also researched the NFEM and mixed FEM for stationary OCPs, and obtained the superconvergence results and optimal order error estimates in [14–16], respectively. But unfortunately, the method does not work in this kind of nonsmooth OCPs.

The aim of this paper is to investigate the nonconforming FEMs for the constrained OCPs governed by the nonlinear elliptic equations and derive the error estimates. The rest of this paper is organized as follows. In Section 2, we present the constructions of the elements and some special properties. In Section 3, by introducing an auxiliary problem, we establish the convergence and superconvergence analysis of the nonsmooth elliptic problem (1.2). In Section 4, we provide the goal-oriented error estimates for the objective function. In the last section, we will extend the results to some other popular nonconforming elements.

2 The Discrete Formulation

Let T_h be a family of rectangular subdivision of Ω, h be the mesh size. For any given $K \in T_h$, let l_i be the four edges of K $(i = 1, 2, 3, 4)$. We define the following FE spaces U_h and V_h as

$$U_h = \{v \in L^2(\Omega),\ v|_K \in \mathrm{span}(1),\ \forall\, K \in T_h\}, \tag{2.1}$$

$$V_h = \Big\{v \in L^2(\Omega),\ v|_K \in \mathrm{span}(1, x_1, x_2, x_1^2, x_2^2),\ \int_F [v]ds = 0,\ F \subset \partial K,\ \forall\, K \in T_h\Big\}, \tag{2.2}$$

where $[v]$ denotes the jump of v across the edge F if F is an internal edge, and it is equal to v itself if F is a boundary edge. Let $K_h = U_{ad} \cap U_h$, $\overline{\Pi}_h$ and Π_h be the associated interpolation operators over U_h and V_h, respectively, where $\overline{\Pi}_h|_K = \overline{\Pi}_K$ and $\Pi_h|_K = \Pi_K$ satisfy

$$\begin{cases} \displaystyle\int_K (v - \overline{\Pi}_K v)dx = 0, & \forall\, v \in L^2(\Omega), \\ \displaystyle\int_K (v - \Pi_K v)dx = 0, & \forall\, v \in H^1(\Omega), \\ \displaystyle\int_{l_i} (v - \Pi_K v)ds = 0, \quad i = 1, 2, 3, 4, & \forall\, v \in H^1(\Omega). \end{cases} \tag{2.3}$$

By the interpolation theory, we have

$$\|u - \overline{\Pi}_h u\|_0 \le ch\|u\|_1, \qquad \forall\, u \in H^1(\Omega) \tag{2.4}$$

and

$$\|y - \Pi_h y\|_0 + h\|y - \Pi_h y\|_h \le ch^2|y|_2, \qquad \forall\, y \in H^2(\Omega), \tag{2.5}$$

here and later, $\|\cdot\|_h = (\sum\limits_K |\cdot|_{1,K})^{1/2}$ is a broken energy norm on V_h.

It is easy to check that if we set $a_h(y, v) = \sum\limits_K \int_K \nabla y \nabla v dx$, there holds the following property:

$$a_h(y - \Pi_h y, v_h) = 0, \qquad \forall\, v_h \in V_h. \tag{2.6}$$

In fact, because Δv_h is a constant on K and $\int_K (y - \Pi_h y)dx = \int_{\partial K}(y - \Pi_h y)ds = 0$, we can derive that

$$a_h(y - \Pi_h y, v_h) = \sum_K \int_{\partial K} \frac{\partial v_h}{\partial n}(y - \Pi_h y)ds - \sum_K \int_K \Delta v_h (y - \Pi_h y)dx = 0.$$

Lemma 2.1[29]**.** *Let $y \in H^2(\Omega) \cap H_0^1(\Omega)$, then for all $v_h \in V_h$, we have*

$$\Big|\sum_K \int_{\partial K} \frac{\partial y}{\partial n} v_h ds\Big| \le \mathrm{ch}\|y\|_2\|v_h\|_h. \tag{2.7}$$

Furthermore, assume that $y \in H^3(\Omega) \cap H_0^1(\Omega)$, there holds

$$\Big|\sum_K \int_{\partial K} \frac{\partial y}{\partial n} v_h ds\Big| \le \mathrm{ch}^2\|y\|_3\|v_h\|_h. \tag{2.8}$$

3 Convergence and Superconvergence Analysis of the Nonsmooth Elliptic Problem

The standard weak variational form of (1.2) reads as: find $y \in Y$, such that

$$a(y,v) + \beta(\max(0,y),v) = (u,v), \qquad \forall\, v \in H_0^1(\Omega), \tag{3.1}$$

where $a(y,v) = \int_\Omega \nabla y \nabla v dx,\ \ (u,v) = \int_\Omega uv dx$.

The corresponding nonconforming FE discrete form reads as: find $(y_h,u_h) \in V_h \times K_h$, such that

$$J_h(y_h,u_h) = \min_{u_h \in K_h} \Big\{ \frac{1}{2}\int_\Omega (y_h - y_0)^2 dx + \frac{\alpha}{2}\int_\Omega u_h^2 dx \Big\}, \tag{3.2}$$

subject to

$$a_h(y_h,v_h) + \beta(\max(0,y_h),v_h) = (u_h,v_h), \qquad \forall\, v_h \in V_h. \tag{3.3}$$

With the purpose of estimating the errors between the solutions of (1.1)–(1.2) and (3.2)–(3.3), we consider the FE analysis of the following nonsmooth elliptic problem: find $w \in H_0^1(\Omega) \cap H^2(\Omega)$, such that

$$\begin{cases} -\Delta w + \beta \max(0,w) = f, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where $f \in L^2(\Omega)$ is the source term.

The variational form of (3.4) reads as: find $w \in H_0^1(\Omega)$, such that

$$a(w,v) + (\beta \max(0,w),v) = (f,v), \qquad \forall\, v \in H_0^1(\Omega). \tag{3.5}$$

The nonconforming EQ_1^{rot} FE discrete form of (3.5) reads as: find $w_h \in V_h$, such that

$$a_h(w_h,v_h) + (\beta \max(0,w_h),v_h) = (f,v_h), \qquad \forall\, v_h \in V_h. \tag{3.6}$$

To discuss the errors between w and w_h, we introduce the following auxiliary problem: find $\widetilde{w}_h \in V_h$, such that

$$a_h(\widetilde{w}_h,v_h) + (\beta \max(0,w),v_h) = (f,v_h), \qquad \forall\, v_h \in V_h. \tag{3.7}$$

Lemma 3.1. *Let $w \in H_0^1(\Omega) \cap H^2(\Omega)$, $w_h \in V_h$ and $\widetilde{w}_h \in V_h$ be the solutions of (3.5), (3.6) and (3.7), respectively, then we have*

$$\|\widetilde{w}_h - w_h\|_h \le \sqrt{\beta}\|w - \widetilde{w}_h\|_0. \tag{3.8}$$

Proof. Subtracting (3.6) from (3.7), we have

$$\begin{aligned} &\|\widetilde{w}_h - w_h\|_h^2 + \beta\|\max(0,w) - \max(0,w_h)\|_0^2 \\ =&\beta(\max(0,w_h) - \max(0,w), \widetilde{w}_h - w_h) \\ &+ \beta(\max(0,w) - \max(0,w_h), \max(0,w) - \max(0,w_h)). \end{aligned} \tag{3.9}$$

Noticing that

$$\begin{aligned} &\beta(\max(0,w) - \max(0,w_h), \max(0,w) - w) \\ &+ \beta(\max(0,w) - \max(0,w_h), w_h - \max(0,w_h)) \end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} (w-w_h,0)+(w-w_h,0)=0, & \text{if } y\geq 0 \quad \text{and} \quad y_h\geq 0;\\ (w,0)+(w,w_h)=(w,w_h), & \text{if } y\geq 0 \quad \text{and} \quad y_h\leq 0;\\ (-w_h,-w)+(-w_h,0)=(w,w_h)\leq 0, & \text{if } y<0 \quad \text{and} \quad y_h\geq 0;\\ (0,-w)+(0,w_h)=0, & \text{if } y<0 \quad \text{and} \quad y_h<0 \end{cases}\\
&\leq 0, \qquad (3.10)
\end{aligned}
$$

we have

$$
\begin{aligned}
&\beta(\max(0,w)-\max(0,w_h),\max(0,w)-\max(0,w_h)) \qquad (3.11)\\
\leq&\beta(\max(0,w)-\max(0,w_h),w-w_h). \qquad (3.12)
\end{aligned}
$$

Substituting (3.11) to (3.9), we can derive that

$$
\begin{aligned}
&\|\widetilde{w}_h-w_h\|_h^2+\beta\|\max(0,w)-\max(0,w_h)\|_0^2\\
\leq&\beta(\max(0,w_h)-\max(0,w),\widetilde{w}_h-w_h)+\beta(\max(0,w)-\max(0,w_h),w-w_h)\\
=&\beta(\max(0,w)-\max(0,w_h),w-\widetilde{w}_h)\\
\leq&\beta\|\max(0,w)-\max(0,w_h)\|_0\|w-\widetilde{w}_h)\|_0\\
\leq&\beta\|\max(0,w)-\max(0,w_h)\|_0^2+\beta\|w-\widetilde{w}_h\|_0^2, \qquad (3.13)
\end{aligned}
$$

which completes the proof. □

Lemma 3.2. *Under the assumptions of Lemma 3.1, we have*

$$\|w-\widetilde{w}_h\|_h\leq \mathrm{ch}|w|_2 \qquad (3.14)$$

and

$$\|w-\widetilde{w}_h\|_0\leq \mathrm{ch}^2|w|_2. \qquad (3.15)$$

Proof. Subtracting (3.5) from (3.7), there yields

$$a_h(w,v_h)-a_h(\widetilde{w}_h,v_h)=\sum_K\int_{\partial K}\frac{\partial w}{\partial n}v_h ds, \qquad \forall\, v_h\in V_h. \qquad (3.16)$$

Noticing that (2.7), we have

$$
\begin{aligned}
\|w-\widetilde{w}_h\|_h^2=&a_h(w-\widetilde{w}_h,w-\widetilde{w}_h)\\
=&\sum_K\int_{\partial K}\frac{\partial w}{\partial n}(\Pi_h w-\widetilde{w}_h)ds\leq \mathrm{ch}|w|_2\|\Pi_h w-\widetilde{w}_h\|_h, \qquad (3.17)
\end{aligned}
$$

which leads to the result of (3.14).

Now we start to prove (3.15). Let $\varphi\in H^2(\Omega)\cap H_0^1(\Omega)$ be the solution of the following auxiliary problem:

$$a(\varphi,v)=(\Pi_h w-\widetilde{w}_h,v), \qquad \forall\, v\in H_0^1(\Omega) \qquad (3.18)$$

satisfying

$$|\varphi|_2\leq c\|\Pi_h w-\widetilde{w}_h\|_0. \qquad (3.19)$$

Then, using (2.6)–(2.7) and (3.16), there holds

$$
\begin{aligned}
&\|\Pi_h w-\widetilde{w}_h\|_0^2=(\Pi_h w-\widetilde{w}_h,\Pi_h w-\widetilde{w}_h)\\
=&a_h(\varphi,\Pi_h w-\widetilde{w}_h)=a_h(\varphi-\Pi_h\varphi,\Pi_h w-\widetilde{w}_h)+a_h(\Pi_h\varphi,\Pi_h w-\widetilde{w}_h)
\end{aligned}
$$

$$
\begin{aligned}
&= a_h(\Pi_h\varphi, w-\widetilde{w}_h) = a_h(\Pi_h\varphi-\varphi, w-\widetilde{w}_h) \le \mathrm{ch}\,|w|_2\|\Pi_h\varphi-\varphi\|_h \le \mathrm{ch}^2|w|_2|\varphi|_2 \\
&\le \mathrm{ch}^2|w|_2\|\Pi_h w-\widetilde{w}_h\|_0,
\end{aligned} \tag{3.20}
$$

which together with (2.5) leads to (3.15). The proof is thus completed. □

Lemma 3.3. *Let $w \in Y$, $w_h \in V_h$ and $\widetilde{w}_h \in V_h$ be the solutions of (3.5), (3.6) and (3.7), respectively, then we have*

$$
\|\Pi_h w-\widetilde{w}_h\|_h \le ch^2|w|_3. \tag{3.21}
$$

Proof. Subtracting (3.5) from (3.7), and using the results of (2.6) and (2.8), we can obtain that

$$
\begin{aligned}
&\|\Pi_h w-\widetilde{w}_h\|_h^2 = a_h(\Pi_h w-\widetilde{w}_h, \Pi_h w-\widetilde{w}_h) \\
&= a_h(w-\widetilde{w}_h, \Pi_h w-\widetilde{w}_h) = \sum_K \int_{\partial K} \frac{\partial w}{\partial n}(\Pi_h w-\widetilde{w}_h)ds \\
&\le ch^2|w|_3\|\Pi_h w-\widetilde{w}_h\|_h,
\end{aligned} \tag{3.22}
$$

the desired result follows. □

Theorem 3.1. *Under the assumptions of Lemma 3.1, there hold:*

$$
\|w-w_h\|_h \le c(h+\sqrt{\beta}h^2)|w|_2 \tag{3.23}
$$

and

$$
\|w-w_h\|_0 \le c(h^2+\sqrt{\beta}h^2)|w|_2. \tag{3.24}
$$

Proof. By Lemmas 3.1–3.2, we have

$$
\begin{aligned}
\|w-w_h\|_h \le& \|w-\widetilde{w}_h\|_h + \|\widetilde{w}_h-w_h\|_h \\
&\le \mathrm{ch}\,|w|_2 + c\sqrt{\beta}\|w-\widetilde{w}_h\|_0 \le \mathrm{ch}\,|w|_2 + c\sqrt{\beta}h^2|w|_2
\end{aligned} \tag{3.25}
$$

and

$$
\begin{aligned}
\|w-w_h\|_0 \le& \|w-\widetilde{w}_h\|_0 + \|\widetilde{w}_h-w_h\|_0 \\
&\le \mathrm{ch}^2|w|_2 + c\sqrt{\beta}\|w-\widetilde{w}_h\|_0 \le \mathrm{ch}^2|w|_2 + c\sqrt{\beta}h^2|w|_2.
\end{aligned} \tag{3.26}
$$

The proof is thus completed. □

Theorem 3.2. Under the assumptions of Lemma 3.3, there holds the following superclose result:

$$
\|\Pi_h w-w_h\|_h \le ch^2|w|_3. \tag{3.27}
$$

Proof. By Lemmas 3.1-3.3, and the triangle inequality, we have

$$
\begin{aligned}
\|\Pi_h w-w_h\|_h \le& \|\Pi_h w-\widetilde{w}_h\|_h + \|\widetilde{w}_h-w_h\|_h \\
&\le \|\Pi_h w-\widetilde{w}_h\|_h + \sqrt{\beta}\|w-\widetilde{w}_h\|_0 \le \mathrm{ch}^2|w|_3.
\end{aligned} \tag{3.28}
$$

The proof is completed. □

In order to obtain the global superconvergence result, we combine four neighbouring elements $K_1,\ K_2,\ K_3,\ K_4 \in T_h$ into a big rectangular element K_0 (see Figure 1). T_{2h} respects the corresponding new partition[21]. We construct the interpolated postprocessing operator Π_{2h} as follows.

$$\begin{cases} \Pi_{2h} w|_{K_0} \in P_2(K_0), \qquad \forall\, K_0 \in T_{2h}, \\ \displaystyle\int_{L_i} (\Pi_{2h} w - w) ds = 0, \qquad i = 1,2,3,4, \\ \displaystyle\int_{K_1 \cup K_3} (\Pi_{2h} w - w) dx = 0, \qquad \int_{K_2 \cup K_4} (\Pi_{2h} w - w) dx = 0, \qquad \forall\, K_0 \in T_{2h}, \end{cases}$$

in which L_i $(i = 1, 2, 3, 4)$ are the four edges of K_0, P_2 denotes the set of polynomials of degree 2.

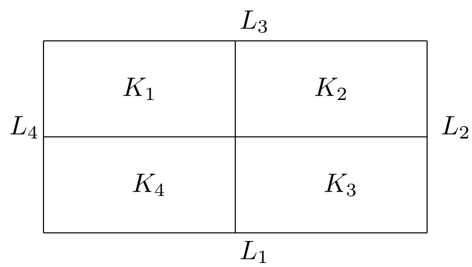

Figure 1. Big element K_0

It can be validated that Π_{2h} is well-posed and has the following properties[33]:

$$\begin{cases} \Pi_{2h} \Pi_h w = \Pi_{2h} w, & \forall\, w \in H^2(\Omega), \\ \|\Pi_{2h} w - w\|_h \le ch^r |w|_{r+1}, & \forall\, w \in H^{r+1}(\Omega), \quad 0 \le r \le 2, \\ \|\Pi_{2h} v_h\|_h \le c\|v_h\|_h, & \forall\, v_h \in V_h. \end{cases} \tag{3.29}$$

Theorem 3.3. *Under the assumptions of Theorem 3.2, we have the following superconvergence result*

$$\|w - \Pi_{2h} w_h\|_h \le \mathrm{ch}^2 |w|_3. \tag{3.30}$$

Proof. By (3.29), we get

$$\|\Pi_{2h} \Pi_h w - \Pi_{2h} w_h\|_h = \|\Pi_{2h}(\Pi_h w - w_h)\|_h \le c\|\Pi_h w - w_h\|_h \le \mathrm{ch}^2 |w|_3$$

and

$$\|w - \Pi_{2h} \Pi_h w\|_h = \|w - \Pi_{2h} w\|_h \le \mathrm{ch}^2 |w|_3^2.$$

So we have

$$\begin{aligned} \|w - \Pi_{2h} w_h\|_h &= \|w - \Pi_{2h} \Pi_h w + \Pi_{2h} \Pi_h w - \Pi_{2h} w_h\|_h \\ &\le \|w - \Pi_{2h} \Pi_h w\|_h^2 + \|\Pi_{2h} \Pi_h w - \Pi_{2h} w_h\|_h \le \mathrm{ch}^2 |w|_3. \end{aligned}$$

□

4 Goal-oriented Error Estimates

As mentioned in the previous sections that analysis of the standard error estimate under H^1-norm or L_2-norm is not proper, we resort to the goal-oriented error estimates with the form of

$|J(y,u)-J_h(y_h,u_h)|$ by studying $J(y,u)-J_h(y_h,u_h)$ and $J_h(y_h,u_h)-J(y,u)$ in the following two lemmas.

Lemma 4.1. *Let (y,u) and (y_h,u_h) be the solutions of (1.1) and (3.2), respectively, then we have*

$$J(y,u)-J_h(y_h,u_h)\le \mathrm{ch}^2. \tag{4.1}$$

Proof. Noting the fact that (y,u) is the solution of (1.1), it is obviously right that

$$J(y,u)\le J(y(u_h),u_h), \tag{4.2}$$

where $y(u_h)$ is the solution of

$$a(y(u_h),v)+\beta(\max(0,y(u_h)),v)=(u_h,v), \qquad \forall\, v\in H_0^1(\Omega). \tag{4.3}$$

By Theorem 3.1, we have

$$\|y(u_h)-y_h\|_0\le \mathrm{ch}^2. \tag{4.4}$$

Thus,

$$\begin{aligned} J(y,u)-J_h(y_h,u_h) \le& J(y(u_h),u_h)-J_h(y_h,u_h)\\ =&\frac{1}{2}\int_\Omega (y(u_h)-y_0)^2dx-\frac{1}{2}\int_\Omega (y_h-y_0)^2dx\\ =&\frac{1}{2}\int_\Omega (y(u_h)-y_h)(y(u_h)+y_h-2y_0)dx\\ \le&\frac{1}{2}\|y(u_h)-y_h\|_0\|y(u_h)+y_h-2y_0\|_0\le \mathrm{ch}^2, \end{aligned} \tag{4.5}$$

where (4.4) has been used in the last step. The proof is thus completed. ☐

Lemma 4.2. Under the assumptions of Lemma 4.1, we have

$$J_h(y_h,u_h)-J(y,u)\le \mathrm{ch}^2. \tag{4.6}$$

Proof. Let $y_h(u)$ and $y_h(\overline{\Pi}_h u)$ be the solutions of the following two auxiliary equations:

$$a_h(y_h(u),v_h)+(\beta\max(0,y_h(u)),v_h)=(u,v_h), \qquad \forall\, v_h\in V_h \tag{4.7}$$

and

$$a_h(y_h(\overline{\Pi}_h u),v_h)+(\beta\max(0,y_h(\overline{\Pi}_h u)),v_h)=(\overline{\Pi}_h u,v_h), \qquad \forall\, v_h\in V_h, \tag{4.8}$$

respectively. Then setting $v_h=y_h(u)-y_h(\overline{\Pi}_h u)$ in (4.7) and (4.8), we can get the following error equation

$$\begin{aligned} &\|y_h(u)-y_h(\overline{\Pi}_h u)\|_h^2+\beta(\max(0,y_h(u))-\max(0,y_h(\overline{\Pi}_h u)),y_h(u)-y_h(\overline{\Pi}_h u))\\ =\ &(u-\overline{\Pi}_h u,y_h(u)-y_h(\overline{\Pi}_h u)). \end{aligned} \tag{4.9}$$

In fact,

$$\begin{aligned} &\beta(\max(0,y_h(u))-\max(0,y_h(\overline{\Pi}_h u)),(y_h(u)-y_h(\overline{\Pi}_h u))\\ =&\begin{cases} (y_h(u)-y_h(\overline{\Pi}_h u))^2\ge 0, & \text{if } y_h(u)\ge 0 \text{ and } y_h(\overline{\Pi}_h u)\ge 0;\\ y_h(u)(y_h(u)-y_h(\overline{\Pi}_h u))\ge 0, & \text{if } y_h(u)\ge 0 \text{ and } y_h(\overline{\Pi}_h u)\le 0;\\ -y_h(\overline{\Pi}_h u)(y_h(u)-y_h(\overline{\Pi}_h u))\ge 0, & \text{if } y_h(u)< 0 \text{ and } y_h(\overline{\Pi}_h u)\ge 0;\\ 0, & \text{if } y_h(u)< 0 \text{ and } y_h(\overline{\Pi}_h u)< 0 \end{cases} \end{aligned}$$

$$\geq 0. \tag{4.10}$$

So

$$\begin{aligned}\|y_h(u)-y_h(\overline{\Pi}_h u)\|_h^2 \leq&(u-\overline{\Pi}_h u, y_h(u)-y_h(\overline{\Pi}_h u))\\ \leq& c\|u-\overline{\Pi}_h u\|_{-1}\|y_h(u)-y_h(\overline{\Pi}_h u)\|_h.\end{aligned} \tag{4.11}$$

It is easy to check that

$$\|u-\overline{\Pi}_h u\|_{-1}=\sup_{\Phi\in H^1(\Omega)}\frac{(u-\overline{\Pi}_h u,\Phi)}{\|\Phi\|_1}=\sup_{\Phi\in H^1(\Omega)}\frac{(u-\overline{\Pi}_h u,\Phi-\overline{\Pi}_h\Phi)}{\|\Phi\|_1}\leq \mathrm{ch}^2|u|_1. \tag{4.12}$$

Thus, substituting (4.12) to (4.11) yields

$$\|y_h(u)-y_h(\overline{\Pi}_h u)\|_h\leq \mathrm{ch}^2|u|_1. \tag{4.13}$$

Because (y_h,u_h) is the solution of (3.2), there holds

$$J_h(y_h,u_h)\leq J_h(y(\overline{\Pi}_h u),\overline{\Pi}_h u). \tag{4.14}$$

Thus,

$$\begin{aligned}
&J_h(y_h,u_h)-J(y,u) \qquad (4.15)\\
\leq& J_h(y(\overline{\Pi}_h u),\overline{\Pi}_h u)-J(y(u_h),u_h)\\
=&\frac{1}{2}\int_\Omega (y_h(\overline{\Pi}_h u)-y_0)^2dx-\frac{1}{2}\int_\Omega (y-y_0)^2+\frac{\alpha}{2}\int_\Omega(\overline{\Pi}_h u)^2dx-\frac{\alpha}{2}\int_\Omega u^2dx\\
=&\frac{1}{2}\int_\Omega (y_h(\overline{\Pi}_h u)-y)(y_h(\overline{\Pi}_h u)+y-2y_0)dx+\frac{\alpha}{2}\int_\Omega(\overline{\Pi}_h u+u)(\overline{\Pi}_h u-u)dx\\
=&\frac{1}{2}\int_\Omega (y_h(\overline{\Pi}_h u)-y)(y_h(\overline{\Pi}_h u)+y-2y_0)dx\\
&+\frac{\alpha}{2}\int_\Omega(\overline{\Pi}_h u+u-\overline{\Pi}_h u)(\overline{\Pi}_h u-u)dx\\
=&\frac{1}{2}\int_\Omega (y_h(\overline{\Pi}_h u)-y)(y_h(\overline{\Pi}_h u)+y-2y_0)dx+\frac{\alpha}{2}\int_\Omega(u-\overline{\Pi}_h u)(\overline{\Pi}_h u-u)dx\\
\leq&\frac{1}{2}\int_\Omega (y_h(\overline{\Pi}_h u)-y)(y_h(\overline{\Pi}_h u)+y-2y_0)dx\\
\leq&\|y_h(\overline{\Pi}_h u)-y\|_0\|y_h(\overline{\Pi}_h u)+y-2y_0\|_0\\
\leq&(\|y_h(\overline{\Pi}_h u)-y_h(u)\|_0+\|y_h(u)-y\|_0)\|y_h(\overline{\Pi}_h u)+y-2y_0\|_0\\
\leq&\mathrm{ch}^2(\|u\|_1+|y|_2), \qquad (4.16)
\end{aligned}$$

where, (4.13) and (3.24) have been used in the last step. The proof is thus completed. □

Therefore, we have the following goal-oriented error estimate:

Theorem 4.1. *Under the assumptions of Lemmas 4.1–4.2, respectively, it can be obtained directly that*

$$|J_h(y_h,u_h)-J(y,u)|\leq \mathrm{ch}^2. \tag{4.17}$$

Remark. Theorem 4.1 indicates that the FE solution tends to an exact solution in the sense of goal function although the error estimates of u_h-u and y_h-y under H^1-norm or L^2-norm can't be obtained directly.

4.1 Goal-oriented Error Estimates

It has been mentioned that the result of Theorem 4.1 follows the convergence result Theorem 3.1. In fact, for some other very popular nonconforming elements cases, the results also hold. It can be checked that for $v_h \in V_h$, if (2.6) is replaced by

$$a_h(u - \Pi_h u, v_h) = O(h^2)\|v_h\|_h, \qquad \forall\, v_h \in V_h, \tag{4.18}$$

or the interpolation operator Π_h in our paper is replaced by Riesz projection operator R_h define by

$$a_h(R_h u - u, v_h) = 0, \qquad \forall\, v_h \in V_h, \tag{4.19}$$

and (2.7) is satisfied, then Theorems 3.1–3.3 and Theorem 4.1 also hold true.

(a) Q_1^{rot} element on square meshes

It has been shown in [24] that (2.6) and Lemma 2.1 hold true. Thus Theorems 3.1-3.3 and Theorem 4.1 are also valid for this element.

(b) Quasi-Wilson quadrilateral element[9], Quasi-Carey triangular element[28] and modified quasi-Wilson quadrilateral element[31].

For any $v_h \in V_h$, let $v_h = \overline{v_h} + v_h^1$, where $\overline{v_h}$ and v_h^1 be the conforming and nonconforming parts of v_h, respectively. So if we replace $\Pi_h u$ by $\overline{\Pi_h u}$ in the proof of our paper, Theorems 3.1–3.3 and Theorem 4.1 are also valid for these two elements.

(c) P_1-nonconforming rectangular finite element[18].

It has been proved in [18] that P_1-nonconforming rectangular finite element is identity to the constrained rotated Q_1 element (CNQ_1^{rot}), and there hold Lemma 2.1 and (4.18). Thus Theorems 3.1–3.3 and Theorem 4.1 are valid for this element.

(d) $P_1^{\rm mod}$-nonconforming triangular element and modified Crouzeix-Raviart type rectangular elements[20].

In [20], it has been proved that the consistency error for $P_1^{\rm mod}$-nonconforming triangular element is of order $O(h^3)$. So we can replace the interpolation operator Π_h with R_h defined by (4.19) to ensure Theorems 3.1–3.3 and Theorem 4.1 to be true.

(e) Wilson rectangular element, Carey triangular element and C-R type nonconforming triangular element

As to these three elements, since their consistency error is of order $O(h)$, i.e., (2.8) does not hold, even if Π_h is replaced by R_h as in above (d), whether the result of Theorems 3.2–3.3 are true or not remains open. But Theorems 3.1 and Theorem 4.1 still hold.

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