Acta Mathematicae Applicatae Sinica, English Series
Vol. 35, No. 4 (2019) 820–829
https://doi.org/10.1007/s10255-019-0855-1
http://www.ApplMath.com.cn & www.SpringerLink.com

Acta Mathematicae Applicatae Sinica, English Series

Regularity of Global Attractor for Atmospheric Circulation Equations with Humidity Effect

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Abstract In this article, regularity of the global attractor for atmospheric circulation equations with humidity effect is considered. It is proved that atmospheric circulation equations with humidity effect possess a global attractor in $H^k(\Omega, \mathbf{R}^4)$ for any $k \geq 0$, which attracts any bounded set of $H^k(\Omega, \mathbf{R}^4)$ in the H^k−norm. The result is established by means of an iteration technique and regularity estimates for the linear semigroup of operator, together with a classical existence theorem of global attractor.

Keywords global attractor; regularity; atmospheric circulation equations; humidity effect
2000 MR Subject Classification 35K20, 35Q35, 35B40, 35B41

1 Introduction

In this article, we concern the following the atmospheric circulation equations

$$\frac{\partial u}{\partial t} = P_r(\Delta u - \nabla p - \sigma u) + P_r(RT - \widetilde{R}q)\vec{\kappa} - (u \cdot \nabla)u, \qquad t > 0, \quad x \in \Omega, \tag{1.1}$$

$$\frac{\partial T}{\partial t} = \Delta T + u_2 - (u \cdot \nabla)T + Q, \qquad t > 0, \quad x \in \Omega, \tag{1.2}$$

$$\frac{\partial q}{\partial t} = L_e \Delta q + u_2 - (u \cdot \nabla)q + G, \qquad t > 0, \quad x \in \Omega, \tag{1.3}$$

$$\operatorname{div} u = 0, \qquad t > 0, \qquad x \in \Omega, \tag{1.4}$$

where the unknown functions $u = (u_1, u_2)$, T, q and p denote velocity field, temperature, humidity and pressure, respectively, and $\vec{\kappa} = (0, 1)$. Here $(x, t) = (x_1, x_2, t) \in \Omega \times (0, \infty)$ ($\Omega = (0, 2\pi) \times (0, 1)$ is a period of C^∞ field $(-\infty, +\infty) \times (0, 1)$), and $Q(x), G(x)$ are given functions. Besides, $P_r > 0$, $R > 0$, $\widetilde{R}$ and $L_e > 0$ are constants, and σ is a matrix

$$\sigma = \begin{pmatrix} \sigma_0 & -\omega \\ \omega & \sigma_1 \end{pmatrix},$$

where σ_0, σ_1 and ω are positive constants.

The Problems (1.1)–(1.4) are supplemented with the following Dirichlet boundary condition at $x_2 = 0, 1$ and periodic condition for x_1,

$$(u, T, q) = 0, \qquad x_2 = 0, 1; \qquad (u, T, q)(0, x_2) = (u, T, q)(2\pi, x_2), \tag{1.5}$$

and initial value condition

$$(u, T, q) = (u_0, T_0, q_0), \qquad t = 0. \tag{1.6}$$

Manuscript received June 21, 2014. Revised August 13, 2019.
Supported by the National Natural Science Foundation of China (No.11701399).
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The partial differential Eqs.(1.1)–(1.6) were presented in atmospheric circulation with humidity effect. The atmosphere and ocean around the earth are rotating geophysical fluids, which are also two important components of the climate system. The phenomena of the atmosphere and ocean are extremely rich in their organization and complexity, therefore a lot of them can not be produced by laboratory experiments. The atmosphere, the ocean or the atmosphere-ocean coupling can be viewed as an initial and boundary value problem[8,9,13,14], or an infinite dimensional dynamical system[1−3]. We deduce atmospheric circulation models (1.1)–(1.6) which are able to show the features of atmospheric circulation and are easy to be studied from the very complex atmospheric circulation model based on the actual background and meteorological data, furthermore, we present global solutions of atmospheric circulation equations in H^k spaces in [4]. In [5], the steady state solution to atmospheric circulation Eqs.(1.1)–(1.6) with humidity effect is studied and a sufficient condition of existence of steady state solution to atmospheric circulation equations is obtained, and regularity of steady state solution is verified. In [6], by C-condition, it is obtained that atmospheric circulation equations have a global attractor in $L^2(\Omega, \mathbf{R}^4)$. In this article, we investigate regularity of attractor to the atmospheric circulation equations by an iteration procedure[7,9,15,17].

The paper is organized as follows. In Section 2 we present preliminary results, especially some properties of semigroups and several lemmas which will be used later. In Section 3, we obtain regularity of the global attractor for the Eqs.(1.1)–(1.6).

2 Preliminaries

Let X and X_1 be two Banach spaces, $X_1 \subset X$ a compact and dense inclusion. Consider the abstract nonlinear evolution equation defined on X, given by

$$\begin{cases} \dfrac{du}{dt} = Lu + G(u), \\ u(x,0) = u_0, \end{cases} \tag{2.1}$$

where $u(t)$ is an unknown function, $L : X_1 \to X$ a linear operator, and $G : X_1 \to X$ a nonlinear operator.

A family of operators $S(t) : X \to X$ $(t \geq 0)$ is called a semigroup generated by (2.1) if $S(t)$ satisfies the properties:

(1) $S(t) : X \to X$ is a continuous map for any $t \geq 0$,

(2) $S(0) = id : X \to X$ is the identity,

(3) $S(t+s) = S(t) \cdot S(s)$, $\forall\, t, s \geq 0$.

Then the solution of (2.1) can be expressed by $u(t, u_0) = S(t)u_0$.

The following Lemma 2.1 is the classical existence theorem of global attractor.

Lemma 2.1[10,11,16]**.** *Let $S(t) : X \to X$ be the semigroup generated by (2.1). Eq.(2.1) has a global attractor, if the following conditions hold*

(1) $S(t)$ has a bounded absorbing set $B \subset X$, i.e., for any bounded set $A \subset X$ there exists a time $t_A \geq 0$ such that $S(t)u_0 \in B$ for $\forall\, u_0 \in A$ and $t > t_A$,

(2) $S(t)$ is uniformly compact, i.e., for any bounded set $U \subset X$ and some $T > 0$ sufficiently large, the set $\overline{\bigcup_{t \geq T} S(t)U}$ is compact in X.

We assume that the linear operator L in (2.1) is a sectorial operator which generates an analytic semigroup e^{tL}. It is known that there exists a constant $\lambda \geq 0$ such that $L - \lambda I$ generates the fractional power operators $\mathcal{L}^\alpha$ and fractional order spaces X_α for $\alpha \in \mathbf{R}^1$, where $\mathcal{L} = -(L - \lambda I)$. Without loss of generality, we assume that L generates the fractional power operators $\mathcal{L}^\alpha$ and fractional order spaces X_α as follows:

$$\mathcal{L}^\alpha = (-L)^\alpha : X_\alpha \to X, \alpha \in \mathbf{R}^1,$$

where $X_\alpha = D(\mathcal{L}^\alpha)$ is the domain of $\mathcal{L}^\alpha$. By the semigroup theory of linear operators[12], we know that $X_\beta \subset X_\alpha$ is a compact inclusion for any $\beta > \alpha$.

Therefore, Lemma 2.1 can be equivalently expressed by the following Lemma.

Lemma 2.2[9,10,15,17]**.** *Let $u(t,u_0) = S(t)u_0(u_0 \in X,\ t \geq 0)$ be a solution of (2.1) and X_α be the fractional order space generated by L. Assume*

(1) for some $\alpha \geq 0$ there is a bounded set $B \subset X_\alpha$, such that for any $u_0 \in X_\alpha$ there exists a $t_{u_0} > 0$ such that

$$u(t,u_0) \in B, \qquad \forall\, t > t_{u_0},$$

(2) there is a $\beta > \alpha$, for any bounded set $U \subset X_\beta$ there are constants $T > 0$ and $C > 0$ such that

$$\|u(t,u_0)\|_{X_\beta} \leq C, \qquad \forall\, t > T, \quad u_0 \in U.$$

Then Eq.(2.1) has a global attractor $\mathcal{A} \subset X_\alpha$ attracting any bounded set of X_α in the X_α-norm.

Lemma 2.3. *The eigenvalue equation*

$$\begin{cases} -\Delta T(x_1,x_2) = \lambda T(x_1,x_2), & (x_1,x_2) \in (0,2\pi)\times(0,1), \\ T = 0, & x_2 = 0,1, \\ T(0,x_2) = T(2\pi,x_2), & \end{cases} \tag{2.2}$$

has eigenvalue $\{\lambda_k\}_{k=1}^\infty$, and $0 < \lambda_1 \leq \lambda_2 \leq \cdots,\ \lambda_k \to \infty$, as $k \to \infty$.

Let $\phi = (u,T,q)$, and

$$H = \{(u,T,q) \in L^2(\Omega,\mathbf{R}^4) | (u,T,q) \ \text{ satisfies } \ (1.4)-(1.5)\},$$

$$H_1 = \{(u,T,q) \in H^2(\Omega,\mathbf{R}^4) | (u,T,q) \ \text{ satisfies } \ (1.4)-(1.5)\}.$$

We introduce the operators $L : H_1 \to H$ and $F : H_1 \to H$ defined by

$$L(\phi) = \begin{pmatrix} L_1(\phi) \\ L_2(\phi) \\ L_3(\phi) \end{pmatrix} = P \begin{pmatrix} P_r(\Delta u - \nabla p) \\ \Delta T \\ Le\Delta q \end{pmatrix},$$

$$F(\phi) = \begin{pmatrix} F_1(\phi) \\ F_2(\phi) \\ F_3(\phi) \end{pmatrix} = P \begin{pmatrix} -P_r\sigma u + P_r(RT - \widetilde{R}q)\vec{\kappa} - (u\cdot\nabla)u \\ u_2 - (u\cdot\nabla)T + Q \\ u_2 - (u\cdot\nabla)q + G \end{pmatrix},$$

where $P : L^2(\Omega,\mathbf{R}^4) \to H$ is a Leray projection. It is well known that the operator $L : H_1 \to H$ is a sectorial operator, and the associated space $H_{\frac{1}{2}}$ is $H_{\frac{1}{2}} = H^1(\Omega,\mathbf{R}^4) \cap H$. Then the Eqs.(1.1)–(1.6) can be rewritten as an abstract equation

$$\frac{d\phi}{dt} = L\phi + F(\phi).$$

Lemma 2.4[4]**.** *If $\phi_0 = (u_0,T_0,q_0) \in H$, and $Q,G \in L^2(\Omega)$, then the global solution ϕ of the Eqs.(1.1)–(1.6) can be read as*

$$\phi(x,t) = \Phi(t)\phi_0 + \int_0^t \Phi(t-\tau)F(\phi)d\tau, \tag{2.3}$$

where $\Phi(t)$ is an analytic semigroup generated by L.

Suppose λ_1 is the first eigenvalue of elliptic Eq.(2.2) and $\widetilde{\sigma} = \max\{\sigma_0, \sigma_1, \omega\}$. Assume

$$\widetilde{\sigma}\lambda_1 \geq \max\Big\{(R+1)^2, \frac{(\widetilde{R}-1)^2}{L_e}\Big\}. \tag{2.4}$$

Lemma 2.5[6]**.** *If (2.4) holds, then Eqs.(1.1)–(1.6) have an absorbing set in $L^2(\Omega, \mathbf{R}^4)$, and*

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (u^2 + T^2 + q^2)dx\\ \leq &C_1\int_\Omega [-|\nabla u|^2 - |\nabla T|^2 - |\nabla q|^2 dx + C_2\int_\Omega (|Q|^2 + |G|^2)dx.\end{aligned} \tag{2.5}$$

Lemma 2.6[10−12]**.** *Let $\Omega \subset \mathbf{R}^n$ be a Lipschitz field, $L : W^{m,p} \to L^p(\Omega)$ be a sectorial operator, $m \geq 2$ and $1 \leq p < \infty$. Then for $0 \leq \alpha \leq 1$, the fractional order spaces $H_\alpha = D(\mathcal{L}^\alpha)$ satisfy the following relations*

$$\begin{aligned}&H_\alpha \subset W^{k,q}, \quad \text{if} \quad k - \frac{n}{q} \leq m\alpha - \frac{n}{p},\\ &H_\alpha \subset C^{k,\beta}, \quad \text{if} \quad 0 \leq k+\beta < m\alpha - \frac{n}{p},\end{aligned}$$

and the inequalities

$$\begin{aligned}&\|u\|_{W^{k,q}} \leq C\|u\|_{H_\alpha}, \quad \text{if} \quad k - \frac{n}{q} \leq m\alpha - \frac{n}{p},\\ &\|u\|_{C^{k,\beta}} \leq C\|u\|_{H_\alpha}, \quad \text{if} \quad 0 \leq k+\beta < m\alpha - \frac{n}{p}.\end{aligned}$$

For sectorial operators, we also have the following properties.

Lemma 2.7[9,10] *Let $L : H_1 \to H$ be a sectorial operator which generates an analytic semigroup $T(t) = e^{tL}$. If all eigenvalues λ of L satisfy $Re\,\lambda < -\lambda_0$ for some real number $\lambda_0 > 0$, then for $\mathcal{L}^\alpha (\mathcal{L} = -L)$ we have*

(1) $T(t) : H \to H_\alpha$ is bounded for all $\alpha \in \mathbf{R}^1$ and $t > 0$,

(2) $T(t)\mathcal{L}^\alpha x = \mathcal{L}^\alpha T(t)x, \ \forall x \in H_\alpha$,

(3) for each $t > 0, \mathcal{L}^\alpha T(t) : H \to H$ is bounded, and

$$\|\mathcal{L}^\alpha T(t)\| \leq C_\alpha t^{-\alpha} e^{-\delta t},$$

where some $\delta > 0$, $C_\alpha > 0$ is a constant only depending on α,

(4) the $H_\alpha-$norm can be defined by

$$\|x\|_{H_\alpha} = \|\mathcal{L}^\alpha x\|_H, \tag{2.6}$$

(5) if $\mathcal{L}$ is symmetric, for any $\alpha, \beta \in \mathbf{R}^1$ we have $\langle \mathcal{L}^\alpha u, v\rangle_H = \langle \mathcal{L}^{\alpha-\beta} u, \mathcal{L}^\beta v\rangle_H$.

3 Main Theorem

We present our main result of the article by the following theorem.

Theorem 3.1. *If (2.4) holds and $Q, G \in C^\infty(\bar{\Omega})$, then there exists a global attractor $\mathcal{A} \in H^k(\Omega, \mathbf{R}^4) \cap H$ for Eqs.(1.1)–(1.6) and $\mathcal{A}$ attracts all bounded sets of $\mathcal{A} \in H^k(\Omega, \mathbf{R}^4) \cap H$ in the H^k-norm.*

Proof. Firstly, we verify that the solutions of Eqs. (1.1)–(1.6) are bounded in $H_{\frac{1}{2}}$.

From (1.1)–(1.6), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\Big(\frac{1}{P_r}|\nabla u|^2+|\nabla T|^2+|\nabla q|^2\Big)dx\\
=&\int_\Omega\Big[-|\triangle u|^2-|\triangle T|^2-L_e|\triangle q|^2-\sigma\nabla u\cdot\nabla u+(R+1)\nabla T\nabla u_2\\
&-(\widetilde{R}-1)\nabla q\nabla u_2-Q\triangle T-G\triangle q+\frac{1}{P_r}(u\cdot\nabla)u\triangle u+(u\cdot\nabla)T\triangle T+(u\cdot\nabla)q\triangle q\Big]dx\\
\le&\int_\Omega\Big[-|\triangle u|^2-|\triangle T|^2-L_e|\triangle q|^2-\widetilde{\sigma}|\nabla u|^2+\widetilde{\sigma}|\nabla u_2|^2+\frac{(R+1)^2}{2\widetilde{\sigma}}|\nabla T|^2\\
&+\frac{(\widetilde{R}-1)^2}{2\widetilde{\sigma}}|\nabla q|^2+\varepsilon_1|\triangle T|^2+\varepsilon_2|\triangle q|^2+C_{\varepsilon_1,\varepsilon_2}(|Q|^2+|G|^2)\\
&+\frac{1}{P_r}(u\cdot\nabla)u\triangle u+(u\cdot\nabla)T\triangle T+(u\cdot\nabla)q\triangle q\Big]dx\\
\le&\int_\Omega\Big[-|\triangle u|^2-\frac{1}{2}|\triangle T|^2-\frac{L_e}{2}|\triangle q|^2+\varepsilon_1|\triangle T|^2+\varepsilon_2|\triangle q|^2\\
&+C_{\varepsilon_1,\varepsilon_2}(|Q|^2+|G|^2)+\frac{1}{P_r}(u\cdot\nabla)u\triangle u+(u\cdot\nabla)T\triangle T+(u\cdot\nabla)q\triangle q\Big]dx.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\frac{1}{P_r}\|\nabla u\|_{L^2}^2+\|\nabla T\|_{L^2}^2+\|\nabla q\|_{L^2}^2\Big)\\
\le&-\|\triangle u\|_{L^2}^2-\frac{1}{2}\|\triangle T\|_{L^2}^2-\frac{L_e}{2}\|\triangle q\|_{L^2}^2+\varepsilon_1\|\triangle T\|_{L^2}^2+\varepsilon_2\|\triangle q\|_{L^2}^2\\
&+C_{\varepsilon_1,\varepsilon_2}(\|Q\|_{L^2}^2+\|G\|_{L^2}^2)+\frac{C}{P_r}\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}\|\triangle u\|_{L^2}^{\frac{3}{2}}\\
&+C\|u\|_{L^2}^{\frac{1}{2}}\|\triangle u\|_{L^2}^{\frac{1}{2}}\|\nabla T\|_{L^2}\|\triangle T\|_{L^2}+C\|u\|_{L^2}^{\frac{1}{2}}\|\triangle u\|_{L^2}^{\frac{1}{2}}\|\nabla q\|_{L^2}\|\triangle q\|_{L^2}\\
\le&-\|\triangle u\|_{L^2}^2-\frac{1}{2}\|\triangle T\|_{L^2}^2-\frac{L_e}{2}\|\triangle q\|_{L^2}^2+\varepsilon_1\|\triangle T\|_{L^2}^2+\varepsilon_2\|\triangle q\|_{L^2}^2\\
&+C_{\varepsilon_1,\varepsilon_2}(\|Q\|_{L^2}^2+\|G\|_{L^2}^2)+\frac{C}{P_r}\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}\|\triangle u\|_{L^2}^{\frac{3}{2}}+C\|u\|_{L^2}^{\frac{1}{2}}\|\triangle u\|_{L^2}^{\frac{3}{2}}\|\nabla T\|_{L^2}\\
&+C\|u\|_{L^2}^{\frac{1}{2}}\|\nabla T\|_{L^2}\|\triangle T\|_{L^2}^{\frac{3}{2}}+C\|u\|_{L^2}^{\frac{1}{2}}\|\triangle u\|_{L^2}^{\frac{3}{2}}\|\nabla q\|_{L^2}+\|u\|_{L^2}^{\frac{1}{2}}\|\nabla q\|_{L^2}\|\triangle q\|_{L^2}^{\frac{3}{2}}\\
\le&-\|\triangle u\|_{L^2}^2-\frac{1}{2}\|\triangle T\|_{L^2}^2-\frac{L_e}{2}\|\triangle q\|_{L^2}^2+\varepsilon_3\|\triangle u\|_{L^2}^2+\varepsilon_1\|\triangle T\|_{L^2}^2+\varepsilon_2\|\triangle q\|_{L^2}^2\\
&+C_{\varepsilon_1,\varepsilon_2}(\|Q\|_{L^2}^2+\|G\|_{L^2}^2)+C_{\varepsilon_3}(\|u\|_{L^2}^2\|\nabla u\|_{L^2}^4+\|u\|_{L^2}^2\|\nabla T\|_{L^2}^4+\|u\|_{L^2}^2\|\nabla T\|_{L^2}^4\\
&+\|u\|_{L^2}^2\|\nabla q\|_{L^2}^4+\|u\|_{L^2}^2\|\nabla q\|_{L^2}^4).
\end{aligned}
$$

Assuming $\varepsilon_1=\frac{1}{2},\ \varepsilon_2=\frac{Le}{2},\ \varepsilon_3=1$, we have

$$
\begin{aligned}
&\frac{d}{dt}\big(\|\nabla u\|_{L^2}^2+\|\nabla T\|_{L^2}^2+\|\nabla q\|_{L^2}^2\big)\\
\le&C_5(\|Q\|_{L^2}^2+\|G\|_{L^2}^2)+C_6(\|u\|_{L^2}^2\|\nabla u\|_{L^2}^4+\|u\|_{L^2}^2\|\nabla T\|_{L^2}^4+\|u\|_{L^2}^2\|\nabla T\|_{L^2}^4\\
&+\|u\|_{L^2}^2\|\nabla q\|_{L^2}^4+\|u\|_{L^2}^2\|\nabla q\|_{L^2}^4).
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla T\|_{L^2}^2+\|\nabla q\|_{L^2}^2)\\
\le&C_5(\|Q\|_{L^2}^2+\|G\|_{L^2}^2)+C_6\|u\|_{L^2}^2(\|\nabla u\|_{L^2}^2+\|\nabla T\|_{L^2}^2+\|\nabla q\|_{L^2}^2)^2. \qquad (3.1)
\end{aligned}
$$

Integrate (2.5) over $[t, t+r]$, it follows that

$$C_1\int_t^{t+r}(\|\nabla u\|_{L^2}^2+\|\nabla T\|_{L^2}^2+\|\nabla q\|_{L^2}^2)dt$$
$$\leq C_2(\|Q\|_{L^2}^2+\|G\|_{L^2}^2)r+(\|u(t)\|_{L^2}^2+\|T(t)\|_{L^2}^2+\|q(t)\|_{L^2}^2).$$

Since (u, T, q) has an absorbing set in L^2, we have

$$\int_t^{t+r}(\|\nabla u\|_{L^2}^2+\|\nabla T\|_{L^2}^2+\|\nabla q\|_{L^2}^2)dt$$
$$\leq C_7(\|Q\|_{L^2}^2+\|G\|_{L^2}^2)r+C_8M^2, \qquad t>t_*.$$

Let

$$\begin{cases} g=C_6\|u\|_{L^2}^2(\|\nabla u\|_{L^2}^2+\|\nabla T\|_{L^2}^2+\|\nabla q\|_{L^2}^2), \\ h=C_5(\|Q\|_{L^2}^2+\|G\|_{L^2}^2), \\ y=\|\nabla u\|_{L^2}^2+\|\nabla T\|_{L^2}^2+\|\nabla q\|_{L^2}^2, \end{cases} \tag{3.2}$$

and furthermore, we assume

$$\begin{cases} a_1=C_6M[C_7(\|Q\|_{L^2}^2+\|G\|_{L^2}^2)r+C_8M^2], \\ a_2=C_5(\|Q\|_{L^2}^2+\|G\|_{L^2}^2)r, \\ a_3=C_7(\|Q\|_{L^2}^2+\|G\|_{L^2}^2)r+C_8M^2. \end{cases} \tag{3.3}$$

Applying the uniform Gronwall Lemma[16] for Eq.(3.1), we have

$$\big\|(\nabla u,\nabla T,\nabla q)(t)\big\|_{L^2}^2\leq\Big(\frac{a_3}{r}+a_2\Big)e^{a_1}, \qquad \text{for any } \ t\geq t_*+r.$$

Clearly, there exists a bounded constant M_1 such that $M_1^2>\big(\frac{a_3}{r}+a_2\big)e^{a_1}$, for any $(u_0, T_0, q_0)\in B$, where B is a bounded set in H. Then there exists $t_*+r>0$ satisfying

$$S(t)(u_0,T_0,q_0)=(u(t),T(t),q(t))\in B_{M_1}, \qquad t>t_*+r, \tag{3.4}$$

where B_{M_1} is a ball in $H_{\frac{1}{2}}$, at 0 of radius M_1. Thus Eqs.(1.1)–(1.6) have a bounded set B_{M_1} in $H_{\frac{1}{2}}$.

Next, we need to prove that the solution $\phi(t)=(u,T,q)$ to Eqs.(1.1)–(1.6) is uniformly bounded in H_α $(\alpha<1)$, i.e.,

$$\|\phi(t,\phi_0)\|_{H_\alpha}\leq C_\alpha, \qquad \frac{1}{2}\leq\alpha<1, \qquad \forall\,\phi_0\in H_\alpha. \tag{3.5}$$

It is known that, for any θ, H_θ and $H_{-\theta}$ are dual, and if, for $v\in H_\theta$, we have $<u,v>\leq C\|v\|_{H_\theta}$, then $u\in H_{-\theta}$ and $\|u\|_{-\theta}\leq C$.

For $\widetilde{\phi}=(\widetilde{u},\widetilde{T},\widetilde{q})\in H_\theta$, we have

$$\langle F\phi,\widetilde{\phi}\rangle=\int_\Omega[-P_r\sigma u\cdot\widetilde{u}+P_r(RT-\widetilde{R}q)\widetilde{u}_2+(\nabla\cdot u)u\cdot\widetilde{u}$$
$$+u_2\widetilde{T}-(u\cdot\nabla)T\widetilde{T}+Q\widetilde{T}+u_2\widetilde{q}-(u\cdot\nabla)q\widetilde{q}+G\widetilde{q}]dx$$
$$\leq\int_\Omega[|P_r\sigma u\cdot\widetilde{u}|+P_rR|T||\widetilde{u}_2|+|\widetilde{R}||q||\widetilde{u}_2|+|u_2||\widetilde{T}|+|u_2||\widetilde{q}|$$
$$+|Q||\widetilde{T}|+|G||\widetilde{q}|]dx+\int_\Omega[|(u\cdot\nabla)u\cdot\widetilde{u}|+|(u\cdot\nabla)T\widetilde{T}|+|(u\cdot\nabla)q\widetilde{q}|]dx$$

$$
\begin{aligned}
\leq & C(\|u\|_{L^2}\|\widetilde{u}\|_{L^2}+\|T\|_{L^2}\|\widetilde{u}_2\|_{L^2}+\|q\|_{L^2}\|\widetilde{u}_2\|_{L^2}+\|u_2\|_{L^2}\|\widetilde{T}\|_{L^2} \\
& +\|u_2\|_{L^2}\|\widetilde{q}\|_{L^2}+\|Q\|_{L^2}\|\widetilde{T}\|_{L^2}+\|G\|_{L^2}\|\widetilde{q}\|_{L^2}) \\
& +\int_\Omega [|u||Du||\widetilde{u}|+|u||DT||\widetilde{T}|+|u||Dq||\widetilde{q}|]dx \\
\leq & C(\|u\|_{L^2}+\|T\|_{L^2}+\|q\|_{L^2})(\|\widetilde{u}\|_{L^2}+\|\widetilde{T}\|_{L^2}+\|\widetilde{q}\|_{L^2}) \\
& +C(\|Q\|_{L^2}+\|G\|_{L^2})(\|\widetilde{u}\|_{L^2}+\|\widetilde{T}\|_{L^2}+\|\widetilde{q}\|_{L^2}) \\
& +C(\|u\|_{H^1}+\|T\|_{H^1}+\|q\|_{H^1})\|u\|_{L^{2q}}(\|\widetilde{u}\|_{L^{2p}}+\|\widetilde{T}\|_{L^{2p}}+\|\widetilde{q}\|_{L^{2p}}),
\end{aligned}
$$

where $p>1$ is arbitrary, $q=p/(p-1)$. By Lemma 2.6, we have

$$\|u\|_{L^{2p}} \leq C\|u\|_{H_\theta}, \qquad \forall\, \theta \geq \frac{p-1}{2p}.$$

Hence,

$$
\begin{aligned}
\langle F\phi, \widetilde{\phi}\rangle \leq & C\left[\|u\|_{L^2}+\|T\|_{L^2}+\|q\|_{L^2}+\|Q\|_{L^2}+\|G\|_{L^2}\right. \\
& \left.+(\|u\|_{H^1}+\|T\|_{H^1}+\|q\|_{H^1})\|u\|_{H^1}\right](\|\widetilde{u}\|_{H_\theta}+\|\widetilde{T}\|_{H_\theta}+\|\widetilde{q}\|_{H_\theta}) \\
\leq & C(\|\phi\|_{H^1}+1)\|\widetilde{\phi}\|_{H_\theta}, \qquad \forall\, \theta>0.
\end{aligned}
$$

Then the mapping

$$F: H_{\frac{1}{2}} \to H_{-\theta} \quad \text{is bounded for any } \theta>0. \tag{3.6}$$

So it follows that,

$$\|L^{-\theta}F(\phi(t,\phi_0))\|_H \leq C, \qquad \forall \theta>0, \quad \phi_0\in H_\alpha, \quad \alpha\geq \frac{1}{2}.$$

From (2.3) and Lemma 2.7, we find

$$
\begin{aligned}
\|\phi(t,\phi_0)\|_{H_\alpha} \leq & \|\Phi(t)\phi_0\|_{H_\alpha}+\|\int_0^t \Phi(t-\tau)F(\phi)d\tau\|_{H_\alpha} \\
\leq & \|\Phi(t)\phi_0\|_{H_\alpha}+\int_0^t \|L^\alpha\Phi(t-\tau)F(\phi)\|_H d\tau \\
\leq & C+\int_0^t \|L^{\alpha+\theta}\Phi(t-\tau)\|\|L^{-\theta}F(\phi)\|_H d\tau \\
\leq & C+\int_0^t \tau^{-(\alpha+\theta)}e^{-\delta\tau}d\tau, \qquad \forall\, 0<\alpha+\theta<1.
\end{aligned}
$$

which implies (3.5).

In the following, we prove that for $\frac{1}{2}<\alpha\leq\frac{k}{2}$ $(k\geq 2)$, there exists a number β with $0<\beta\leq\frac{k-1}{2}$ such that

$$F: H_\alpha\to H_\beta \text{ bounded, and } \beta\to\frac{k-1}{2} \text{ as } \alpha\to\frac{k}{2}. \tag{3.7}$$

It suffices to prove (3.7) only for bilinear operator

$$
\begin{cases}
F_0: H_{\alpha_1}\times H_{\alpha_2}\to H_\beta, \\
F_0(\phi,\widetilde{\phi})=(P[(u\cdot\nabla)\widetilde{u}],(u\cdot\nabla)\widetilde{T},(u\cdot\nabla)\widetilde{q}),
\end{cases} \tag{3.8}
$$

where $\phi=(u,T,q)\in H_{\alpha_1}$, $\widetilde{\phi}=(\widetilde{u},\widetilde{T},\widetilde{q})\in H_{\alpha_2}$.

It is known that $H_\alpha\big(\frac{k-1}{2}\le\alpha\le\frac{k}{2}\big)$ are interpolations between $H_{\frac{k}{2}}$ and $H_{\frac{k-1}{2}}$ (see [16]):

$$\begin{aligned}&H_\alpha=[H_{\frac{k}{2}},H_{\frac{k-1}{2}}]_\theta, &&\theta=\frac{k-1+\alpha}{2}, && 0\le\theta\le1,\\ &H_{\frac{k-1}{2}}=[H_{\frac{k}{2}},H_{\frac{k-1}{2}}]_{\theta=0}, && H_{\frac{k}{2}}=[H_{\frac{k}{2}},H_{\frac{k-1}{2}}]_{\theta=1}.\end{aligned}$$

Moreover, if

$$\begin{aligned}&L:H_{\frac{k}{2}}\to H_{\frac{k-1}{2}}\quad\text{linear bounded},\\ &L:H_{\frac{k-1}{2}}\to H_{\frac{k-2}{2}}\quad\text{linear bounded},\end{aligned}$$

then

$$L:[H_{\frac{k}{2}},H_{\frac{k-1}{2}}]_\theta\to[H_{\frac{k-1}{2}},H_{\frac{k-2}{2}}]_\theta\quad\text{linear bounded},\quad\forall 0\le\theta\le1. \tag{3.9}$$

For the bilinear operator (3.8), it is readily to verify that for any bounded set $U\in H_{\frac{k-1}{2}}$, there is a constant $C>0$, such that for all $\phi\in U$, we have

$$\begin{aligned}&F_0(\phi,\cdot):H_{\frac12}\to H_{-\gamma}\quad\text{linear bounded},\qquad\forall\,\gamma>0,\\ &F_0(\phi,\cdot):H_{\frac{k}{2}}\to H_{\frac{k-1}{2}}\quad\text{linear bounded},\qquad\forall\,k\ge2,\\ &\|F_0(\phi,\cdot)\|\le C,\qquad\forall\,\phi\in U\subset H_{\frac{k}{2}},\qquad k\ge1.\end{aligned}$$

By the interpolation relation (3.9) for linear bounded operators, we infer

$$\begin{aligned}&F_0(\phi,\cdot):[H_{\frac{k}{2}},H_{\frac{k-1}{2}}]_\theta\to[H_{\frac{k-1}{2}},H_{\frac{k-2}{2}}]_\theta,\qquad\forall\,k\ge3,\\ &F_0(\phi,\cdot):[H_1,H_{\frac12}]_\theta\to[H_{\frac12},H_{-\gamma}]_\theta,\qquad\forall\,\gamma>0,\\ &\|F_0(\phi,\cdot)\|\le C,\qquad\forall\,\phi\in U\subset H_{\frac{k-1}{2}},\quad k\ge3,\qquad\text{or}\ \ \forall\,\phi\in U\subset[H_1,H_{\frac12}]_\theta.\end{aligned}$$

We denote

$$\alpha=\frac{k-1+\theta}{2},\qquad k\ge2,\qquad \beta=\begin{cases}\dfrac{k-2+\theta}{2}, & k\ge3,\\ -\gamma+(\frac12+\gamma)\theta, & k=2.\end{cases}$$

Then,

$$F_0:H_\alpha\to H_\beta\ \text{bounded and}\ \beta\to\frac{k-1}{2}\ \text{as}\ \alpha\to\frac{k}{2}.$$

Thus (3.7) holds.

Next, we shall verify that any solution $\phi(t)=(u,T,q)$ to Eqs.(1.1)–(1.6) is uniformly bounded in $H_\alpha\big(\frac{k-1}{2}\le\alpha<\frac{k}{2}\big)$, i.e.,

$$\|\phi(t,\phi_0)\|_{H_\alpha}\le C_\alpha,\qquad\frac{k-1}{2}\le\alpha<\frac{k}{2},\qquad\forall\,\phi_0\in H_\alpha. \tag{3.10}$$

From (3.7), for $\frac{k-1}{2}\le\alpha<\frac{k}{2}$ there is $\frac{k-2}{2}\le\beta<\frac{k-1}{2}$ satisfying

$$\|F(\phi(t,\phi_0))\|_{H_\beta}=\|\mathcal{L}^\beta F(\phi(t,\phi_0))\|_H\le C,\qquad\forall t>0,\ \ \phi_0\in H_\alpha.$$

Utilizing (2.3) and Lemma 2.7, we obtain that

$$\|\phi(t,\phi_0)\|_{H_\alpha}\le\|\Phi(t)\phi_0\|_{H_\alpha}+\Big\|\int_0^t\Phi(t-\tau)F(\phi)d\tau\Big\|_{H_\alpha}$$

$$\begin{aligned}
&\le \|\Phi(t)\phi_0\|_{H_\alpha} + \int_0^t \|L^\alpha \Phi(t-\tau)F(\phi)\|_H d\tau \\
&\le C + \int_0^t \|L^{\alpha-\beta}\Phi(t-\tau)\|\|L^\beta F(\phi)\|_H d\tau \\
&\le C + \int_0^t \tau^{-(\alpha-\beta)} e^{-\delta\tau} d\tau, \qquad \forall\, 0<\alpha-\beta<1.
\end{aligned}$$

Thus (3.10) holds. Then for any $\alpha \ge 0$, any solution $\phi(t) = (u, T, q)$ to Eqs.(1.1)–(1.6) is uniformly bounded in H_α, i.e.,

$$\|\phi(t,\phi_0)\|_{H_\alpha} \le C_\alpha, \qquad 0 \le \alpha, \qquad \forall\, \phi_0 \in H_\alpha. \tag{3.11}$$

Finally, we shall prove that for any $\alpha \ge 0$, Eqs.(1.1)–(1.6) have a bounded absorbing set in H_α. From Theorem 4.1 in [6] and (3.4), the conclusion holds for the case $\frac{1}{2} \ge \alpha \ge 0$. We only proceed with the case $\alpha > \frac{1}{2}$.

By (2.3) we find

$$\phi(t,\phi_0) = \Phi(t-T)\phi(T,\phi_0) + \int_T^t \Phi(t-\tau)F(\phi)d\tau. \tag{3.12}$$

Let $D \subset H_{\frac{1}{2}}$ be the bounded absorbing set of Eqs.(1.1)–(1.6) in $H_{\frac{1}{2}}$ and $T_0 > 0$ such that

$$\phi(t,\phi_0) \in D, \qquad \forall\, t > T_0, \qquad \forall\, \phi_0 \in U \subset H_\alpha, \qquad \alpha \ge \frac{1}{2}.$$

By Lemma 2.7, for all $\phi_0 \in U$ and $T > 0$, we have that

$$\lim_{t\to\infty} \|\Phi(t-T)\phi(T,\phi_0)\|_{H_\alpha} \to 0. \tag{3.13}$$

By Lemma 2.7, it follows from (3.12) and (3.13) that, for any $\alpha < 1$ and $T > T_0$. Then

$$\begin{aligned}
&\|\phi(t,\phi_0)\|_{H_\alpha} \le \|\Phi(t-T)\phi(T_0,\phi_0)\|_{H_\alpha} + \Big\| \int_T^t \Phi(t-\tau)F(\phi)d\tau \Big\|_{H_\alpha} \\
&\le \|\Phi(t-T)\phi(T_0,\phi_0)\|_{H_\alpha} + \int_T^t \|L^\alpha\Phi(t-\tau)F(\phi)\|_H d\tau \\
&\le C + \int_T^t \|L^{\alpha+\theta}\Phi(t-\tau)\|\|L^{-\theta}F(\phi)\|_H d\tau \\
&\le C + \int_T^t \tau^{-(\alpha+\theta)} e^{-\delta\tau} d\tau, \qquad \forall\, 0 < \alpha+\theta < 1.
\end{aligned}$$

For $\forall\, \alpha < 1$ and $U \subset H_\alpha$, there is a $T > 0$, such that

$$\|\phi(t,\phi_0)\|_{H_\alpha} \le C, \qquad \forall\, t \ge T, \quad \phi_0 \in U,$$

where $C > 0$ is independent of ϕ_0. Then Eqs.(1.1)–(1.6) have a bounded absorbing set in $H_\alpha (\alpha < 1)$.

By iteration procedures, we find that Eqs.(1.1)–(1.6) have a bounded absorbing set in $H_\alpha(\alpha \ge 0)$, i.e., for $\forall\, \alpha \ge 0$ and $U \subset H_\alpha$, there is a $T > 0$ such that

$$\|\phi(t,\phi_0)\|_{H_\alpha} \le C, \qquad \forall\, t \ge T, \quad \phi_0 \in U,$$

where $C > 0$ is independent of ϕ_0.

From Lemma 2.2, (3.11) and (3.14), Eqs.(1.1)–(1.6) possess a global attractor $\mathcal{A}$ in $H^k(\Omega, \mathbf{R}^4) \cap H$ and $\mathcal{A}$ attracts any bounded set of $H^k(\Omega, \mathbf{R}^4)$ in the H^k-norm. $\square$

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