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Infinitely Many Solutions to a Class of p-Laplace Equations

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Abstract In this paper, we study a class of p-Laplace equations. Using variational methods, we prove that there are two solutions and one of these solutions is nonnegative. Using recurrence method, we prove that there are infinitely many solutions to this class of equations.

Keywords p-Laplace equation; variational methods; infinitely many solutions; nonnegative solution
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1 Introduction

It is well known that problems involving the p-Laplacian operator appear in many areas of applied mathematics and physics (for more details, please see [8]). Recently, some people have studied the following problem

$$\begin{cases} -\triangle u = u - |u|^{-2\theta}u, \\ u \in H^1(\mathbb{R}^N) \cap L^{2(1-\theta)}(\mathbb{R}^N). \end{cases} \tag{1.1}$$

In [2], Balabane et al. proved that for each integer k, there exists a radial compactly supported solution of (1.1) which has k-zeros in it's support. In [9], Ounaies proved that there exists a ground state solution of (1.1) which is non-negative radial compactly supported. In [3], Benrhouma and Ounaies studied the existence of two solutions for the following nonhomogeneous problem

$$\begin{cases} -\triangle u = u - |u|^{-2\theta}u + f, \\ u \in H^1(\mathbb{R}^N) \cap L^{2(1-\theta)}(\mathbb{R}^N), \end{cases} \tag{1.2}$$

where $f \in L^2(\mathbb{R}^N) \cap L^{\frac{2(1-\theta)}{1-2\theta}}(\mathbb{R}^N)$, $f \geq 0$, $f \neq 0$, $N \geq 3$ and $\theta \in (0, \frac{1}{2})$.

In [10], Su discussed a class of p-Laplace equations with negative power of the unknown function on unbounded domains in $\mathbb{R}^N$. Motivated by [3] and [10], we study the following p-Laplace nonhomogeneous problem

$$\begin{cases} -\triangle_p u = |u|^{p-2}u - |u|^{-2\theta}u^{2m-1} + f, \\ u \in W^{1,p}(\mathbb{R}^N) \cap L^{2(m-\theta)}(\mathbb{R}^N), \end{cases} \tag{1.3}$$

where m is a positive integer, $\triangle_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $N > p \geq 2$, $m - \frac{p}{2} < \theta < m - \frac{1}{2}$, f satisfies $||f||_q^q < c_\theta$ for some constants $c_\theta > 0$ and

$$\text{(H)} \quad f \in L^q(\mathbb{R}^N) \cap L^{\frac{2(m-\theta)}{2m-1-2\theta}}(\mathbb{R}^N), \qquad f \geq 0, \quad f \neq 0, \qquad \text{where } q \text{ satisfies } \frac{1}{p} + \frac{1}{q} = 1.$$

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If $f \equiv 0$, then Problem (1.3) becomes the following p-Laplace homogeneous problem

$$\begin{cases} -\triangle_p u = |u|^{p-2}u - |u|^{-2\theta}u^{2m-1}, \\ u \in W^{1,p}(\mathbb{R}^N) \cap L^{2(m-\theta)}(\mathbb{R}^N), \end{cases} \tag{1.4}$$

In [5], Deng proved that there exists a nonnegative ground state solution to a class of general p-Laplace equations which include Problem (1.4) as a special case.

2 Preliminaries

Let $E = W^{1,p}(\mathbb{R}^N) \cap L^{2(m-\theta)}(\mathbb{R}^N)$, and endow E with the following norm

$$||u|| = ||\nabla u||_p + ||u||_{2(m-\theta)}.$$

It is easy to verify that $(E, \|\cdot\|)$ is a Banach space. Define the functionals I^∞ and I on E by

$$I^\infty(u) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p - |u|^p)dx + \frac{1}{2(m-\theta)}\int_{\mathbb{R}^N}|u|^{2(m-\theta)}dx,$$
$$I(u) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p - |u|^p)dx + \frac{1}{2(m-\theta)}\int_{\mathbb{R}^N}|u|^{2(m-\theta)}dx - \int_{\mathbb{R}^N} fudx.$$

Obviously, I^∞ and I are C^1 functions on E. It is well known that the critical points of I and I^∞ are weak solutions to Problems (1.3) and (1.4) respectively.

Lemma 2.1. *There exists a constant $c_0 > 1$ such that for any $u \in E$ we have*

$$\frac{2}{p}||u||_p^p \le \frac{1}{4(m-\theta)}||u||_{2(m-\theta)}^{2(m-\theta)} + c_0||\nabla u||_p^{p^*}.$$

Proof. If $s \neq 0$, we consider the following function

$$h(s) = \frac{\frac{2}{p}|s|^p - \frac{1}{4(m-\theta)}|s|^{2(m-\theta)}}{|s|^{\frac{Np}{N-p}}}, \qquad s \neq 0.$$

It is easy to prove that there exists $c_1 > 1$ such that

$$\frac{2}{p}|s|^p \le \frac{1}{4(m-\theta)}|s|^{2(m-\theta)} + c_1|s|^{p^*}. \tag{2.1}$$

If $s = 0$, it is obvious that (2.1) holds. Let $s = |u(x)|$, by (2.1) we have

$$\frac{2}{p}|u(x)|^p \le \frac{1}{4(m-\theta)}|u(x)|^{2(m-\theta)} + c_1|u(x)|^{p^*}, \qquad \forall\, x \in \mathbb{R}^N. \tag{2.2}$$

Integrate (2.2), we get

$$\frac{2}{p}||u||_p^p \le \frac{1}{4(m-\theta)}||u||_{2(m-\theta)}^{2(m-\theta)} + c_1||u||_{p^*}^{p^*}.$$

By the Gagliardo-Nirenberg inequality we conclude that Lemma 2.1 is true. □

In the following, we fix the constant c_0 in Lemma 2.1.

Lemma 2.2. *Suppose that $0 < \rho \le (\frac{1}{2pc_0})^{\frac{1}{p^*-p}}$, then there exist $c_\theta > 0$ and $a > 0$ such that $I(u) \ge a$ provided $||f||_q^q < c_\theta$ and $||u|| = \rho$.*

Proof. By the Cauchy-Young inequality, we get

$$I(u) \geq \frac{1}{p}(||\nabla u||_p^p - ||u||_p^p) + \frac{1}{2(m-\theta)}||u||_{2(m-\theta)}^{2(m-\theta)} - \frac{1}{p}||u||_p^p - \frac{1}{q}||f||_q^q. \tag{2.3}$$

By Lemma 2.1 and (2.3), we have

$$I(u) \geq \frac{1}{p}||\nabla u||_p^p - c_0||\nabla u||_p^{p^*} + \frac{1}{4(m-\theta)}||u||_{2(m-\theta)}^{2(m-\theta)} - \frac{1}{q}||f||_q^q. \tag{2.4}$$

Note that if $||\nabla u||_p \leq \left(\frac{1}{2pc_0}\right)^{\frac{1}{p^*-p}}$, then $\frac{1}{p}||\nabla u||_p^p - c_0||\nabla u||_p^{p^*} \geq \frac{1}{2p}||\nabla u||_p^p$. Thus, according to (2.4) we conclude that

$$I(u) \geq \frac{1}{2p}||\nabla u||_p^p + \frac{1}{4(m-\theta)}||u||_{2(m-\theta)}^{2(m-\theta)} - \frac{1}{q}||f||_q^q. \tag{2.5}$$

Since $\rho \leq \left(\frac{1}{2pc_0}\right)^{\frac{1}{p^*-p}}$, then (2.5) is true for $||u|| = \rho$. On the other hand, we observe that if $||u|| = \rho$ then $||\nabla u||_p \geq \frac{1}{2}\rho$ or $||u||_{2(m-\theta)} \geq \frac{1}{2}\rho$. Therefore, there are two cases to be discussed.

Case 1. $||\nabla u||_p \geq \frac{1}{2}\rho$. In this case, by (2.5) we have $I(u) \geq \frac{1}{2p}(\frac{1}{2\rho})^p - \frac{1}{q}||f||_q^q$. Let $c_\theta = \frac{q}{2p}(\frac{1}{2\rho})^p$ and $a = \frac{q}{2p}(\frac{1}{2\rho})^p - ||f||_q^q$, then $I(u) \geq a$ provided $||f||_q^q \leq c_\theta$ and $||u|| = \rho$.

Case 2. $||u||_{2(m-\theta)} \geq \frac{1}{2}\rho$. In this case, we have $I(u) \geq \frac{1}{4(m-\theta)}(\frac{1}{2\rho})^{2(m-\theta)} - \frac{1}{q}||f||_q^q$. Let $c_\theta = \frac{q}{4(m-\theta)}(\frac{1}{2\rho})^{2(m-\theta)}$ and $a = \frac{q}{4(m-\theta)}\left(\frac{1}{2\rho}\right)^{2(m-\theta)} - ||f||_q^q$. It is obvious that $I(u) \geq a$ provided $||f||_q^q \leq c_\theta$ and $||u|| = \rho$.

The proof is complete. □

Fix the constant ρ in Lemma 2.2 and set $B_\rho = \{u \in E,\ ||u|| \leq \rho\}$. We have

Lemma 2.3. *Let* $c^\infty = \inf_{u \in B_\rho} I^\infty(u)$, *then* $c^\infty = 0$.

Proof. Suppose $c^\infty < 0$. If $(v_n) \subseteq B_\rho$ is a minimizing sequence of $\inf_{u\in B_\rho} I^\infty(u)$, then $I^\infty(v_n) < 0$ for n large enough. Therefore we have

$$\frac{1}{p}||\nabla v_n||_p^p + \frac{1}{2(m-\theta)}||v_n||_{2(m-\theta)}^{2(m-\theta)} \leq \frac{1}{p}||v_n||_p^p.$$

Since $0 < \rho \leq \left(\frac{1}{2pc_0}\right)^{\frac{1}{p^*-p}}$ and $(v_n) \subseteq B_\rho$, then $||\nabla v_n||_p \leq \left(\frac{1}{2pc_0}\right)^{\frac{1}{p^*-p}}$. It follows that

$$\frac{1}{p}||\nabla v_n||_p^p - c_0||\nabla v_n||_p^{p^*} \geq \frac{1}{2p}||\nabla v_n||_p^p.$$

Using Lemma 2.1, similar to [3], we can get a contradiction and consequently we arrive at the conclusion $c^\infty = 0$. □

Lemma 2.4 (see [11], Lemma 2.1). *Let* $(u_n) \subseteq W_0^{1,p}(\Omega)$ *a bounded sequence and* $p \geq 2$. *Going if necessary to a subsequence, one may assume that* $u_n \rightharpoonup u$ *in* $W_0^{1,p}(\Omega)$, $u_n \to u$ *a.e., where* $\Omega \subseteq \mathbb{R}^N$ *is an open subset. Then,*

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n|^p dx \geq \lim_{n\to\infty}\int_\Omega |\nabla u_n - \nabla u|^p dx + \lim_{n\to\infty}\int_\Omega |\nabla u|^p dx. \tag{2.6}$$

Lemma 2.5[4]**.** *Suppose* $f_n \to f$ *a.e. and* $||f_n||_p \leq c < \infty$ *for all* n *and for some* $0 < p < \infty$. *Then*

$$\lim_{n\to\infty}||f_n||_p^p = \lim_{n\to\infty}||f_n - f||_p^p + ||f||_p^p. \tag{2.7}$$

Let $f_n = g_n + f$. If $g_n \to 0$ a.e., $0 < p < \infty$ and $g_n \rightharpoonup 0$ in $L^p(\Omega)$ for open subset $\Omega \subseteq \mathbb{R}^N$, similar to the proof of Theorem 2 in [4] we have

$$\lim_{n\to\infty} ||g_n + f||_p^p = \lim_{n\to\infty} ||g_n||_p^p + ||f||_p^p. \tag{2.8}$$

Lemma 2.6. *Suppose that (u_n) is a sequence in $W^{1,p}(\mathbb{R}^N)$ such that (u_n) converges weakly to U_0. Let $v_n = u_n - U_0$, going if necessary to a subsequence, we conclude that there exists a constant $k > 0$ such that*

$$\frac{1}{p}\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla v_n + \nabla U_0|^p dx \geq k \lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla v_n|^p + \frac{1}{p}\int_{\mathbb{R}^N} |\nabla U_0|^p dx. \tag{2.9}$$

Proof. If (u_n) converges strongly to U_0 in $W^{1,p}(\mathbb{R}^N)$, then $\lim\limits_{n\to\infty}\int_{\mathbb{R}^N} |\nabla v_n|^p = 0$. Thus (2.9) holds. If (u_n) converges weakly to U_0 in $W^{1,p}(\mathbb{R}^N)$, then

$$k = \frac{\lim\limits_{n\to\infty}\int_{\mathbb{R}^N} |\nabla v_n + \nabla U_0|^p dx - \int_{\mathbb{R}^N} |\nabla U_0|^p dx}{2p\lim\limits_{n\to\infty}\int_{\mathbb{R}^N} |\nabla v_n|^p}.$$

is the constant satisfies (2.9). The proof is complete. □

3 Existence of the Nonnegative Solution

In this section, we prove that there exists a nonnegative solution to Problem (1.3). For this purpose, we set $B_\rho = \{u \in E, ||u|| \leq \rho\}$ and consider the following problem

$$c = \inf_{u\in B_\rho} I(u). \tag{3.1}$$

If ρ is small enough, similar to [3] and some arguments in Lemma 2.2, we can easily get $-\infty < c < 0$. Furthermore, we have

Theorem 3.1. *There exists $U_0 \in B_\rho$ such that $c = I(U_0)$ and U_0 is a nonnegative nontrivial solution of Problem (1.3).*

Proof. Let $(u_n) \subseteq B_\rho$ be a minimizing sequence of Problem (3.1). We can extract a subsequence of (u_n), also denoted by (u_n) such that $u_n \rightharpoonup U_0$ in E, $u_n \to U_0$ in $L^s_{\text{loc}}(\mathbb{R}^N)$, $\forall\, 1 \leq s < p^*$ and $u_n \to U_0$ a.e in $\mathbb{R}^N$. Let $v_n = u_n - U_0$. By (2.8) and (2.9) we conclude that there exists a constant $k > 0$ such that

$$\begin{aligned} c = \lim_{n\to\infty} I(u_n) &\geq k\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla v_n|^p + \frac{1}{p}\int_{\mathbb{R}^N} |\nabla U_0|^p dx - \frac{1}{p}\lim_{n\to\infty}\int_{\mathbb{R}^N} (|v_n|^p + |U_0|^p) dx \\ &+ \frac{1}{2(m-\theta)}\lim_{n\to\infty}\int_{\mathbb{R}^N} (|v_n|^{2(m-\theta)} + |U_0|^{2(m-\theta)}) dx - \lim_{n\to\infty}\int_{\mathbb{R}^N} f v_n dx - \int_{\mathbb{R}^N} f U_0 dx. \end{aligned} \tag{3.2}$$

Since $\int_{\mathbb{R}^N} f v_n dx \to 0$, then

$$c \geq I(U_0) + \lim_{n\to\infty}\Big[k||\nabla v_n||_p^p - \frac{1}{p}||v_n||_p^p + \frac{1}{2(m-\theta)}||v_n||_{2(m-\theta)}^{2(m-\theta)}\Big]. \tag{3.3}$$

On the other hand, by (2.6) we get

$$\rho^{\,p} \geq \lim_{n\to\infty} ||\nabla u_n||_p^p \geq \lim_{n\to\infty} ||\nabla v_n||_p^p + ||\nabla U_0||_p^p.$$

Thus, we have $\lim_{n\to\infty}||\nabla v_n||_p < \rho$ which means that $||\nabla v_n||_p \le \rho$ for n large enough. If ρ is small enough, similar to the discussions in Lemma 2.2, we have

$$k||\nabla v_n||_p^p - \frac{1}{p}||v_n||_p^p + \frac{1}{2(1-\theta)}||v_n||_{2(m-\theta)}^{2(m-\theta)} \ge \frac{k}{2}||\nabla v_n||_p^p + \frac{1}{4(m-\theta)}||v_n||_{2(m-\theta)}^{2(m-\theta)}. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$c \ge I(U_0) + \lim_{n\to\infty}\frac{k}{2}||\nabla v_n||_p^p + \frac{1}{4(m-\theta)}\lim_{n\to\infty}||v_n||_{2(m-\theta)}^{2(m-\theta)}. \tag{3.5}$$

Since $u_n \rightharpoonup U_0$ in E, then $||U_0|| \le \liminf_{n\to\infty}||u_n|| \le \rho$. This means that $U_0 \in B_\rho$ and consequently we have $c - I(U_0) \le 0$. Therefore, by (3.5) we conclude that $v_n \to 0$ in E. Thus $I(u_n) \to I(U_0) = c < 0$, $I'(U_0) = 0$ and U_0 is a nonnegative nontrivial solution of problem (1.3). The proof of Theorem 3.1 is complete. □

By the same discussions of Theorem 3.3 in [3], we have

Theorem 3.2. *If $f \to 0$ in L^q then $U_0(f) \to 0$ in E.*

4 Existence of the Second Solution

In this section, we use the mountain-pass theorem of Ambrosetti Rabinowitz (see [1]) to prove that there is another solution for Problem (1.3). Let U_0 be the nonnegative solution of Problem (1.3) given in Lemma 3.1 and choose a nonnegative ground state w of Problem (1.4).

Lemma 4.1 (see [6], p.322). *Suppose that $0 \le a \le b$ and $p > 1$, then*

$$(a+b)^p \le b^p + (2^p - 1)b^{p-1}a.$$

According to Lemma 4.1, we can easily get

Lemma 4.2. *If $p > 1$, then*

$$(a+b)^p \le (|a|+|b|)^p \le |b|^p + |a|^p + (2^p-1)|b|^{p-1}|a| + (2^p-1)|a|^{p-1}|b|. \tag{4.1}$$

Lemma 4.3. *If w is a nonnegative ground state of Problem (1.4), then*

$$\begin{aligned} I(U_0 + tw) <& I(U_0) + \Big(\frac{t^{2(m-\theta)}}{2(m-\theta)} - \frac{t^p}{p}\Big)||w||_{2(m-\theta)}^{2(m-\theta)} + (2^p-1)t^{p-1}\int_{\mathbb{R}^N}|\nabla U_0||\nabla w|^{p-1}dx \\ &+ (2^p-1)t\int_{\mathbb{R}^N}|\nabla U_0|^{p-1}|\nabla w|dx + \frac{2^{2(m-\theta)}-1}{2(m-\theta)}t^{2(m-\theta)-1}\int_{\mathbb{R}^N}w^{2(m-\theta)-1}U_0dx \\ &+ \frac{t^{2(m-\theta)}}{2(m-\theta)}\int_{\mathbb{R}^N}w^{2(m-\theta)}dx + \frac{2^{2(m-\theta)}-1}{2(m-\theta)}t\int_{\mathbb{R}^N}wU_0^{2(m-\theta)-1}dx \\ &+ \frac{1}{2(m-\theta)}\int_{\mathbb{R}^N}|U_0|^{2(m-\theta)}dx - t\int_{\mathbb{R}^N}fwdx. \end{aligned} \tag{4.2}$$

Proof. It is obvious that if $a > 0$ and $b > 0$ then

$$(a+b)^p > a^p + b^p. \tag{4.3}$$

By Lemma 4.2 and (4.3), we have

$$\begin{aligned}I(U_0+tw)<&\frac{1}{p}\int_{\mathbb{R}^N}(|\nabla U_0|^p-U_0^p)dx+\frac{t^p}{p}\int_{\mathbb{R}^N}(|\nabla w|^p-w^p)dx\\&+(2^p-1)t^{p-1}\int_{\mathbb{R}^N}|\nabla U_0||\nabla w|^{p-1}dx+(2^p-1)t\int_{\mathbb{R}^N}|\nabla U_0|^{p-1}|\nabla w|dx\\&+\frac{1}{2(m-\theta)}\int_{\mathbb{R}^N}|U_0|^{2(m-\theta)}dx+\frac{2^{2(m-\theta)}-1}{2(m-\theta)}t^{2(m-\theta)-1}\int_{\mathbb{R}^N}w^{2(m-\theta)-1}U_0dx\\&+\frac{t^{2(m-\theta)}}{2(m-\theta)}\int_{\mathbb{R}^N}w^{2(m-\theta)}dx+\frac{2^{2(m-\theta)}-1}{2(m-\theta)}t\int_{\mathbb{R}^N}wU_0^{2(m-\theta)-1}dx\\&-\int_{\mathbb{R}^N}fU_0dx-t\int_{\mathbb{R}^N}fwdx.\end{aligned}\tag{4.4}$$

Since w is a critical point of I^∞, then

$$\int_{\mathbb{R}^N}(|\nabla w|^p-w^p)dx=-\int_{\mathbb{R}^N}|w|^{2(m-\theta)}dx\tag{4.5}$$

Combining (4.4) and (4.5), we conclude that (4.2) is true. The proof is complete. □

According to (4.2), we can choose t_0 large enough such that $I(U_0+tw)<I(U_0)$ for all $t\geq t_0$. Put $\varphi_0=U_0+t_0w$ and $\varphi_1=\varphi_0+w$. Let

$$d=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I(\gamma(t)),\tag{4.6}$$

where $\Gamma=\{\gamma\in C([0,1],E);\ \gamma(0)=\varphi_0,\ \gamma(1)=\varphi_1\}\neq\emptyset$.

Lemma 4.4. $-\infty<d<I(U_0)$.

Proof. Let $\gamma_0\in\Gamma$ defined by $\gamma_0(t)=\varphi_0+tw$. Then, there exists $T\in[0,1]$ such that

$$d=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I(\gamma(t))\leq\max_{t\in[0,1]}I(\gamma_0(t))=I(\gamma_0(T))=I(\varphi_0+Tw).\tag{4.7}$$

Since $I(\varphi_0+Tw)=I(U_0+(T+t_0)w)$ and $T+t_0\geq t_0$, by (4.7) we have $d<I(U_0)$. It is obvious that $d>-\infty$, then we complete the proof. □

Lemma 4.5. $-\infty<d<I(U_0)+I^\infty(w)$

Proof. It follows from (4.5) that

$$I^\infty(w)=\Big(\frac{1}{2(m-\theta)}-\frac{1}{p}\Big)||w||_{2(m-\theta)}^{2(m-\theta)}>0.\tag{4.8}$$

Therefore, by Lemma 4.4 we have $d<I(U_0)+I^\infty(w)$. the proof is complete. □

Lemma 4.6. *Let $(u_n)\subseteq E$ such that $(I(u_n))$ is bounded and $I'(u_n)\to 0$ in E', then (u_n) is bounded in E.*

Proof. There exists $M>0$ such that for n large enough, $|I(u_n)|<M$ and $|\langle I'(u_n),u_n\rangle|\leq ||u_n||$. By the Hölder inequality and the Young inequality, we have

$$I(u_n)-\frac{1}{p}\langle I'(u_n),u_n\rangle\geq\Big(\frac{1}{2p(m-\theta)}-\frac{1}{p}\Big)||u_n||_{2(m-\theta)}^{2(m-\theta)}-c||f||_{\frac{2(m-\theta)}{2m-1-2\theta}}^{\frac{2(m-\theta)}{2m-1-2\theta}}.\tag{4.9}$$

According to (4.9), we get

$$\Big(\frac{1}{2p(m-\theta)}-\frac{1}{p}\Big)||u_n||_{2(m-\theta)}^{2(m-\theta)}-c_1<M+\frac{1}{p}||u_n||.\tag{4.10}$$

Since $||u_n|| = ||\nabla u_n||_p + ||u_n||_{2(m-\theta)}$, by (4.10) we get

$$\Big(\frac{1}{2p(m-\theta)} - \frac{1}{p}\Big)\,||u_n||_{2(m-\theta)}^{2(m-\theta)} - \frac{1}{p}||u_n||_{2(m-\theta)} < \frac{1}{p}||\nabla u_n||_p + c_2.$$

We suppose by contrary $||u_n||_{2(m-\theta)} \to +\infty$. If $1 < \lambda < 2(m-\theta)$, then for n large enough we can suppose

$$||u_n||_{2(m-\theta)}^{\lambda} < \Big(\frac{1}{2p(m-\theta)} - \frac{1}{p}\Big)\,||u_n||_{2(m-\theta)}^{2(m-\theta)} - \frac{1}{p}||u_n||_{2(m-\theta)} < \frac{1}{p}||\nabla u_n||_p + c_2.$$

Therefore, we get

$$||u_n||_{2(m-\theta)} < c||\nabla u_n||_p^{\frac{1}{\lambda}} + c_3. \tag{4.11}$$

Choosing r such that

$$\frac{2r}{p^*} + \frac{2(1-r)}{2(1-\theta)} = 1,$$

we have

$$||u_n||_p^p \le ||u_n||_{p*}^{2r}||u_n||_{2(1-\theta)}^{2(1-r)}. \tag{4.12}$$

By the Gagliardo-Nirenberg inequality and (4.12), we get

$$||u_n||_p^p \le c_4||\nabla u_n||_p^{2r}||u_n||_{2(m-\theta)}^{2(1-r)}. \tag{4.13}$$

Similar to the proof of Lemma 4.4 in [3], we can get a contradiction. Thus $||u_n||_{2(m-\theta)}$ is bounded and consequently by (4.13) we conclude that $||\nabla u_n||_p$ is also bounded. The proof is complete. □

Theorem 4.7. *There exists V_0 which is the second solution of Problem (1.3) and V_0 is a critical point of I.*

Proof. By the mountain-pass theorem of Ambrosetti-Rabinowitz in [1], we conclude that there exists (PS) sequence (u_n) in E such that $I(u_n) \to d$ and $I'(u_n) \to 0$ in E'. According to Lemma 4.6, (u_n) is bounded in E and then up to a subsequence there is a function V_0 such that $u_n \rightharpoonup V_0$ in E, $u_n \to V_0$ in $L^q_{\text{loc}}(\mathbb{R}^N)$ for all $1 \le q \le p^*$ and $u_n \to V_0$ a.e in $\mathbb{R}^N$. Similar to [3], we have

$$\langle I'(u_n), \varphi\rangle \to \langle I'(V_0), \varphi\rangle = 0, \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N)$$

It follows that V_0 is a weak solution of problem (1.3) and V_0 is a critical point of I.

Now, we prove that $V_0 \ne U_0$, where U_0 is the nonnegative solution to problem (1.3) given in Theorem 3.1. By the fact that $u_n \rightharpoonup V_0$, we have $||V_0|| \le \liminf_{n\to\infty} ||u_n||$. Since $(||u_n||)$ is bounded, then there exists a subsequence, also denoted by $(||u_n||)$ such that $\lim_{n\to\infty} ||u_n||$ exists. We distinguish two cases to discuss and only give details which is different from that in [3].

Case 1. $\lim_{n\to\infty} ||u_n|| = ||V_0||$. Similar to [3], we can easily verify that $u_n \to V_0$ in E. Therefore, $I(u_n) \to d = I(V_0)$. By Lemma 4.4, we have $d < I(U_0)$. Thus, we conclude that $V_0 \ne U_0$.

Case 2. $\lim_{n\to\infty} ||u_n|| > ||V_0||$. In this case, we set $v_n(x) = u_n(x) - V_0(x)$, then $v_n \rightharpoonup 0$ in E. Furthermore, we affirm that there exists $(y_n^1) \subset \mathbb{R}^N$ such that (y_n^1) is not bounded, $v_n(\cdot + y_n^1) \rightharpoonup V_1 \ne 0$ in E, and V_1 is a solution of the homogeneous Problem (1.4). We separate three steps to prove this affirmation.

Step 1. Suppose that $v_n(\cdot + y_n^1) \rightharpoonup 0$ in E for all $(y_n^1) \subset \mathbb{R}^N$, then

$$\sup_{y\in\mathbb{R}^N} \int_{B(y,R)} |v_n|^{2(m-\theta)} dx \to 0, \qquad \text{for all } \ R > 0.$$

By Lemma 1.1 in [7], we have $v_n \to 0$ in $L^s(\mathbb{R}^N)$ if $2(m-\theta) < s < p^*$. According to (2.8), (2.9) and the fact that $u_n(x) = v_n(x) + V_0(x)$, we conclude that there exists a constant k such that

$$0 = \lim_{n\to\infty} \langle I'(u_n), u_n\rangle \geq \langle I'(V_0), V_0\rangle + \lim_{n\to\infty}\Big(k\int_{\mathbb{R}^N} |\nabla v_n|^p dx + \int_{\mathbb{R}^N} |v_n|^{2(m-\theta)} dx\Big). \tag{4.14}$$

According to (4.14) and $\langle I'(V_0), V_0\rangle = 0$, we conclude that $v_n \to 0$ in E. This contradicts $\lim_{n\to\infty} ||u_n|| > ||V_0||$.

Step 2. Suppose that (y_n^1) is bounded. Similar to the Step 2 in [3], we can get $V_1 = 0$ a.e in $\mathbb{R}^N$. Therefore, we obtain a contradiction.

Step 3. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$. Direct calculation shows that

$$\begin{aligned}\langle I'(u_n), \varphi(x-y_n^1)\rangle = &\int_{\mathbb{R}^N} (|\nabla u_n(x+y_n^1)|^{p-1}\nabla\varphi(x) - |u_n(x+y_n^1)|^{p-1}\varphi(x))dx \\ &+ \int_{\mathbb{R}^N} |u_n(x+y_n^1)|^{2m-2\theta-2}(u_n(x+y_n^1))\varphi(x)dx - \int_{\mathbb{R}^N} f(x)\varphi(x-y_n^1)dx.\end{aligned}$$

Since $|y_n^1| \to +\infty$, then $\int_{\mathbb{R}^N} f(x)\varphi(x-y_n^1)dx \to 0$. Recall that $v_n(\cdot + y_n^1) \rightharpoonup V_1$ in E, we get

$$\int_{\mathbb{R}^N} (|\nabla u_n(x+y_n^1)|^{p-1}\nabla\varphi(x) - |u_n(x+y_n^1)|^{p-1}\varphi(x))dx \to \int_{\mathbb{R}^N} (|\nabla V_1|^{p-1}\nabla\varphi - |V_1|^{p-1}\varphi)dx$$

and $\langle I'(u_n), \varphi(x-y_n^1)\rangle \to 0$. Similar to the proof of proposition 4.6 in [3], we have

$$\int_{\mathbb{R}^N} (|\nabla V_1|^{p-1}\nabla\varphi - |V_1|^{p-1}\varphi + |V_1|^{2m-2\theta-2}V_1\varphi)dx = 0.$$

Therefore, we conclude that V_1 is a weak solution of the homogeneous Problem (1.4).

Now, we prove that in this case, we also have $U_0 \neq V_0$. In fact, since $v_n(\cdot + y_n^1) \rightharpoonup V_1$ in E we have $||V_1|| \leq \liminf_{n\to\infty} ||v_n(\cdot + y_n^1)||$. If $||V_1|| = \liminf_{n\to\infty} ||v_n(\cdot + y_n^1)|$, similar to [3] we have $u_n - V_0 - V_1(\cdot - y_n^1) \to 0$ in E. By (2.6) and (2.7), we get

$$\begin{aligned}\lim_{n\to\infty} I(u_n - V_0) \leq &\frac{1}{p}\lim_{n\to\infty}\int_{\mathbb{R}^N} (|\nabla u_n|^p - |\nabla V_0|^p)dx - \frac{1}{p}\lim_{n\to\infty}\int_{\mathbb{R}^N}(|u_n|^p - |V_0|^p)dx \\ &+ \frac{1}{2(m-\theta)}\lim_{n\to\infty}\int_{\mathbb{R}^N}(|u_n|^{2(m-\theta)} - |V_0|^{2(m-\theta)})dx \\ &- \int_{\mathbb{R}^N} fV_0 dx + \int_{\mathbb{R}^N} fV_0 dx.\end{aligned} \tag{4.15}$$

According to (4.15), we conclude that

$$\lim_{n\to\infty} I(u_n - V_0) \leq \lim_{n\to\infty} I(u_n) - I(V_0) = d - I(V_0). \tag{4.16}$$

Since $u_n - V_0 \rightharpoonup 0$ and $u_n - V_0 - V_1(\cdot - y_n^1) \to 0$ in E, then we conclude that $V_1(\cdot - y_n^1) \rightharpoonup 0$ in E and consequently we have

$$\lim_{n\to\infty} I(u_n - V_0) = \lim_{n\to\infty} I(V_1(\cdot - y_n^1)) = \lim_{n\to\infty} I^\infty(V_1(\cdot - y_n^1)). \tag{4.17}$$

After a scale change $z = x - y_n^1$ in the integral, we find that $\lim_{n\to\infty} I^\infty(V_1(\cdot - y_n^1)) = I^\infty(V_1)$. Since w is a ground state of problem (1.4), by (4.16), (4.17) and Lemma 4.5 we get

$$I(V_0) + I^\infty(w) \leq I(V_0) + I^\infty(V_1) \leq d < I(U_0) + I^\infty(w).$$

Then we arrive at $U_0 \neq V_0$.

If $||V_1|| < \liminf_{n\to\infty} ||v_n(\cdot + y_n^1)|$, we put $v_n^1 = u_n - V_0 - V_1(\cdot - y_n^1)$. Similar to [3], we restart the analysis with the sequence (v_n^1) while reiterating the process as many time as necessary. Similar to the above Step 3 and the proof of (4.8), (4.16) and (4.17), we conclude that there is a general decomposition of the following form

$$\begin{aligned} &u_n(x) - \Big(V_0(x) + \sum_{k=1}^{m} V_k(x - y_n^k)\Big) \to 0 \qquad \text{in} \ \ E, \\ &\forall\, k \geq 1, \qquad |y_n^{(k)}| \to +\infty, \qquad |y_n^{(k)} - y_n^{(l)}| \to +\infty \qquad \text{if } k \neq l, \\ &I(u_n) \to d \geq I(V_0) + \sum_{k=1}^{s_1} I^\infty(V_k) \end{aligned}$$

and $(V_k)_{1\leq k\leq s_1}$ are solutions of Problem (1.4) which satisfies $I^\infty(V_k) > 0$, $\forall\, 1 \leq k \leq s_1$. Therefore, we have

$$I(U_0) + I^\infty(w) > d \geq I(V_0) + I^\infty(V_1) \geq I(V_0) + I^\infty(w).$$

Thus we also have $V_0 \neq U_0$. The proof is complete. □

According to the proof of Theorem 4.7, we have

Proposition 4.8. *$(V_k)_{1\leq k\leq s_1}$ are solutions of the Problem (1.4), where (V_k), $1 \leq k \leq s_1$ are mentioned in the proof of Theorem 4.7.*

5 Existence of Infinitely Many Solutions

Theorem 5.1. *There exist infinitely many solutions to Problem (1.3).*

Proof. Let V_0 be the solution mentioned in Theorem 4.7. Let w be a nonnegative ground stat of Problem (1.4). By the same discussions in Lemma 4.3 we have

$$\begin{aligned} I(V_0 + tw) <& I(V_0) + \Big(\frac{t^{2(m-\theta)}}{2(m-\theta)} - \frac{t^p}{p}\Big) ||w||_{2(m-\theta)}^{2(m-\theta)} + (2^p - 1)t^{p-1} \int_{\mathbb{R}^N} |\nabla V_0||\nabla w|^{p-1} \\ &+ (2^p - 1)t \int_{\mathbb{R}^N} |\nabla V_0|^{p-1}|\nabla w| + \frac{2^{2(m-\theta)} - 1}{2(m-\theta)} t^{2(m-\theta)-1} \int_{\mathbb{R}^N} w^{2(m-\theta)-1} V_0 dx \\ &+ \frac{t^{2(m-\theta)}}{2(m-\theta)} \int_{\mathbb{R}^N} w^{2(m-\theta)} dx + \frac{2^{2(m-\theta)} - 1}{2(m-\theta)} t \int_{\mathbb{R}^N} w V_0^{2(m-\theta)-1} dx \\ &+ \frac{1}{2(m-\theta)} \int_{\mathbb{R}^N} |V_0|^{2(m-\theta)} dx - t \int_{\mathbb{R}^N} f w dx. \end{aligned} \tag{5.1}$$

By (5.1), we can choose t_0 large enough such that $I(V_0 + tw) < I(V_0)$ for all $t \geq t_0$. Put $\varphi_2 = V_0 + t_0 w$ and $\varphi_3 = \varphi_2 + w$. Let $d = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t))$ where $\Gamma = \{\gamma \in C([0, 1], E);\ \gamma(0) = \varphi_2, \gamma(1) = \varphi_3\} \neq \emptyset$. Similar to the discussions in Lemma 4.4 and Lemma 4.5, we have

$$-\infty < d < I(V_0), \qquad -\infty < d < I(V_0) + I^\infty(w). \tag{5.2}$$

By the mountain-pass theorem of Ambrosetti-Rabinowitz, we conclude that there exists (PS) sequence (u_n) in E such that $I(u_n) \to d$ and $I'(u_n) \to 0$ in E'. According to Lemma 4.6, (u_n) is bounded in E and then up to a subsequence there is a function U_1 such that $u_n \rightharpoonup U_1$ in E,

$u_n \to U_1$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for all $1 \le q \le p^*$ and $u_n \to U_1$ a.e in $\mathbb{R}^N$. Similar to the discussions in Theorem 4.7 we have

$$\langle I'(u_n), \varphi\rangle \to \langle I'(U_1), \varphi\rangle = 0, \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N).$$

Therefore, U_1 is a weak solution of Problem (1.3).

If $\lim\limits_{n\to\infty} ||u_n|| = ||U_1||$. Similar to the Case 1 in the proof of Theorem 4.7, we have $I(u_n) \to d = I(U_1)$. By (5.2), we conclude that $V_0 \ne U_1$.

If $\lim\limits_{n\to\infty} ||u_n|| > ||U_1||$. Set $v_n(x) = u_n(x) - U_1(x)$, then $v_n \rightharpoonup 0$ in E. Similar to the Case 2 in the proof of Theorem 4.7, we affirm that there exists $(y_n^1) \subset \mathbb{R}^N$ such that (y_n^1) is not bounded, $v_n(\cdot + y_n^1) \rightharpoonup U_2 \ne 0$ in E, and U_2 is a solution of the homogeneous Problem (1.4).

If $||U_2|| = \liminf\limits_{n\to\infty} ||v_n(\cdot + y_n^1)|$, then $u_n - U_1 - U_2(\cdot - y_n^1) \to 0$ in E. Similar to the proof of Theorem 4.7, we have

$$I(U_1) + I^\infty(w) \le I(U_1) + I^\infty(U_2) \le d < I(V_0) + I^\infty(w).$$

Then we arrive at $V_0 \ne U_1$.

If $||U_2|| < \liminf\limits_{n\to\infty} ||v_n(\cdot + y_n^1)|$, Similar to the proof of Theorem 4.7 we conclude that there exist $(U_k)_{2\le k\le s_2}$ which are solutions of problem (1.4). In this case, we also have $I(V_0)+I^\infty(w) > d \ge I(U_1) + I^\infty(U_2) \ge I(U_1) + I^\infty(w)$. Therefore, $V_0 \ne U_1$. Similarly, we can prove that $U_0 \ne U_1$. Repeat the above processes, we conclude that (1.3) has infinitely many solutions. The proof is complete. □

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