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Cubic Semisymmetric Graphs of Order $2qp^2$

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Abstract A regular edge-transitive graph is said to be *semisymmetric* if it is not vertex-transitive. Let p be a prime. By Folkman [J. Combin. Theory 3 (1967), 215–232], there is no cubic semisymmetric graph of order $2p$ or $2p^2$, and by Hua *et al.* [Science in China A 54 (2011), 1937–1949], there is no cubic semisymmetric graph of order $4p^2$. Lu *et al.* [Science in China A 47 (2004), 11–17] classified connected cubic semisymmetric graphs of order $6p^2$. In this paper, for $p > q \geq 5$ two distinct odd primes, it is shown that the sufficient and necessary conditions which a connected cubic edge transitive bipartite graph of order $2qp^2$ is semisymmetric.

Keywords Bi-Cayley graph; edge-transitive graph; semisymmetric graph; regular covering
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1 Introduction

Throughout this paper graphs are assumed to be finite, simple and undirected. For a graph X, let $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ be the vertex set, the edge set, the arc set and the automorphism group of X, respectively. For $u, v \in V(X)$, denote by uv the edge incident to u and v in X, and by $N_X(u)$ the neighborhood of u in X, the set of vertices adjacent to u in X. A graph $\widetilde{X}$ is called a *covering* of a graph X with projection $p : \widetilde{X} \to X$ if there is a surjection $p : V(\widetilde{X}) \to V(X)$ such that $p|_{N_{\widetilde{X}}(\widetilde{v})} \to N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\widetilde{v} \in V(\widetilde{X})$. A covering $\widetilde{X}$ of X with a projection p is said to be *regular* (or K-*covering)* if there is a semiregular subgroup K of the automorphism group $\mathrm{Aut}(\widetilde{X})$ such that graph X is isomorphic to the quotient graph $\widetilde{X}/K$, say by h, and the quotient map $\widetilde{X} \to \widetilde{X}/K$ is the composition ph of p and h (for the purpose of this paper, all functions are composed from left to right). If K is cyclic or elementary abelian then $\widetilde{X}$ is called a *cyclic* or an *elementary abelian covering* of X, and if $\widetilde{X}$ is connected K becomes the covering transformation group. The *fibre* of an edge or a vertex is its preimage under p. An automorphism of $\widetilde{X}$ is said to be *fibre-preserving* if it maps a fibre to a fibre, while every covering transformation maps a fibre on to itself.

An s-*arc* in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$, in other words, a directed walk of length s which never includes a backtracking. For a graph X and a subgroup G of $\mathrm{Aut}(X)$, X is said to be *G-vertex-transitive, G-edge-transitive* or *G-s-arc-transitive* if G is transitive on the sets of vertices, edges or s-arcs of X respectively, and *G-s-regular* if G acts regularly on the set of s-arcs of X. Similarly, a graph is *G-semisymmetric* if it is G-edge-transitive but not G-vertex-

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transitive. A graph X is said to be *vertex-transitive, edge-transitive, s-arc-transitive* or *s-regular* if X is $\mathrm{Aut}(X)$- vertex-transitive, $\mathrm{Aut}(X)$-edge-transitive, $\mathrm{Aut}(X)$-s-arc-transitive or $\mathrm{Aut}(X)$-s-regular respectively. In particular, 1-arc-transitive means *arc-transitive* or *symmetric*. It can be shown that a G-edge-transitive but not G-vertex-transitive graph is necessarily bipartite, where the two partite parts of the graph are orbits of G. Moreover, if X is regular, then these two partite sets $L(X)$ and $R(X)$ have equal cardinality. Let A^+ be the subgroup of $\mathrm{Aut}(X)$ fixing $L(X)$ and $R(X)$ setwise. Clearly, if X is connected then either $|\mathrm{Aut}(X) : A^+| = 2$ or $\mathrm{Aut}(X) = A^+$, depending on whether or not there exists an automorphism which interchanges the two parts. Suppose G is a subgroup of A^+. Then X is said to be G-*semitransitive* if G acts transitively on both $L(X)$ and $R(X)$, and *semitransitive* if X is A^+-semitransitive. We called a graph *semisymmetric* if it is regular and edge- but not vertex-transitive graph.

The class of semisymmetric graphs was first introduced by Folkman [11], where several in finite families of such graphs were constructed and eight open problems were posed which spurred the interest in this topic (see for example [1,2,13-16,26,31,33]). A beautiful recent result on the automorphism groups of cubic semisymmetric graphs of twice odd order was given by Parker [28]. Marušič [25] constructed the first in finite family of cubic semisymmetric graphs and as one of the first applications of covering techniques, Malnič, et al.[23] classified cubic semisymmetric cyclic coverings of the bipartite graph $K_{3,3}$ when the fibre-preserving group contains an edge- but not vertex-transitive subgroup. Semisymmetric elementary abelian coverings of the Heawood graph were considered in [5,22]. Using the method developed in [21,22], Malnič, et al.[20] determined all pairwise nonisomorphic minimal semisymmetric elementary abelian regular covering projections of the Möbius-Kantor graph, and Feng and Zhou[9] proved that the coverings corresponding to these covering projections are indeed semisymmetric. Malnič, et al.[24] classified cubic semisymmetric graphs of order $2p^3$ for a prime p, while Folkman [10] proved there is no cubic semisymmetric graphs of order $2p$ or $2p^2$. Du and Xu [7] classified connected cubic semisymmetric graphs of order $2pq$, one has there is no connected cubic semisymmetric graph of order $6p$, and Lu, et al.[18] classified connected cubic semisymmetric graphs of order $6p^2$. Feng, et al.[8] and Hua, et al.[11] classified connected cubic semisymmetric graphs of order $4p^3$, $6p^3$ and $8p^3$.

In this paper, we consider that whether a connected cubic edge-transitive bipartite graph of order $2qp^t(t = 2, 3)$ is semisymmetric. If $q = 2$ and $p = 3$, then there are no cubic semisymmetric graph of order 36 or 108 by Conder [5]. If $q = 2$ and $p > 3$, by [11, Lemma 3.1], $\mathrm{Aut}(X)$ has a normal sylow-p-subgroup, say P, and by Proposition 2.1, the quotient graph X_P of X relative to P has order 4. It follows that $X_P \cong K_4$, a contradiction because K_4 is not bipartite. If $q = 3$, by [18] and [8], there exists cubic semisymmetric graphs. Thus, we only consider a connected cubic edge-transitive bipartite graph of order $2qp^t(t = 2, 3)$ for $p > q \geq 5$.

Theorem 1.1. *Let $p > q \geq 5$ be two odd prime, and let X be a connected cubic edge transitive bipartite graph of order $2qp^t$, where $t = 2, 3$. If X is semisymmetric, then either X is a regular covering of Heawood graph for $q = 7$ or X is a connected cubic normal bi-Cayley graph on non-abelian group G of order qp^t for $q \neq 7$ and $3|q-1$.*

Let $G_7 = \langle a, b, c \mid a^p = b^p = c^q = [a, b] = 1, a^c = b, b^c = a^{-1}b^{l+l^p}, l^q \equiv 1 \pmod p\rangle$ be a nonabelian group of order qp^2. Further, one has the following theorem.

Theorem 1.2. *Let $p > q \geq 5$ be two odd prime, and let X be a connected cubic edge-transitive bipartite graph of order $2qp^2$. Then X is semisymmetric if and only if there exists an element of order three $\alpha \in \mathrm{Aut}(G_7)$ such that $G_7 = \langle c, c^\alpha\rangle$, and $c^\alpha = a^s b^t c^w$, $s, t \in \mathbb{Z}_p$, $w^2 + w + 1 = 0 (\mathrm{mod}\ q)$, and there does not exist an element of order two $\beta \in \mathrm{Aut}(G_7)$ such that $c^\beta = c^{-1}, c^{\alpha\beta} = (c^\alpha)^{-1}$. Furthermore, $X = B\mathrm{Cay}(G_7, S)$ is a normal bi-Cayley graph on G_7, where $S = \{1, c^{-1}, c^\alpha\}$.*

2 Preliminary Results

For a connected cubic G-semisymmetric graph, the action of a normal subgroup of G was considered by Lu, Wang and Xu in [18].

Proposition 2.1 *Let X be a connected cubic G-semisymmetric graph with bipartite sets $L(X)$ and $R(X)$, and let N be a normal subgroup of G. If N is intransitive on both $L(X)$ and $R(X)$ then N acts semiregularly on $L(X)$ and $R(X)$. The kernel of G on the quotient graph X_N is N, and X is an N-covering of X_N, which is a G/N-semisymmetric graph.*

Let X be a connected bipartite graph and H an abelian subgroup of $\mathrm{Aut}(X)$ acting regularly on each bipartite set of X. Then we may identify $R(X) = \{R(h) \mid h \in H\}$ and $L(X) = \{L(h) \mid h \in H\}$ as the two bipartition sets of X, and the action of $h \in H$ on $R(X)$ and $L(X)$ is just the left multiplication by h, that is, $R(g)^h = R(hg)$ and $L(g)^h = L(hg)$ for any $g, h \in H$. It is easy to check that the map $R(h) \mapsto L(h^{-1}),\ L(h) \mapsto R(h^{-1}),\ h \in H$, is an automorphism of X interchanging $R(X)$ and $L(X)$.

Proposition 2.2. *Let X be a connected G-semisymmetric graph with bipartition sets $L(X)$ and $R(X)$, and let $H \le G$ be abelian acting regularly on $L(X)$ and $R(X)$, respectively. Then X is symmetric.*

Now let us mention several definitions which will be used in the following sections. For a group T, and a subset S (possibly, containing the identity element 1_T) of T, the *bi-Cayley graph* $\mathrm{BCay}(T, S)$ of T with respect to S is bipartite graph with vertex set $T \times \{0, 1\}$ and edge set $\{\{(g, 0), (sg, 1)\} | g \in T,\ s \in S\}$.

Each

$$R(g) : (x, 0) \mapsto (xg, 0), (x, 1) \mapsto (xg, 1), \qquad \forall\, x \in T$$

is an automorphism of $\mathrm{BCay}(T, S)$. Set $R(T) = \{R(g) | g \in T\}$. Furthermore, a bi-Cayley graph $\mathrm{BCay}(T, S)$ is said to be *normal* if $R(T)$ is normal in $\mathrm{Aut}(\mathrm{BCay}(T, S))$.

Analogous to a theorem for Cayley graphs, the following can be gained from [7, Lemma 2.5].

Proposition 2.3. *A semitransitive graph X is bi-Cayley if and only if $(\mathrm{Aut}\, X)^+$ has a subgroup which acts regular on each of the two parts of $V(X)$.*

Proposition 2.4 ([17, Theorem 2.3]). *Let S be subset of finite non-cyclic group G, containing the identity element 1 of G, such that $|S| = 3$ and $\langle S\rangle = G$. Set $X = \mathrm{BCay}(G, S)$ and A^+ be the maximal subgroup of $\mathrm{Aut}(X)$ fixing two bipartition sets of X setwise. Then $N_{A^+}(R(G))$ acts transitively on the edge set $E(X)$ if and only if X is isomorphic to $\mathrm{BCay}(G, S_1)$, where $S_1 = \{1, a^{-1}, a^{\alpha}\}$, α is an automorphism of group G of order three such that $aa^{\alpha}a^{\alpha^2} = 1$.*

Proposition 2.5. ([19, Theorem 3.1]). *Let T be a finite nonabelian group and $S = \{1_T, a, b\}$ be a subset of T such that $T = \langle a, b\rangle$, and let $X = \mathrm{BCay}(T, S)$. Suppose that $R(T)$ is normal in $\mathrm{Aut}(X)$. If X is edge-transitive, then X is symmetric if and only if there exists $\alpha \in \mathrm{Aut}(T)$ such that $S^{\alpha} = S^{-1}$.*

We introduce the so called coset graph (see [27,29]) constructed from a finite group G relative to a subgroup H of G and a union D of some double cosets of H in G such that $D^{-1} = D$. The coset graph $\mathrm{Cos}(G, H, D)$ of G with respect to H and D is defined to have vertex set $[G : H]$, the set of right cosets of H in G, and edge set $\{\{Hg, Hdg\} | g \in G, d \in D\}$. The graph $\mathrm{Cos}(G, H, D)$ has valency $|D|/|H|$ and is connected if and only if D generates the group G.

By Zhou and Feng [34], we have the following proposition.

Proposition 2.6. *Let X be a connected cubic symmetric graph of order $2pq$, where $p > q$ are odd primes. Then X can be s-regular for each $1 \le s \le 5$. Furthermore,*

(1) *X is 1-regular if and only if either $q = 3$ and $3|(p-1)$ or $3|(p-1)$ and $3|(q-1)$.*

(2) *X is 2-regular if and only if it is isomorphic to $\mathcal{NC}^1_{182}$, where $\mathcal{NC}^1_{182}$ is isomorphic to the coset graph of $\mathrm{PSL}(2,13)$ relative to the subgroup S_3, and $\mathrm{Aut}(\mathcal{NC}^1_{182}) = \mathrm{PSL}(2,13)$.*

(3) *X is 3-regular if and only if it is isomorphic to $\mathcal{NC}^2_{182}$, $\mathcal{NC}_{506}$, or Coxer-Frucht graph CF_{110}, where $\mathcal{NC}^2_{182}$ is isomorphic to the coset graph of $\mathrm{PGL}(2,13)$ relative to the subgroup D_{12}, $\mathcal{NC}_{506}$ is isomorphic to the coset graph of $\mathrm{PSL}(2,23)$ relative to the subgroup D_{12}, and $\mathrm{Aut}(\mathcal{NC}^2_{182}) = \mathrm{PGL}(2,13)$, $\mathrm{Aut}(\mathcal{NC}_{506}) = \mathrm{PSL}(2,23)$, $\mathrm{Aut}(CF_{110}) = \mathrm{PGL}(2,11)$.*

(4) *X is 4-regular if and only if it is isomorphic to $\mathcal{C}_{506}$, $\mathcal{NC}_{2162}$, or the Smith-Biggs graph SB_{102}, where $\mathcal{C}_{506}$ and $\mathcal{NC}_{2162}$ is isomorphic to the Coset graph of $\mathrm{PGL}(2,23)$ and $\mathrm{PSL}(2,47)$ relative to the subgroup S_4, respectively, and $\mathrm{Aut}(\mathcal{C}_{506}) = \mathrm{PGL}(2,23)$,*

$\mathrm{Aut}(\mathcal{NC}_{2162}) = \mathrm{PSL}(2,47)$, $\mathrm{Aut}(SB_{102}) = \mathrm{PSL}(2,17)$.

(5) *X is 5-regular if and only if it is isomorphic to the Levi graph L_{30}, where $\mathrm{Aut}(L_{30}) = S_6 \rtimes \mathbb{Z}_2$.*

By [3, IV chapter], one has the following proposition.

Proposition 2.7. *Let G be a non-abelian group of order qp^2. Then G is one of the following presentations:*

$$G_1 = \langle a, b \mid a^q = b^{p^2} = 1, a^b = a^r, r^p \equiv 1 \pmod q\rangle,$$

$$G_2 = \langle a, b \mid a, b \mid a^q = b^{p^2} = 1, a^b = a^r, r^{p^2} \equiv 1 \pmod q, r^p \not\equiv 1 \pmod q\rangle,$$

$$G_3 = \langle a, b \mid a^{p^2} = b^q = 1, a^b = a^r, r^q \equiv 1 \pmod{p^2}\rangle,$$

$$G_4 = \langle a, b, c \mid a^p = b^p = c^q = [a,b] = [a,c] = 1, c^b = c^r, r^p \equiv 1 \pmod q\rangle \cong \mathbb{Z}_p \times F_{qp},$$

$$G_5 = \langle a, b, c \mid a^p = b^p = c^q = [a,b] = [a,c] = 1, b^c = b^r, r^q \equiv 1 \pmod p\rangle \cong \mathbb{Z}_p \times F_{pq},$$

$$G_{6x} = \langle a, b, c \mid a^p = b^p = c^q = [a,b] = 1, a^c = a^r, b^c = b^{r^x}, r^q \equiv 1 \pmod p, x \neq 0\rangle;$$

$$G_7 = \langle a, b, c \mid a^p = b^p = c^q = [a,b] = 1, a^c = b, b^c = a^{-1}b^{l+l^p}, l^q \equiv 1 \pmod p\rangle.$$

Note: Obviously, if $q < p$, then G is not G_1, G_2 or G_4. For G_{6x}, set $u = b, v = a, w = c^y$ and $xy \equiv 1 \pmod q$. we have $\langle a, b, c \mid a^p = b^p = c^q = [a,b] = 1, a^c = a^r, b^c = b^{r^x}\rangle \cong \langle u, v, w \mid u^p = v^p = w^q = [u,v] = 1, u^w = u^r, v^c = v^{r^y}\rangle$. So x has $1 + 1 + \frac{q-3}{2} = \frac{q+1}{2}$ options. It is easy to check that they are nonisomorphic each other for each x, that is, there are $\frac{1}{2}(q+1)$ non-isomorphic groups. For G_7, l is a complex number, and $l^q \equiv 1 \pmod p$ and $l^p + l$ are interge numbers. □

Lemma 2.8. *Let α be a group automorphism of G_{6x}. If $x = 1$, then α is the following form:*

$$\alpha := \begin{cases} a \mapsto a^i b^j, \\ b \mapsto a^k b^m, \\ c \mapsto a^s b^t c. \end{cases}$$

where $i, j, k, m, s, t \in \mathbb{Z}_p$.

If $x \neq 1$, then either

$$\alpha := \begin{cases} a \mapsto a^i, \\ b \mapsto b^m, \\ c \mapsto a^s b^t c. \end{cases} \qquad \alpha := \begin{cases} a \mapsto b^j, \\ b \mapsto a^k, \\ c \mapsto a^s b^t c^{-1}. \end{cases}$$

where $i, j, k, m, s, t \in \mathbb{Z}_p$*, and the last case occurs only when* $x = -1$.

Proof. Suppose that $\alpha : a \mapsto a^i b^j,\ b \mapsto a^k b^m,\ c \mapsto a^s b^t c^y$, where $i, j, k, m, s, t \in \mathbb{Z}_p$ and $y \in \mathbb{Z}_q$. By Proposition 2.7, one has $a^c = a^r$. Considering the images of $a^c = a^r$ under α, one has $a^{ir^y} b^{jr^{xy}} = a^{ir} b^{jr}$, implying that $ir^y = ir$ and $jr^{xy} = jr$.

Thus, if $x = 1$, then $y = 1 \pmod q$. If $x \neq 1$, then $y = 1 \pmod q$, $j = 0 \pmod p$ or $xy = 1 \pmod q$, $i = 0 \pmod p$. For the last case, Considering the images of $b^c = b^{r^x}$ under α, one has $a^{kr^y} = a^{kr^x}$, implying that $kr^y = kr^x$, that is, $y = x \pmod q$. Combining $xy = 1 \pmod q$, one has $y = x = \pm 1 \pmod q$. □

Finally we introduce some concepts. Let G be a simple group and Z an abelian group. We call an extension E of Z by G a *central extension* of G if $Z \leq Z(E)$. If E is perfect, that is, the derived group $E' = E$, we call E a *covering group* of G. Schur proved that for every simple group G there is a unique maximal covering group M such that every covering group of G is a factor group of M. This group M is called the *full covering group* of G, and the center of M is called the *Schur multiplier* of G, denoted by $\mathrm{Mult}(G)$. □

3 Main Result

Lemma 3.1. *Let* X *be a connected cubic edge-transitive bipartite graph of order* $2qp^t$*, where* $t \geq 2$ *and* $p > q \geq 5$*. Then one of the following holds:*

(1) $\mathrm{Aut}(X)$ *has normal Sylow* p*-subgroups;*

(2) $\mathrm{Aut}(X)$ *has normal* p*-subgroups of order* p^{t-1}*. Further,* X *is a regular covering of cubic edge-transitive bipartite graph of order* $2qp$.

Proof. Denote by $L(X)$ and $R(X)$ the bipartite sets of X. Clearly, $|L(X)| = |R(X)| = qp^t$. Let A^+ be the subgroup of $\mathrm{Aut}(X)$ fixing $L(X)$ and $R(X)$ setwise. Then $|\mathrm{Aut}(X) : A^+| \leq 2$ and X is A^+-semisymmetric. Since X has valency 3, the connectivity of X implies that $|A^+| = 2^m \cdot 3 \cdot q \cdot p^t$ for some integer $t \geq 2$. For any prime divisor of $|A^+|$, let $O_r(A^+)$ be the maximal normal r-subgroup of A^+. We show that $|O_p(A^+)| = p^t$ or $|O_p(A^+)| = p^{t-1}$.

Suppose that $|O_p(A^+)| = p^s$ with $0 \leq s < t$. To finish the proof, it is sufficient to show that $s = t - 1$. Let $H/O_p(A^+) = O_r(A^+/O_p(A^+))$. Clearly, $O_p(A^+/H) = 1$. Since $|L(X)| = |R(X)| = qp^t$ and $H \trianglelefteq A^+$, H can not be transitive on $L(X)$ and $R(X)$. By Proposition 2.1, H is semiregular on $L(X)$ and $R(X)$, implying $r = q$, and $|H| = p^s$ or qp^s, depending on $|H/O_p(A^+)| = 1$ or q, respectively. Since $A^+/O_p(A^+)/H/O_p(A^+) \cong A^+/H$, one has $O_q(A^+/H) = 1$. It follows that $O_2(A^+/H) = O_3(A^+/H) = O_q(A^+/H) = O_p(A^+/H) = 1$.

Let $B = A^+/H$ and denote by $Y = X_H$ the quotient graph of X relative to the orbits of H. By proposition 2.1, one may assume that $B \leq \mathrm{Aut}(Y)$, and Y is a cubic B-semisymmetric graph. Then Y is bipartite and let $L(Y)$ and $R(Y)$ be the two partite sets of Y. Depending on $|H/O_p(A^+)| = 1$ or q, one has $|L(Y)| = |R(Y)| = p^{t-s}$ or qp^{t-s}, and $|B| = 2^m \cdot 3 \cdot p^{t-s}$ or $|B| = 2^m \cdot 3 \cdot q \cdot p^{t-s}$, where $t - s \geq 1$. The connectivity of Y implies that $3^2 \nmid |B_u|$ for any $u \in V(Y)$. Let N be a minimal normal subgroup of B. Since $O_2(B) = O_3(B) = O_q(B) = O_p(B) = 1$, N is non-solvable. If N is not transitive on both $R(Y)$ and $L(Y)$, then by Propositon 2.1, N is semiregular. Since $|R(Y)| = |L(Y)| = qp^{t-s}$ or p^{t-s}, it contradicts that the nonsolvability of N. Thus, N is transitive on $R(Y)$ or $L(Y)$. Without loss of any generality, let N be transitive on $L(Y)$. For any $x \in V(Y)$, B_x is transitive on $N_Y(x)$ because of the B-semisymmetric of Y. Clearly B_x is primitive on $N_Y(x)$. Since $N_x \trianglelefteq B_x$, N_x is transitive on $N_Y(x)$, implying that $3 \mid |N_x|$. Note that $3^2 \nmid |N_x|$. Let $u \in L(Y)$ and $v \in R(Y)$. Then $|N| = |u^N||N_u| = |v^N||N_v|$, that is, $|L(Y)||N_u| = |v^N||N_v|$. Suppose that N is not transitive on $R(Y)$. Then $|v^N| \neq |R(Y)|$, and the normality of N in B, implies that $|v^N|$ is a divisor of $|R(Y)|$. It follows that $|N_v| = \frac{|L(Y)|}{|v^N|} \cdot |N_u|$, which forces $q = 2$ and $|N_v| = 2|N_u|$ because

$|L(Y)| = p^{t-s}$ or qp^{t-s}, and $3^2 \nmid |N_v|$. Thus, N has two orbits on $R(Y)$, say $R_1(Y)$ and $R_2(Y)$. Since Y has valency 3, u has two neighbors not only in $R_1(Y)$ but also in $R_2(Y)$, forcing that Y has valency at least 4, a contradiction. Thus N is transitive both on $L(Y)$ and $R(Y)$, hence Y is N-semisymmetric,

Since N is nonsolvable, $N = T^m$, where T is a non-abelian simple group. Recall that $N \leq B$ and $|B| = 2^m \cdot 3 \cdot p^{t-s}$ or $|B| = 2^m \cdot 3 \cdot q \cdot p^{t-s}$, and by [12, p.134–136] or [30], N is one of the following groups:

$\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,16)$, $\mathrm{PSL}(2,p)$, where $p^2 - 1 = 2^a \cdot 3 \cdot q$ with $p > q > 3$ primes, and $\mathrm{PSL}(2,2^m)$, where $2^m - 1 = p$, $2^m + 1 = 3q$ with $p > q > 3$ primes. In these cases, one has $p||N|$ and $p^2 \nmid |N|$. Note that Y is N-semisymmetric graph, one may easily get $|V(Y)| = 2p$ or $2qp$, implying that $t - s = 1$. □

Lemma 3.2. *Let X be a connected cubic edge-transitive bipartite graph of order $2qp^t$, where $t = 2, 3$ and $p > q \geq 5$. Then $\mathrm{Aut}(X)$ has normal Sylow p-subgroups.*

Proof. By Lemma 3.1, $|Q| = |O_p(A^+)| \geq p^{t-1}$. Suppose $|Q| = p^{t-1}$, we show that this leads to a contradiction.

Denote by X_Q the quotient graph of X relative to the orbits of Q. Then $|V(X_Q)| = 2qp$. By [7], there are three cubic semisymmetric graphs of order 110, 182 and 506. They are bi-Cayley graphs, and $\mathrm{PGL}(2,11), \mathrm{PGL}(2,13), \mathrm{PSL}(2,23)$ act bi-primitively on their vertex set, respectively. By Proposition 2.6, a cubic connected vertex transitive graphs of order $2qp$ were classified and their autmorphism of graphs had also determined. Thus, if $\mathrm{Aut}(X_Q)$ is nonsolvable, there exists a normal subgroup of $\mathrm{PSL}(2,p) = M/Q \leq \mathrm{Aut}(X_Q)$, respectively. Obviously, $C_M(Q) \geq Q$. If $C_M(Q) = Q$, then $M/C_M(Q) = M/Q \lesssim \mathrm{Aut}(Q)$, that is, $\mathrm{PSL}(2,p) \leq \mathrm{Aut}(Q)$. By the simplify of $\mathrm{PSL}(2,p)$, Q is not a cyclic group. It follows that $\mathrm{PSL}(2,p) \leq \mathrm{GL}(2,p)$, implying that $\mathrm{PSL}(2,p) \leq \mathrm{SL}(2,p)$. Since $\mathrm{SL}(2,p)/(Z \cap \mathrm{SL}(2,p)) = \mathrm{PSL}(2,p)$, where $Z = Z(\mathrm{GL}(2,p))$, $\mathrm{SL}(2,p) = \mathrm{PSL}(2,p) \times \mathbb{Z}_2$. Note that $\mathrm{SL}(2,p)$ has only one involution, a contradiction. If $C_M(Q) > Q$, then $C_M(Q)/Q$ is a normal subgroup of $M/Q = \mathrm{PSL}(2,p)$. By the simplify of M/Q, $C_M(Q)/Q = M/Q$. Thus, $Q = Z(M)$. Let M' be the derived group of M. Then $(M/Q)' = M'Q/Q = M/Q$, and $M'/M' \cap Q = M/Q = \mathrm{PSL}(2,p)$. So $|M' \cap Q| \leq p^{t-1}$. If $|M' \cap Q| < p^{t-1}$, then $|p^t| \nmid |M'|$. By Proposition 2.1, M' is semiregular on every bipartite of X. It imply that $|M'| \mid qp^t$, which is contradict to the nonsolvablity of M'. If $|M' \cap Q| = p^{t-1}$, then $M' = M$. Thus M is a covering group of $\mathrm{PSL}(2,p)$, and $Q = Z(M)$ is a Schur multiple of $\mathrm{PSL}(2,p)$. By [6], we have that for all $\mathrm{PSL}(2,p)$, p does not divide the order of their Schur multiplier, a contradiction. If $\mathrm{Aut}(X_Q)$ is solvable, then by [34], X_Q is 1-regular and $3|p-1, q-1$, implying $|\mathrm{Aut}(X_Q)| = 6qp$. Since X is a connected Q-covering X/Q, we have $|\mathrm{Aut}(X)| = 6qp^t$. Let G be a subgroup of $\mathrm{Aut}(X)$ with order qp^t. Set K the kernel of $\mathrm{Aut}(X)$ on $[\mathrm{Aut}(X) : G]$. Then $\mathrm{Aut}(X)/K \lesssim S_6$. Since $3|p-1, q-1$ and $5 \leq q < p$, implying that $|K| \geq qp^t$. Thus $K = G$, that is, $G \trianglelefteq \mathrm{Aut}(X)$. Let P be a Sylow-p subgroup of $\mathrm{Aut}(X)$. Obviously, P is the Sylow-p subgroup of G. By Sylow-Theorem, $P \lhd G$, implying P is characteristic in G. Thus P is a normal subgroup of $\mathrm{Aut}(X)$ because G is normal in $\mathrm{Aut}(X)$. This completes the proof. □

Proof of Theorem 1.1. Let X be a connected cubic edge-transitive bipartitie graph of order $2qp^t$ for two primes $p > q \geq 5$ and $t = 2, 3$. By lemma 3.2, $\mathrm{Aut}(X)$ has a normal Sylow p-subgroup, say P. Considering the quotient graph X_P of X relative to the orbits of P. By proposition 2.1, X_P is a cubic edge-transitive bipartite graph of order $2q$, and by [4], X_P is isomorphic to Petersen graph for $q = 5$, or X_P is isomorphic to Heawood graph for $q = 7$, or $X_P \cong G(2q, 3)$ with $3|q-1$ for $q > 7$. Obviously, for $q = 5$, Petersen graph is not bipartite, a contradiction. For $q = 7$, X is a regular P-covering of Heawood graph. For $q > 7$, assume X is a connected cubic semisymmetric graph. Then X is a regular P-covering of $G(2q, 3)$ with a lift of a semisymmetric group of $\mathrm{Aut}(\mathrm{G}(2q,3))$. By [4], $\mathrm{Aut}(G(2q,3)) = (\mathbb{Z}_q \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, and $\mathrm{Aut}(\mathrm{G}(2q,3))$ has a semisymmetric subgroup N, where $N = \mathbb{Z}_q \rtimes \mathbb{Z}_3$. If N is abelian, then the vertex stabilizer of $\mathrm{Aut}(\mathrm{G}(2q,3))$ $\mathbb{Z}_3$ is characteristic in N, implying that $\mathbb{Z}_3 \lhd \mathrm{Aut}(\mathrm{G}(2q,3))$, a contradiction. Thus N is nonabelian. Let $T = P \rtimes N$. Then X is T-semisymmetric. Set $G \leq T$ and $|G| = qp^t$. Let K be the kernel of T on $[T:G]$. Then $T/K \lesssim S_3$ becuase $[T:G] = 3$, implying that $K = G$, that is, $G \unlhd T$. We claim that G is regular on $R(X)$ and $L(X)$. For any $x \in V(X)$, T_x is transitive on $N_X(x)$ because T is semisymmetric on X. Clearly T_x is primitive on $N_X(x)$. Since $G_x \unlhd T_x$, we have $3 \mid |G_x|$ or $G_x = 1$. Note that 2 and 3 don't divide $|G|$. It follows that G is semiregular on $R(X)$ and $L(X)$, and then G is regular on $L(X)$ and $R(X)$ because $|G| = |R(X)| = |L(X)|$.

By Proposition 2.3, X is a bi-Cayley graph of G. Hence X is a normal bi-Cayley of G. □

Proof of Theorem 1.2. Let X be a connected cubic edge-transitive bipartitie graph of order $2qp^2$ for two primes $p > q \geq 5$. By Lemma 3.2, $\mathrm{Aut}(X)$ has a normal Sylow-p-subgroup, say P. It follows that X is a regular covering of X_P. By Theorem 1.1, one has $q \neq 5$. For $q = 7$, X is a regular covering of Heawood graph with a transformation group P of order p^2 because a connected cubic edge-transitive bipartite graph of order 14 is Heawood graph. Whether $P = \mathbb{Z}_{p^2}$ or $P = \mathbb{Z}_p^2$, there are not semisymmetric graphs which satisfy these conditions by [32] and [22], respectively. For $q > 7$, by Theorem 1.1, X is a connected cubic normal edge-transitive bi-Cayley graph on group G of order qp^2. If X is semisymmetric, one has G is non-ablelian by Proposition 2.2. Set $X = \mathrm{BCay}(G,S)$. By Proposition 2.7, one has $G = G_i (i = 3,5,6x,7)$. Suppose that the Sylow p-subgroup P of G is cyclic. Recall that X is T-semisymmetric, where $T = P \rtimes N$ and $N = \mathbb{Z}_q \rtimes \mathbb{Z}_3$. Let $C = C_T(P)$, obviously, $C \geq P$. If $C = P$, then $N = T/C \lesssim \mathrm{Aut}(P)$. Thus, N is abelian, a contradiction. If $C > P$, then $|C| = qp^2$ because C is semiregular on $V(X)$. Thus, X is the bi-Cayley graph on C, implying that X is symmetric graph, a contradiction. Thus, the Sylow p-subgroup P of G is not cyclic, that is $G \neq G_3$. Suppose that G has a center subgroup $Z = \mathbb{Z}_p$. Since G is normal in T, one has Z is normal in T. Considering that the quotient graph X_Z, X_Z is a edge-transitive graph of order $2qp$. Note that $p > q > 7$. By [7], there is a cubic semisymmetric graph of order 506, which is a bi-Cayley graph, and $\mathrm{PSL}(2,23)$ is bi-primitive on its vertex set. By Proposition 2.6, a cubic connected vertex transitive graphs of order $2qp$ are classified and their autmorphism of graphs had also determined. Thus, if $\mathrm{Aut}(X_Z)$ is nonsolvable, there exists a normal subgroup of $\mathrm{PSL}(2,p) = M/Z \leq \mathrm{Aut}(X_Z)$, respectively. Obviously, $C_M(Z) \geq Z$. If $C_M(Z) = Z$, then $M/C_M(Z) = M/Z \lesssim \mathrm{Aut}(Z)$, that is, $\mathrm{PSL}(2,p) \leq \mathrm{Aut}(Z)$, a contradiction. If $C_M(Z) > Z$, then $C_M(Z)/Z$ is a normal subgroup of $M/Z = \mathrm{PSL}(2,p)$. By the simplify of M/Z, $C_M(Z)/Z = M/Z$. Thus, $Z = Z(M)$. Let M' be the derived group of M. Then $(M/Z)' = M'Z/Z = M/Z$, and $M'/M' \cap Z = M/Z = \mathrm{PSL}(2,p)$. So $M' \cap Z = Z$ or 1. If $M' \cap Z = 1$, then $|p^2| \nmid |M'|$. By Proposition 2.1, M' is semiregular on every bipartite of X. It imply that $|M'| \mid qp^2$, which is contradict to the nonsolvablity of M'. If $M' \cap Z = Z$, then $M' = M$. Thus M is a covering group of $\mathrm{PSL}(2,p)$, and $Z = Z(M)$ is a Schur multiple of $\mathrm{PSL}(2,p)$. By [6], we have that for all $\mathrm{PSL}(2,p)$, p does not divide the order of their Schur multiplier, a contradiction. Thus $\mathrm{Aut}(X_Z)$ is solvable. By [34], X_Z is 1-regular symmetric graph and $\mathrm{Aut}(X_Z) = D_{2pq} \rtimes \mathbb{Z}_3$, implying that the subgroup of order pq of $\mathrm{Aut}(X_Z)$ is cyclic, a contradiction because $G_5/Z = F_{pq} \leq \mathrm{Aut}(X_Z)$. Thus, $G = G_{6x}$ or G_7. Obviously, both groups can be generated by two elements of order q. By Proposition 2.4, one has $S = \{1, c^{-1}, c^\alpha\}$, where $\alpha \in \mathrm{Aut}(G)$ is an element of order 3 such that $cc^\alpha c^{\alpha^2} = 1$. Without loss of generality, we may assume that

$$\alpha := \begin{cases} a \mapsto a^i b^j, \\ b \mapsto a^k b^m, \\ c \mapsto a^s b^t c^w. \end{cases}$$

where $i,j,k,m,s,t \in \mathbb{Z}_p$ and $w^3 = 1 \pmod q$. Further, $w = 1$ or $w^2 + w + 1 = 0 \pmod q$.

Suppose that $G = G_{6x}$. Since $w^3 = 1$, one has $w = 1$ by Lemma 2.8, that is, $c^\alpha = a^s b^t c$. It follows that

$$\begin{aligned}1 =&cc^{\alpha}c^{\alpha^2} = ca^sb^tc(a^sb^tc)^{\alpha} = ca^sb^tc(a^ib^j)^s(a^kb^m)^ta^sb^tc\\ =&ca^sb^tca^{is+kt+s}b^{js+mt+t}c = c^2(c^{-1}a^sc)(c^{-1}b^tc)a^{is+kt+s}b^{js+mt+t}c\\ =&c^2a^{rs}b^{r^xt}a^{is+kt+s}b^{js+mt+t}c.\end{aligned}$$

Thus, $a^{rs+is+kt+s}b^{r^xt+js+mt+t} = c^{-3}$. Obviously, a contradiction.

Thus, $G = G_7$. When $w = 1$, similarly, a contradiction. If there is three order element $\alpha \in \mathrm{Aut}(G_7)$ such that $1 = cc^{\alpha}c^{\alpha^2}$ and $w^2 + w + 1 = 0 \pmod q$. By Proposition 2.5, X is symmetric if and only if there exists a two order element $\beta \in \mathrm{Aut}(G_7)$ such that $S^{\beta} = S^{-1}$. It follows that $\{(c^{-1})^{\beta}, c^{\alpha\beta}\} = \{c, (c^{\alpha})^{-1}\}$. Suppose that $(c^{-1})^{\beta} = (c^{\alpha})^{-1}, c^{\alpha\beta} = c$, one has $w^2 = 1$, a contradiction. Thus $(c^{-1})^{\beta} = c, (c^{\alpha})^{-1} = c^{\alpha\beta}$. It follows that X is a cubic semisymmetric graph of order $2qp^2$ if and only if there is three order element $\alpha \in \mathrm{Aut}(G_7)$ such that $1 = cc^{\alpha}c^{\alpha^2}$ and $w^2 + w + 1 = 0 \pmod q$, and there is not two order element $\beta \in \mathrm{Aut}(G_7)$ such that $(c^{-1})^{\beta} = c, (c^{\alpha})^{-1} = c^{\alpha\beta}$. Further, $X = \mathrm{BCay}(G_7, S)$ and $S = \{1, c^{-1}, c^{\alpha}\}$. □

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