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Standing Waves for Discrete Nonlinear Schrödinger Equations with Nonperiodic Bounded Potentials

Tie-shan HE, Meng ZHANG, Kai-hao LIANG, Peng-fei GUO[†]

College of Computation Science, Zhongkai University of Agriculture and Engineering, Guangzhou 510225, China
([†]E-mail: luyimin68@163.com)

Abstract In this paper, we investigate standing waves in discrete nonlinear Schrödinger equations with nonperiodic bounded potentials. By using the critical point theory and the spectral theory of self-adjoint operators, we prove the existence and infinitely many sign-changing solutions of the equation. The results on the exponential decay of standing waves are also provided.

Keywords Discrete nonlinear Schrödinger equation; Standing wave; Nonperiodic bounded potential; Sign-changing solution; Critical point theory

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Introduction

We consider the discrete nonlinear Schrödinger equation (DNLS)

$$\mathrm{i}\dot{\psi_n} + \Delta\psi_n - \widetilde{v}_n\psi_n + \gamma_n|\psi_n|^{p-2}\psi_n = 0, \qquad n \in \mathbb{Z}, \quad 2 < p < \infty, \tag{1}$$

where $\Delta\psi_n = \psi_{n+1} + \psi_{n-1} - 2\psi_n$ is the discrete Laplacian operator, and $\widetilde{v}_n$ and γ_n are real-valued sequences−the potential and anharmonic parameters, respectively. We consider the special solutions of (1) of the form $\psi_n = e^{-\mathrm{i}t\omega}u_n$. These solutions are called breather solutions or standing waves due to their periodic behavior. Inserting the ansatz of standing waves into (1), we see that any breather solution satisfies the infinite nonlinear system of algebraic equations

$$-\Delta u_n + \widetilde{v}_n u_n - \omega u_n - \gamma_n|u_n|^{p-2}u_n = 0. \tag{2}$$

Looking for solitary waves, we supplement equation (2) with zero boundary condition at infinity:

$$\lim_{n\to\pm\infty} u_n = 0. \tag{3}$$

In the following we will consider a more general equation

$$-\Delta u_n + v_n u_n - \gamma_n f(n, u_n) = 0 \tag{4}$$

with the same boundary Condition (3), where $v_n = \widetilde{v}_n - \omega$ is the potential of equation (4). We are interested in the existence of sign-changing solutions. As usual, we say that a solution $u = \{u_n\}_{n\in\mathbb{Z}}$ of (4) is sign-changing if $\mathrm{sgn}(u_s) = -\mathrm{sgn}(u_t)$ for some $s \in \mathbb{Z}$ and $t \in \mathbb{Z}$. And a solution $u = \{u_n\}_{n\in\mathbb{Z}}$ of (4) is positive if $\mathrm{sgn}(u_s) = 1$ for all $s \in \mathbb{Z}$. We consider (4) as a

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[†]Corresponding author.

nonlinear equation in the space l^2 of two-sided infinite sequences. Note that any element of l^2 automatically satisfies (3).

DNLS equation is a nonlinear lattice system that appears in many areas of physics such as nonlinear optics [2], biomolecular chains [8] and Bose-Einstein condensates [11]. Some reviews on DNLS equations can be found in [3,4,7], especially both theoretical and numerical simulation results can be found in [7]. The discrete Schröinger equation with periodic potentials and periodic nonlinearities has found a great deal of interest in very recent years, see [12–14,17,21,25]. The discrete Schröinger equation with unbounded potential also has been investigated by some authors, see [15,22–24]. In these papers, the condition that the potential $V = \{v_n\}_{n\in\mathbb{Z}}$ satisfies

$$\lim_{|n|\to\infty} v_n = +\infty. \tag{5}$$

plays an essential role. Condition (5) is used to establish compact imbedding theorem (see [23, Theorem 2.2]). So far as the authors are aware, the investigation on the standing wave solutions for DNLS equation with nonperiodic bounded potential cannot be found in the literature. Moreover, there is still no result on the existence and multiplicity of sign-changing solutions for (4).

The main purpose of the present paper is to establish the existence and infinitely many sign-changing decaying solutions for equation (4) with nonperiodic bounded potential. Since condition (5) is not satisfied, to overcome the difficulty caused by the lack of compactness, we construct the positive linear compact operator T (see (9)). But T is not symmetric with respect to the usual inner product of l^2. Therefore, we introduce a new inner product of l^2 to guarantee that T is symmetric. In argument, the operator T and its properties play an important role.

In order to state our results we require the following assumptions:

(V_1) the discrete potential $V = \{v_n\}_{n\in\mathbb{Z}}$ is bounded and satisfies $0 < \underline{v} \le v_n \le \overline{v}$ for all $n \in \mathbb{Z}$ for fixed $\underline{v}$ and $\overline{v}$.

(V_2) $0 < \gamma_n \le \overline{\gamma}$ for all $n \in \mathbb{Z}$ and fixed $\overline{\gamma} > 0$, and $\sum_{n\in\mathbb{Z}} \gamma_n < +\infty$.

(H_1) $f(n,0) = 0$, $f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$, and $\limsup_{|x|\to 0} \frac{|f(n,x)|}{|x|} < \lambda_1$ uniformly in $n \in \mathbb{Z}$, where λ_1 is given in Proposition 10.

(H_2) there are constants $C > 0$ and $p > 2$ such that $|f(n,x)| \le C(1+|x|^{p-1})$ for $n \in \mathbb{Z}, x \in \mathbb{R}$.

(H_3) there is $\eta > 2$ such that $0 < \eta F(n,x) \le f(n,x)x$ for $n \in \mathbb{Z}, x \ne 0$, where $F(n,x) = \int_0^x f(n,s)\mathrm{d}s$.

(H_4) $\lim_{|x|\to\infty} \frac{f(n,x)}{x} = \lambda$ uniformly in $n \in \mathbb{Z}$, and $\lambda > \lambda_2$ is not an eigenvalue of (19).

(H_5) $f(t,x)$ is odd in x, that is, $f(n,-x) = -f(n,x)$ for all $n \in \mathbb{Z}$ and $x \in \mathbb{R}$.

Here are the main results.

Theorem 1. *Assume* (V_1), (V_2), *and* (H_1)–(H_3) *hold. Then problem (4) has at least a positive solution, a negative solution and a sign-changing solution.*

Theorem 2. *Assume* (V_1), (V_2), (H_1) *and* (H_4) *hold. Then Problem (4) has at least a positive solution, a negative solution and a sign-changing solution.*

Theorem 3. *Assume* (V_1), (V_2), (H_1)–(H_3) *and* (H_5) *hold. Then Problem (4) has an unbounded sequence of sign-changing solutions.*

Theorem 4. *Assume that* $\lim_{|x|\to 0} \frac{f(n,x)}{x} = 0$ *uniformly in* $n \in \mathbb{Z}$. *Then any solution* u *obtained in Theorems 1–3 decays exponentially at infinity, i.e., there exist two positive constants* C *and* α *such that* $|u_n| \le Ce^{-\alpha|n|}$ *with* $n \in \mathbb{Z}$.

2 The Eigenvalue Problem

In this section, a difference operator related to Problem (4) is introduced, and some properties of the operator are discussed. We consider the real sequence spaces

$$l^p \equiv l^p(\mathbb{Z}) = \Big\{u = \{u_n\}_{n\in\mathbb{Z}} : \forall\, n \in \mathbb{Z},\ u_n \in \mathbb{R}, \|u\|_p = \Big(\sum_{n\in\mathbb{Z}} |u_n|^p\Big)^{1/p} < \infty\Big\}, \qquad 1 \le p < \infty,$$

with $\|u\|_\infty = \sup_{n\in\mathbb{Z}} |u_n|$ when $p = \infty$. The case $p = 2$ corresponds to the Hilbert space l^2 and the symbol $(\cdot,\cdot)_2$ stands for its standard inner product. The following embedding relation holds:

$$l^q \subset l^p, \qquad \|u\|_p \le \|u\|_q, \qquad 1 \le q \le p \le \infty. \tag{6}$$

Let $P = \{u \in l^2 : u_n \ge 0,\ n \in \mathbb{Z}\}$. Obviously, $P\,(-P)$ is the positive (negative) cone in l^2, and $P\,(-P)$ has empty interior. We introduce the new inner product of l^2 as follows

$$(u, w) = \sum_{n\in\mathbb{Z}} ((u_{n+1} - u_n)(w_{n+1} - w_n) + v_n u_n w_n). \tag{7}$$

The induced norm is $\|u\| = \big(\sum_{n\in\mathbb{Z}} \big(|u_{n+1} - u_n|^2 + v_n|u_n|^2\big)\big)^{\frac{1}{2}}$, which is equivalent to the usual norm of l^2 since

$$\underline{v}\|u\|_2^2 \le \|u\|^2 \le (4 + \overline{v})\|u\|_2^2. \tag{8}$$

Lemma 5. *Let $w \in l^2$ and $\overline{v} > 0$. Then the problem*

$$-\Delta u_n + \overline{v} u_n = w_n, \qquad n \in \mathbb{Z}$$

has a unique solution $u \in l^2$ with $u_n = \sum_{s\in\mathbb{Z}} G(n, s) w_s$, where

$$G(n, s) = \frac{1}{\lambda - \lambda^{-1}} \begin{cases} \lambda^{s-n}, & s \le n, \\ \lambda^{n-s}, & n \le s \end{cases} \qquad \lambda = \frac{\overline{v} + 2 + \sqrt{\overline{v}(\overline{v} + 4)}}{2} > 1.$$

Proof. Let $u_n = \sum_{s\in\mathbb{Z}} G(n, s) w_s$ and $\sigma = (\lambda - \lambda^{-1})^{-1}$. A simple computation shows that

$$-\Delta u_n + \overline{v} u_n = \sigma \sum_{s=-\infty}^{n} \lambda^s w_s(-\Delta\lambda^{-n} + \overline{v}\lambda^{-n}) + \sigma \sum_{s=n+1}^{+\infty} \lambda^{-s} w_s(-\Delta\lambda^n + \overline{v}\lambda^n) + w_n,$$

and $-\Delta\lambda^{-n} + \overline{v}\lambda^{-n} = 0,\ n \in \mathbb{Z}$. Thus, $-\Delta u_n + \overline{v} u_n = w_n$ for all $n \in \mathbb{Z}$.

Now we prove $u \in l^2$. It is easy to see that $c_n := \sum_{s\in\mathbb{Z}} G(n, s) = \frac{1}{\overline{v}}$ and $d_s := \sum_{n\in\mathbb{Z}} G(n, s) = \frac{1}{\overline{v}}$. By the definition of u, we have that $\frac{|u_n}{c_n} \le \sum_{s\in\mathbb{Z}} \frac{G(n,s)}{c_n}|w_s|$. Since the positive numbers $\frac{G(n,s)}{c_n}$ for $n \in \mathbb{Z}$ with a fixed n have the sum 1, the right member is a weighted average of the $|w_s|$. We note that x^2 is a convex function of $x \geqslant 0$, the 2-th power of the the right member does not exceed the weighted average of the $|w_s|^2$ with the same weights [5, p.70]. Thus,

$$\Big(\frac{|u_n|}{c_n}\Big)^2 \le \sum_{s\in\mathbb{Z}} \frac{|G(n, s)|}{c_n}|w_s|^2, \qquad |u_n|^2 \le c_n \sum_{s\in\mathbb{Z}} G(n, s)|w_s|^2.$$

It follows that

$$\|u\|_2^2 \le \frac{1}{\underline{v}} \sum_{s\in\mathbb{Z}} d_s |w_s|^2 \le \frac{1}{\underline{v}^2} \|w\|_2^2. \tag{9}$$

This gives $u \in l^2$. The proof of Lemma 5 is completed. □

Define operators $K, L, \Lambda, \mathbf{f}$, respectively, by

$$(Ku)_n = \sum_{s\in\mathbb{Z}} G(n,s)u_s; \qquad (Lu)_n = (\overline{v} - v_n)u_n; \qquad (\Lambda u)_n = \gamma_n u_n; \qquad (\mathbf{f}u)_n = f(n, u_n),$$

where $u \in l^2, n \in \mathbb{Z}$. It is easy to see from Lemma 5 and (9) that $KL : l^2 \to l^2$ is a linear continuous operator with $\|KL\| \le \|K\|\|L\| \le \frac{\overline{v}-\underline{v}}{\overline{v}} < 1$. This gives that $I - KL$ has a bounded inverse $(I - KL)^{-1}$, where I is the identity operator. Let

$$T := (I - KL)^{-1} K\Lambda = K\Lambda + (KL)K\Lambda + \cdots + (KL)^n K\Lambda + \cdots, \tag{10}$$

where we used the Neumann expansion formula. Then $T : l^2 \to l^2$ is linear continuous.

Lemma 6. *Suppose that* (V_1) *and* (V_2) *hold, then for each* $w \in l^2$*, the following linear problem*

$$-\Delta u_n + v_n u_n = \gamma_n w_n, \qquad n \in \mathbb{Z} \tag{11}$$

has a unique solution $u \in l^2$*, where* $u_n = (Tw)_n$*.*

Proof. It is easy to see that (11) is equivalent to the operator equation $u = KLu + K\Lambda w$. Therefore, System (11) has a unique solution $u = Tw$. □

Lemma 7. *Assume* (H_1) *and* (V_2)*. Then* $\mathbf{f} : l^2 \to l^2$ *is continuous and* $\Lambda\mathbf{f} : l^2 \to l^2$ *is compact.*

Proof. By (H_1), there exists $\delta > 0$ such that $|f(n,x)| \le \lambda_1|x|$ for any $|x| \le \delta$. For any $u \in l^2$, there is a positive integer N such that $|u_n| \le \delta$ for any $|n| > N$. Thus, $\|\mathbf{f}u\|_2^2 = \sum_{|n|\le N} |f(n,u_n)t|^2 + \lambda_1^2 \sum_{|n|>N} |u_n|^2 < \infty$. Then $\mathbf{f}u \in l^2$, which implies that $\mathbf{f} : l^2 \to l^2$.

We now show $\mathbf{f} : l^2 \to l^2$ is continuous. Suppose that $u_k \to u_0$ in l^2 as $k \to +\infty$. Then there exists a positive integer N_1 such that $\sup_{|n|>N_1} |u_{k,n}| \le \delta$ for $k = 0, 1, 2, \cdots$. Thus, we have that $|(\mathbf{f}u_k)_n| = |f(n, u_{k,n})| \le \lambda_1 |u_{k,n}|$, $\forall\, |n| > N_1$. This gives that

$$\big|(\mathbf{f}u_k)_n - (\mathbf{f}u_0)_n big| \le \lambda_1\big(|u_{k,n}| + |u_{0,n}|\big) \le \lambda_1\big(|u_{k,n} - u_{0,n}| + 2|u_{0,n}|\big), \qquad \forall\, |n| > N_1. \tag{12}$$

Now we claim that $\mathbf{f}u_k \to \mathbf{f}u_0$ in l^2 as $k \to +\infty$. Suppose the contrary, then there exist $\varepsilon_0 > 0$ and a subsequence of $\{u_k\}$ (still denoted by $\{u_k\}$) such that

$$\sum_{n\in\mathbb{Z}} \big|(\mathbf{f}u_k)_n - (\mathbf{f}u_0)_n\big|^2 \ge \varepsilon_0, \qquad k = 1, 2, \cdots. \tag{13}$$

Note that $u_k \to u_0$ in l^2. Then, passing to a subsequence if necessary, it can be assumed that $\sum_{k=1}^{+\infty} \|u_k - u_0\|_2 < +\infty$. This means that $u_{k,n} \to u_{0,n}$ for every $n \in \mathbb{Z}$ and

$$w = \Big\{ w_n := \sum_{k=1}^{+\infty} |u_{k,n} - u_{0,n}| \Big\}_{n\in\mathbb{Z}} \in l^2. \tag{14}$$

Combining (12) and (14), we get

$$\big|(\mathbf{f}u_k)_n - (\mathbf{f}u_0)_n\big| \le \lambda_1\big(w_n + 2|u_{0,n}|\big), \qquad \forall\, |n| > N_1. \tag{15}$$

Let $q = \lambda_1(w + 2|u_0|)$. We have $q \in l^2$. So, there is a positive integer $N_2 > N_1$ such that

$$\sum_{|n|>N_2} |q|^2 < \frac{\varepsilon_0}{2}. \tag{16}$$

The continuity of f implies that for sufficiently large k,

$$\sum_{|n|\le N_2} \left|(\mathbf{f}u_k)_n - (\mathbf{f}u_0)_n\right|^2 < \frac{\varepsilon_0}{2}. \tag{17}$$

It follows from (15)–(17) that

$$\sum_{n\in\mathbb{Z}} |(\mathbf{f}u_k)_n - (\mathbf{f}u_0)_n|^2 = \sum_{|n|>N_2} |(\mathbf{f}u_k)_n - (\mathbf{f}u_0)_n|^2 + \sum_{|n|\le N_2} |(\mathbf{f}u_k)_n - (\mathbf{f}u_0)_n|^2 < \varepsilon_0,$$

which contradicts to (13). Thus, the claim above is true and $\mathbf{f} : l^2 \to l^2$ is continuous.

Let $\Theta = \{\theta_n\}_{n\in\mathbb{Z}}$ satisfy $\theta_n = \gamma_n \frac{f(n,u_n)}{u_n}$, if $u_n \neq 0$; $\theta_n = 0$, if $u_n = 0$. By assumptions (H_1) and (V_2), $\theta_n \to 0$ as $|n| \to \infty$. This and Lemma 2.3 in [23] mean that the multiplication by θ_n is a compact operator in l^2. Thus, the operator $\Lambda\mathbf{f}$ is compact. □

Remark 8. By Lemma 6 we know that the solution of (4) is equivalent to the fixed point of $T\mathbf{f}$.

T defined in (10) is an important operator in our later discussion. We present some properties of T. Let us point out that T is not symmetric with respect to the usual inner product of l^2.

Lemma 9. *If* (V_1) *and* (V_2) *are satisfied, then*

(i) $T : l^2 \to l^2$ *is compact symmetric with respect to the inner product* $(\cdot)$ *defined by (7).*

(ii) $T : l^2 \to l^2$ *is positivity preserving, that is,* $(Tu)_n > 0, n \in \mathbb{Z}$ *for all* $u \in P$ *with* $u \neq 0$.

Proof. (i) By (V_2) and Lemma 2.3 in [23], the multiplication operator Λ is a compact operator. Notice that the product of a compact operator with a bounded operator is compact ([6, Theorem 4.8]). Thus, $T = (I - KL)^{-1}K\Lambda : l^2 \to l^2$ is a compact operator. For any given $x, y \in l^2$, let $u = Tx$. It follows from Lemma 6 that

$$(Tx, y) = (u, y) = \sum_{n\in\mathbb{Z}} (-\Delta u_n + v_n u_n) y_n = \sum_{n\in\mathbb{Z}} \gamma_n x_n y_n. \tag{18}$$

This implies that $(Tx, y) = (x, Ty)$. Then T is symmetric.

(ii) From Lemma 5, we see that $G(n, s) > 0$ for all $n, s \in \mathbb{Z}$. This and (V_2) imply that $(K\Lambda u)_n > 0, n \in \mathbb{Z}$ for all $u \in P$ with $u \neq 0$. Obviously, $\mathrm{KL}\,(P) \subset P$, that is, KL is positive. By (10), we obtain that T is positivity preserving. □

Proposition 10. *If* (V_1) *and* (V_2) *are satisfied, then the linear eigenvalue problem*

$$-\Delta u_n + v_n u_n = \lambda \gamma_n u_n, \qquad n \in \mathbb{Z} \tag{19}$$

possesses a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_k \to \infty$ *as* $k \to \infty$, *where each* λ_k *has finite multiplicity, the first eigenvalue* λ_1 *is simple with eigenfunction* $\varphi_1 > 0$ *on* $\mathbb{Z}$ *and the eigenfunctions* φ_k *correspondent to* λ_k $(k \ge 2)$ *are all sign-changing.*

Proof. From Lemma 9 and spectral theory of symmetric compact operators on Hilbert space [6, 20], we obtain the result. □

Remark 11. By Proposition 10, we see that $|(Tx, y)| \le \frac{1}{\lambda_1}\|x\|\|y\|$ for all $x, y \in l^2$. The eigenfunction φ_1 with $\varphi_1 > 0$ is also obtained by the Krein-Rutmann theorem ([18, Theorem 4.2.2]).

3 The Variational Framework

In this section, we shall establish variational framework for the Equation (4). On the Hilbert space l^2, we consider the functional $J : l^2 \to \mathbb{R}$,

$$J(u) = \frac{1}{2}\sum_{n\in\mathbb{Z}}(\left|u_{n+1} - u_n\right|^2 + v_n\left|u_n\right|^2) - \sum_{n\in\mathbb{Z}}\gamma_n F(n,\, u_n) \triangleq J_1(u) - J_2(u). \tag{20}$$

It follows from (H_1) that there exists $\delta > 0$ such that for any $|x| \le \delta$,

$$F(n,x) \le \frac{\lambda_1}{2}\left|x\right|^2. \tag{21}$$

For any fixed $u \in l^2$, take $\widetilde{N} > 0$ large such that $\left|u_n\right| \le \delta$ for every $\left|n\right| > \widetilde{N}$. By (21), one has that

$$J_2(u) \le \sum_{|n|\le\widetilde{N}}\gamma_n F(n,u_n) + \frac{\lambda_1\overline{\gamma}}{2}\sum_{|n|>\widetilde{N}}\left|u_n\right|^2 < \infty.$$

Then we obtain that J_2 and J are both well defined on l^2 if (H_1) holds.

Lemma 12. *If* (V_1), (V_2) *and* (H_1) *hold, then* $J \in C^1(l^2, \mathbb{R})$, *and for any* $u, w \in l^2$,

$$(J'(u), w) = \sum_{n\in\mathbb{Z}}\big((u_{n+1} - u_n)(w_{n+1} - w_n) + v_n u_n w_n - \gamma_n f(n, u_n)w_n\big) = (u - T\mathbf{f}u, w). \tag{22}$$

Moreover, J_2' *is compact, and any nontrivial critical point of* J *is a nontrivial solution of (4).*

Proof. It is easy to check that $J_1 \in C^1(l^2, \mathbb{R})$, and any $u, w \in l^2$,

$$(J_1'(u), w) = (u,\, w) = \sum_{n\in\mathbb{Z}}\big((u_{n+1} - u_n)(w_{n+1} - w_n) + v_n u_n w_n\big), \tag{23}$$

where $J_1'(u)$ is the Fréhet derivative of J_1 at u. Now we show that J_2 is Fréhet differentiable, and

$$(J_2'(u),\, w) = \sum_{n\in\mathbb{Z}}\gamma_n f(n, u_n)w_n = (T\mathbf{f}u, w). \tag{24}$$

For any given $u \in l^2$, let us define $B(u) : l^2 \to \mathbb{R}$ as follows

$$B(u)w = \sum_{n\in\mathbb{Z}}\gamma_n f(n, u_n)w_n.$$

Obviously, $B(u)$ is linear. By (H_1), there exists $\delta > 0$ such that $\left|f(n, x)\right| \le \lambda_1|x|$ for any $|x| \le \delta$. We can easily have that there exists $N > 0$ such that $\left|u_n\right| \le \delta$ for any $|n| > N$. Thus,

$$\left|B(u)w\right| \le D\sum_{|n|\le N}\gamma_n\left|w_n\right| + \lambda_1\sum_{|n|>N}\gamma_n\left|u_n\right|\left|w_n\right| \le \overline{\gamma}(D(2N+1)^{\frac{1}{2}} + \lambda_1\|u\|_2)\|w\|_2,$$

where $D = \max_{|n|\le N}\left|f(n,\, u_n)\right| < \infty$. Then $B(u)$ is bounded. Moreover, for $u,\, w \in l^2$, by the mean value theorem, (18) and Lemma 7, we have

$$\begin{aligned}\left|J_2(u+w) - J_2(u) - B(u)w\right| =& \Big|\sum_{n\in\mathbb{Z}}\gamma_n(f(n\,, u_n + \theta_n w_n) - f(n,\, u_n))w_n\Big| \\ \le& \|T\|\|\mathbf{f}(u+\theta w) - \mathbf{f}u\|\|w\| = \circ(\|w\|),\end{aligned}$$

where $0 < \theta_n < 1$ and $(\theta w)_n = \theta_n w_n$. Hence, J_2 is Fréhet differentiable in l^2, and (24) holds.

Next, it is to show that $J_2'(u)$ is continuous in u. Suppose that $u_k \to u$ in l^2. Owing to Lemma 7, one has that

$$J_2'(u_k) - J_2'(u) = \sup_{\|w\|=1} |(T\mathbf{f}u_k - T\mathbf{f}u,\, w)| \le \|T\|\|\mathbf{f}u_k - \mathbf{f}u\| \to 0 \qquad \text{as } \; k \to \infty.$$

So, $J_2'(u)$ is continuous in u and $J_2 \in C^1(l^2, \mathbb{R})$. It is easy to see that $J \in C^1(l^2, \mathbb{R})$. By (23) and (24), we have that (22) holds. Furthermore, $J_2' : l^2 \to l^2$ is compact by (10) and Lemma 7.

Lastly, we check that nontrivial critical points of J on l^2 are nontrivial solutions of problem (4). Let $u \in l^2$ be a nontrivial critical point of J. Now, by (23) and (24), we have that $0 = (J'(u), w) = (u - T\mathbf{f}u, w)$ for all $w \in l^2$. Hence $u - T\mathbf{f}u = 0$. This, together with Remark 8, implies that u is a nontrivial solution of problem (4). This completes the proof. □

In order to study the critical points of J, we now recall two abstract critical point theorems in [1,10], respectively. Also see [9,16] for related results.

Lemma 13 (See [10, Theorem 3.2]). *Let E be a Hilbert space and J be a C^1 functional defined on E. Assume that J satisfies the PS condition on E and $J'(u)$ has the expression $J'(u) = u - Au$ for $u \in E$. Assume that D_1 and D_2 are two open convex subsets of E with the properties that $A(\partial D_1) \subset D_1$, $A(\partial D_2) \subset D_2$ and $D_1 \cap D_2 \ne \emptyset$. If there exists a path $h : [0,1] \to E$ such that*

$$h(0) \in D_1 \setminus D_2, \qquad h(1) \in D_2 \setminus D_1,$$

] and

$$\inf_{u \in \overline{D_1} \cap \overline{D_2}} J(u) > \sup_{t \in [0,1]} J(h(t)),$$

then J has at least four critical points, one in $D_1 \cap D_2$, one in $D_1 \setminus \overline{D_2}$, one in $D_2 \setminus \overline{D_1}$, and one in $E \setminus (\overline{D_1} \cup \overline{D_2})$.

In order to treat even functionals, we recall the notion of genus for a space B with involution $B \ni b \mapsto \overline{b} \in B$. The genus of B is defined by

$$\operatorname{gen}(B) := \inf\{n \ge 0 : \exists f \in C(B, \mathbb{R}^n \setminus \{0\}),\ f \text{ is odd}\}.$$

Here a map $f : B \to \mathbb{R}^n \setminus \{0\}$ is said to be odd if $f(\overline{b}) = -f(b)$ for all $b \in B$.

Let X be a Banach space, $D^{\pm}$ closed convex subsets of X, $A \in C(X,X)$ an operator and $J \in C^1(X, \mathbb{R})$ a functional. Consider $D^{\pm}$, A and J satisfying the following assumptions.

(D) $\mathcal{O} := \operatorname{int}(\mathcal{D}^+) \cap \operatorname{int}(\mathcal{D}^-) \ne \emptyset$.

(A) the map A is compact, and $A(D^{\pm}) \subset \operatorname{int}(D^{\pm})$.

(J_1) the exist $a_1 > 0$ and $a_2 > 0$ such that for every $u \in X$, $J'(u)(u - A(u)) \ge a_1\|u - A(u)\|^2$ and $\|J'(u)\| \le a_2\|u - A(u)\|$.

(J_2) there exists a path $h : [0,1] \to X$ such that $h(0) \in \operatorname{int}(D^+) \setminus D^-$ and $h(1) \in \operatorname{int} D^- \setminus D^+$, and $\alpha_0 := \inf_{u \in D^+ \cap D^-} J(u) > \sup_{t \in [0,\,1]} J(h(t))$.

(J_3) there exist a number α_1, a sequence $\{X_k\}_{k \in \mathbb{N}}$ of subspaces of X and a sequence $\{R_k\}_{k \in \mathbb{N}}$ of positive numbers satisfying $\dim X_k \ge k$ for $k \in \mathbb{N}$, and $\alpha_0 = \inf_{u \in D^+ \cap D^-} J(u) > \alpha_1 \ge \sup_{u \in X_k \setminus B_k} J(u)$, where $B_k := \{u \in X_k : \|u\| \le R_k\}$.

Lemma 14 (See [1, Theorem 2.5]). *Assume* (D), (A), (J_1) − (J_3). *Assume also that A is odd, J is even and $D^+ = -D^-$. Define*

$$d_k := \inf_{S \in \Gamma_k} \sup_{u \in S} J(u) \qquad \textit{for} \qquad k \ge 2,$$

where $\Gamma_k := \{h(B_m \setminus B) \setminus (D^+ \cup D^-) : h \in G_m,\ m \geq k,\ B = -B \in B_k \text{ open},\ gen(B) \leq m-k\}$ *with* $G_m := \{h \in C(B_m, X) : h \text{ is odd and } h(x) = x \text{ for } x \in \partial_{X_m} B_m\}$. *Then*

$$d_k \to +\infty \quad \text{as} \quad k \to \infty \qquad \text{and} \qquad K^\star_{d_k} \neq \emptyset \quad \text{for} \quad k \geq 2,$$

where $K^\star_{d_k} := K_{d_k} \setminus (D^+ \cup D^-)$ *with* $K_{d_k} := \{u \in X : J(u) = d_k,\ J'(u) = 0\}$.

4 Proof of Theorems

In the following we consider the convex cones $P = \{u \in l^2 : u \geq 0\}$ and $-P = \{u \in l^2 : u \leq 0\}$. The distance in l^2 with respect to $\|\cdot\|$ is denoted by dist. For $\varepsilon > 0$, we define

$$D^+_\varepsilon = \big\{u \in l^2 : \operatorname{dist}(u, P) < \varepsilon\big\}, \qquad D^-_\varepsilon = \big\{u \in l^2 : \operatorname{dist}(u, -P) < \varepsilon\big\}.$$

Obviously, $D^+_\varepsilon \cap D^-_\varepsilon \neq \emptyset$. Note that D^+_ε and D^-_ε are open convex subsets of l^2. Moreover, $l^2 \setminus (D^+_\varepsilon \cup D^-_\varepsilon)$ contains only sign changing functions.

Lemma 15. *Assume that* (V_1), (V_2), (H_1) *and* (H_2) *are satisfied, and that* $f(n,x)x > 0$ *for all* $x \neq 0$ *and* $n \in \mathbb{Z}$. *Then there exists* $\varepsilon_0 > 0$ *and such that for* $0 < \varepsilon < \varepsilon_0$, *one has*

(i) $T\mathbf{f}(\partial D^-_\varepsilon) \subset D^-_\varepsilon$, *and if* $u \in D^-_\varepsilon$ *is a nontrivial critical point of* J, *then* u *is a negative solution of Problem (4);*

(ii) $T\mathbf{f}(\partial D^+_\varepsilon) \subset D^+_\varepsilon$, *and if* $u \in D^+_\varepsilon$ *is a nontrivial critical point of* J, *then* u *is a positive solution of Problem (4).*

Proof. (i) Let $u \in l^2$ and $w = T\mathbf{f}(u)$. We denote by u^+ resp. u^- the positive resp. negative part of u, that is $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$. Then, by (8),

$$\|u^+\|_2 = \inf_{h \in -P} \|u - h\|_2 \leq \frac{1}{\sqrt{\underline{v}}} \inf_{h \in -P} \|u - h\| = \frac{1}{\sqrt{\underline{v}}} \operatorname{dist}(u, -P). \tag{25}$$

It follows from (H_1) and (H_2) that there exist $\tau > 0$, $C_1 > 0$ such that

$$|f(n,x)| \leq (\lambda_1 - \tau)|x| + C_1|x|^{p-1} \quad \text{for } n \in \mathbb{Z},\ x \in \mathbb{R}. \tag{26}$$

The fact that $u^+ = u - u^-$and $u^- \in -P$ implies $\operatorname{dist}(w, -P) \leq \|w - w^-\| = \|w^+\|$. Then by the assumption $f(n,x)x > 0$ for all $x \neq 0$ and $n \in \mathbb{Z}$, (26), (18), Remark 11, (6), (8) and (25), we have

$$\begin{aligned}
\operatorname{dist}(w, -P)\|w^+\| \leq &\|w^+\|^2 = (w, w^+) \leq \sum_{n \in \mathbb{Z}} \gamma_n f(n, u^+_n) w^+_n \\
\leq &(\lambda_1 - \tau)(Tu^+, w^+) + \overline{\gamma} C_1 \|u^+\|_p^{p-1} \|w^+\|_p \\
\leq &\frac{\lambda_1 - \tau}{\lambda_1} \|u^+\| \|w^+\| + \overline{\gamma} C_1 \|u^+\|_2^{p-1} \|w^+\|_2 \\
\leq &\Big(\frac{\lambda_1 - \tau}{\lambda_1} \operatorname{dist}(u, -P) + \frac{\overline{\gamma} C_1}{\sqrt{\underline{v}^p}} (\operatorname{dist}(u, -P))^{p-1}\Big) \|w^+\|.
\end{aligned}$$

Hence, $\operatorname{dist}(w, -P) \leq \frac{\lambda_1 - \tau}{\lambda_1} \operatorname{dist}(u, -P) + \frac{\overline{\gamma} C_1}{\sqrt{\underline{v}^p}} (\operatorname{dist}(u, -P))^{p-1}$. So there exists $\varepsilon_0 > 0$ such that for any $u \in D^-_\varepsilon$ with $0 < \varepsilon < \varepsilon_0$,

$$\operatorname{dist}(T\mathbf{f}(u), -P) \leq \frac{2\lambda_1 - \tau}{2\lambda_1} \operatorname{dist}(u, -P). \tag{27}$$

Then $T\mathbf{f}(\partial D^-_\varepsilon) \subset D^-_\varepsilon$, $\forall u \in \partial D^-_\varepsilon$. If $u \in D^-_\varepsilon$ is a nontrivial critical point of J, then $J'(u) = u - T\mathbf{f}(u)$, that is, $T\mathbf{f}(u) = u$. By (27), $u \in -P \setminus \{0\}$. According to Lemma 9 and the assumption $f(n,x)x > 0$ for all $x \neq 0$ and $n \in \mathbb{Z}$, $u < 0$ on $\mathbb{Z}$. Hence, u is a negative solution of problem (4) and (i) holds. The proof of (ii) is similar and omitted. □

Lemma 16. *Assume* (V_1), (V_2), (H_1), *and either* (H_3) *or* (H_4) *hold. Then the functional* J *(see (20)) satisfies the (PS) condition, i.e., for any sequence* $\{u_k\}$ *such that* $J(u_k)$ *is bounded and* $J'(u_k) \to 0$ *as* $k \to \infty$, *there exists a subsequence of* $\{u_k\}$ *which is convergent in* l^2.

Proof. Let $\{u_k\}$ be a sequence in l^2 such that $J(u_k)$ is bounded and $J'(u_k) \to 0$ as $k \to \infty$. Our aim is to prove that $\{u_k\}$ is bounded and has a convergent subsequence.

First suppose that (H_3) holds; we know, by (20) and (22), that

$$\eta J(u_k) - (J'(u_k), u_k) = \left(\frac{\eta}{2} - 1\right) \|u_k\|^2 - \sum_{n\in\mathbb{Z}} \gamma_n(\eta F(n, u_{k,n}) - f(n, u_{k,n})u_{k,n}).$$

So by (H_3), $\left(\frac{\eta}{2} - 1\right)\|u_k\|^2 \leq \eta J(u_k) - (J'(u_k), u_k)$. Thus, bearing in mind that $\eta > 2$, we conclude that $\|u_k\|$ is bounded. It follows from Lemma 12 that

$$u_k = J'(u_k) + T\mathbf{f}u_k \tag{28}$$

and $T\mathbf{f} : l^2 \to l^2$ is compact. Therefore, one can deduce that $\{u_k\}$ has a convergent subsequence. And the (PS) condition is verified.

Now, suppose that (H_4) holds. We claim that $\{u_k\}$ is bounded. Suppose the contrary, then there exists a subsequence of $\{u_k\}$ (still denoted by $\{u_k\}$) such that $\rho_k = \|u_k\| \to +\infty$ as $k \to \infty$. Set $w_k = \frac{u_k}{\rho_k}$. Obviously, $\|w_k\| = 1$. So passing to a subsequence if necessary, it can be assumed that $w_k \rightharpoonup w_0$. By (6) and (8), there exists $C_1 > 0$ such that $\sup_{n\in\mathbb{Z}} |w_{k,n}| \leq C_1$ for all $k = 0, 1, 2, \cdots$. It is easy to verify that $w_{k,\,n}$ converses to $w_{0,\,n}$ pointwise for all $n \in \mathbb{Z}$, that is,

$$\lim_{k\to\infty} w_{k,n} = w_{0,n}, \qquad \forall\, n \in \mathbb{Z}. \tag{29}$$

For any given number $\varepsilon > 0$, since $\sum\limits_{n\in\mathbb{Z}} \gamma_n < +\infty$, there is a positive integer N such that $\sum\limits_{|n|>N} \gamma_n < \varepsilon$. By ($H_1$) and ($H_4$), there exist $\tau > 0, C_2 > 0$ such that $|f(n,x)| \leq (\lambda_1 - \tau)|x| + C_2|x|^{p-1}$ for $n \in \mathbb{Z}$, $x \in \mathbb{R}$. So we have

$$\Big|\sum_{|n|>N} \gamma_n(f(n, w_{k,n}) - f(n, w_{0,n}))(w_{k,n} - w_{0,n})\Big| \leq C_3 \sum_{|n|>N} \gamma_n < C_3\varepsilon, \tag{30}$$

where $C_3 > 0$. It follows from (29) and the continuity of $f(n,x)$ on x that

$$\sum_{|n|\leq N} \gamma_n(f(n, w_{k,n}) - f(n,\, w_{0,n}))(w_{k,n} - w_{0,n}) \to 0 \quad \text{as} \quad k \to \infty. \tag{31}$$

Since ε is arbitrary, combining (30) with (31), we get

$$\sum_{n\in\mathbb{Z}} \gamma_n(f(n, w_{k,n}) - f(n, w_{0,n}))(w_{k,n} - w_{0,n}) \to 0 \quad \text{as} \quad k \to \infty. \tag{32}$$

It follows from (22) that

$$(J'(w_k) - J'(w_0),\, w_k - w_0) = \|w_k - w_0\|^2 - \sum_{n\in\mathbb{Z}} \gamma_n(f(n,\, w_{k,n}) - f(n,\, w_{0,n})(w_{k,n} - w_{0,n}). \tag{33}$$

Note that $(J'(w_k) - J'(w_0), w_k - w_0) \to 0$. By (32) and (33), we have that $w_k \to w_0$ in l^2 as $k \to \infty$. Put $\widetilde{w}_k = \{\widetilde{w}_{k,n}\}_{n\in\mathbb{Z}}$ with $\widetilde{w}_{k,n} = \frac{f(n,u_{k,n})}{u_{k,n}} w_{k,n}$. Since $\lim\limits_{|x|\to\infty} \frac{f(n,x)}{x} = \lambda$ uniformly in $n \in \mathbb{Z}$, we have

$$\frac{J'(u_k)}{\rho_k} = w_k - \frac{1}{\rho_k} T\mathbf{f}u_k = w_k - T\widetilde{w}_k \to w_0 - \lambda T w_0.$$

Bearing in mind that $\frac{J'(u_k)}{\rho_k} \to 0$ as $k \to \infty$, we get that $w_0 - \lambda T w_0 = 0$. According to Lemma 6 and Proposition 10, λ is an eigenvalue of problem (19), contrary to assumption. Hence, $\{u_k\}$ is bounded. By (28), we know also that $\{u_k\}$ has a convergent subsequence. The proof is complete. □

Lemma 17. *Assume* (V_1)*,* (V_2)*,* (H_1)*, and either* (H_3) *or* (H_4) *hold. Then* $J(u) \to -\infty$ *as* $\|u\| \to +\infty$*, where* $u \in X_2 := \operatorname{span}\{\varphi_1, \varphi_2\}$*, and* φ_1, φ_2 *are eigenfunctions corresponding to eigenvalues* λ_1, λ_2 *of Problem (19).*

Proof. First suppose that (H_3) holds. By (H_3), for each $n \in \mathbb{Z}$, there is $a_n > 0$ such that

$$F(n,x) \geq a_n |x|^\eta, \text{ for each } |x| \geq 1. \tag{34}$$

Hence, for $u \in X_2$, $\sum\limits_{|u_n|>1} F(n,u_n) \geq \sum\limits_{|u_n|>1} a_n |u_n|^\eta$ and

$$J(u) = \frac{1}{2}\|u\|^2 - \sum_{|u_n|>1} \gamma_n F(n,\, u_n) - \sum_{|u_n|\leq 1} \gamma_n F(n,\, u_n) \leq \frac{1}{2}\|u\|^2 - \sum_{|u_n|>1} a_n \gamma_n |u_n|^\eta. \tag{35}$$

Define the functional $\psi : S^\infty \to \mathbb{R}$ as

$$\psi(w) = \sum_{|w_n|>1} a_n \gamma_n |w_n|^\eta,$$

where $S^\infty = \{w \in X_2 : \|w\|_\infty = 2\}$. Obviously, for any $w \in S^\infty$, $\psi(w) > 0$ and the set $\{n \in \mathbb{Z} : |w_n| > 1,\, w \in S^\infty \subset l^2\}$ is finite. We show that $\psi : S^\infty \to \mathbb{R}$ is lower semicontinuous. Suppose that $\{w_k\} \subset S^\infty$, $w_0 \in S^\infty$ and $w_k \to w_0$ as $k \to \infty$. By (6) and (8), we have that $\{n \in \mathbb{Z} : |w_{k,n}| > 1\} \supset \{n \in \mathbb{Z} : |w_{0,n}| > 1\}$ for sufficiently large k. This together with $w_k \to w_0$ as $k \to \infty$ yields that $\psi(w_0) \leq \liminf_{k\to\infty} \psi(w_k)$, that is, $\psi : S^\infty \to \mathbb{R}$ is lower semicontinuous. Since S^∞ is a compact subset of the finite dimension subspace X_2, we can obtain that $\rho := \inf\limits_{w\in S^\infty} \psi(w) > 0$. For any $u \in X_2$ with $\|u\|_\infty > 2$, setting $w = \big\{w_n := \frac{2u_n}{\|u\|_\infty}\big\}_{n\in\mathbb{Z}}$, we have that $w \in S^\infty$ and $\{n \in \mathbb{Z} : |u_n| > 1\} \supset \{n \in \mathbb{Z} : |w_n| > 1\}$. Hence, by (35), one has

$$J(u) \leq \frac{1}{2}\|u\|^2 - \frac{1}{2^\eta}\|u\|_\infty^\eta \sum_{|w_n|>1} a_n \gamma_n |w_n|^\eta \leq \frac{1}{2}\|u\|^2 - \frac{\rho}{2^\eta}\|u\|_\infty^\eta.$$

Since all norms on the finite dimension subspace X_2 are equivalent, this together with $\eta > 2$ implies that $J(u) \to -\infty$ as $\|u\| \to +\infty$, where $u \in X_2$.

Now, suppose that (H_4) holds. For $u \in X_2$, $u = \varepsilon_1\varphi_1 + \varepsilon_2\varphi_2$. Notice that φ_1 and φ_2 are orthogonal, i.e., $(\varphi_1, \varphi_2) = 0$. Then $\|u\|^2 = \varepsilon_1^2\|\varphi_1\|^2 + \varepsilon_2^2\|\varphi_2\|^2$. Choose ε such that $0 < \varepsilon < \min\{\lambda - \lambda_1, \lambda - \lambda_2\}$. By $\lim\limits_{|x|\to\infty} \frac{f(n,x)}{x} = \lambda$ uniformly in $n \in \mathbb{Z}$, we have that there exists $a > 0$ such that for any $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, $F(n,x) \geq \frac{\lambda-\varepsilon}{2}x^2 - a$. Hence, for $u \in X_2$, by (18) and Proposition 10,

$$\begin{aligned} J(u) \leq& \frac{1}{2}(\varepsilon_1^2\|\varphi_1\|^2 + \varepsilon_2^2\|\varphi_2\|^2) - \frac{\lambda-\varepsilon}{2}\Big(\frac{1}{\lambda_1}\varepsilon_1^2\|\varphi_1\|^2 + \frac{1}{\lambda_2}\varepsilon_2^2\|\varphi_2\|^2\Big) + a\sum_{n\in\mathbb{Z}}\gamma_n \\ =& \frac{\lambda_1 - \lambda + \varepsilon}{2\lambda_1}\varepsilon_1^2\|\varphi_1\|^2 + \frac{\lambda_2 - \lambda + \varepsilon}{2\lambda_2}\varepsilon_2^2\|\varphi_2\|^2 + a\sum_{n\in\mathbb{Z}}\gamma_n \to -\infty \end{aligned}$$

as $\|u\| \to \infty$. The proof is complete. □

Proof of Theorem 1. Our aim is to apply Lemma 13. By (H_1) and (H_2), we have that $F(n, x) \le \frac{\lambda_1 - \tau}{2}|x|^2 + \frac{C}{p}|x|^p$ for $n \in \mathbb{Z}, x \in \mathbb{R}$. This, together with (18), Remark 11 and (6), gives that

$$J(u) \ge \frac{1}{2}\|u\|^2 - \frac{\lambda_1 - \tau}{2\lambda_1}\|u\|^2 - \frac{\overline{\gamma} C}{p}\|u\|_p^p \ge \frac{\tau}{2\lambda_1}\|u\|^2 - \frac{\overline{\gamma} C}{p}\|u\|_2^p.$$

We note that (25) implies that for any $u \in \overline{D_\varepsilon^+} \cap \overline{D_\varepsilon^-}$, $\|u^\pm\|_2 \le \frac{1}{\sqrt{\underline{v}}}\operatorname{dist}(u, \mp P) \le \frac{1}{\sqrt{\underline{v}}}\varepsilon_0$. Thus there exists $\alpha_0 > -\infty$ such that $\inf_{u \in \overline{D_\varepsilon^+} \cap \overline{D_\varepsilon^-}} J(u) = \alpha_0$. Lemma 17 yields that there exists $R > 2\varepsilon_0$ such that $J(u) < \alpha_0 - 1$ for $u \in X_2$ and $\|u\| = R$. Define a path $h : [0, 1] \to X_2$ as

$$h(s) = R\frac{\cos(\pi s)\varphi_1 + \sin(\pi s)\varphi_2}{\|\cos(\pi s)\varphi_1 + \sin(\pi s)\varphi_2\|}.$$

Then $h(0) = \frac{R\varphi_1}{\|\varphi_1\|} \in D_\varepsilon^+ \setminus D_\varepsilon^-$, $h(1) = -\frac{R\varphi_1}{\|\varphi_1\|} \in D_\varepsilon^- \setminus D_\varepsilon^+$ and $\inf_{u \in \overline{D_\varepsilon^+} \cap \overline{D_\varepsilon^-}} J(u) > \sup_{t \in [0,1]} J(h(t))$. According to Lemmas 15, 16 and 13, there exists a critical point in $l^2 \setminus (\overline{D_\varepsilon^+} \cup \overline{D_\varepsilon^-})$, which is a sign-changing solution of problem (4). Also we have a critical point in $D_\varepsilon^+ \setminus \overline{D_\varepsilon^-}$ and a critical point in $D_\varepsilon^- \setminus \overline{D_\varepsilon^+}$, which correspond to a positive solution and a negative solution of problem (4), respectively. This completes the proof of Theorem 1. □

Proof of Theorem 2. By assumptions (H_1) and (H_4), we may fixed $m \ge 0$ such that $(f(n, x) + mx)x > 0$ for all $x \ne 0$ and $n \in \mathbb{Z}$. By replacing $f(n, x)$ by $f(n, x) + mx$ and v_n by $v_n + m\gamma_n$ in the preceding three sections, Lemmas 15 and 16 and the proof of Theorem 1 give the results. □

Proof of Theorem 3. For each $k \in \mathbb{N}$, define

$$X_k := \operatorname{span}\{\varphi_1, \varphi_2, \cdots, \varphi_k\},$$

where $\{\varphi_1, \varphi_2, \cdots, \varphi_k\}$ is given in Proposition 10. By the proof of Lemma 17, it is easy to construct the desired number α_1 and the sequence of positive numbers $\{R_k\}_{k \in \mathbb{N}}$. Note also that Lemmas 15, 16 and the proof of Theorem 1. Then the assumptions in Lemma 14 are satisfied. Therefore the functional J has a sequence of critical points $\{\pm u_k\}_{k \in \mathbb{N}}$ in $l^2 \setminus (\overline{D_\varepsilon^+} \cup \overline{D_\varepsilon^-})$ which are sign-changing solutions of problem (4) and which satisfy $J(u_k) = d_k = \inf_{S \in \Gamma_k} \sup_{u \in S} J(u) \to +\infty$ as $k \to \infty$. Since $f(n, x)x > 0$ for all $x \ne 0$ and $n \in \mathbb{Z}$, we have that $J(u_k) \le \frac{1}{2}\|u_k\|^2$. Hence $\|u_k\| \to +\infty$. This completes the proof. □

Proof of Theorem 4. It follows from Lemmas 6 and 9, and Proposition 10 that $T^{-1} : l^2 \to l^2$, the inverse mapping of T, exists and is given by $T^{-1}u_n = \frac{-\Delta u_n + v_n u_n}{\gamma_n}$, and the essential spectrum, $\sigma_{ess}(T^{-1})$, of T^{-1} satisfies $\sigma_{\mathrm{ess}}(T^{-1}) = \emptyset$. Let $\tau_n = -\frac{f(n, u_n)}{u_n}$ and $Ku_n = T^{-1}u_n + \tau_n u_n$. Then equation (4) is equivalent to

$$Ku_n = 0. \tag{36}$$

By assumptions, $\tau_n \to 0$ as $|n| \to \infty$. Thus, the multiplication by τ_n is a compact operator in l^2, which implies that $\sigma_{\mathrm{ess}}(K) = \sigma_{\mathrm{ess}}(T^{-1}) = \emptyset$. Equation (36) means that $u \in l^2$ is an eigenvector of K, with eigenvalue $0 \notin \sigma_{\mathrm{ess}}(K)$. Therefore, the result follows from the standard theorem on the exponential decay for such eigenfunctions, see, for Example [19, Lemma 2.5]. □

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