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A Riemann-Hilbert Approach to the Chen-Lee-Liu Equation on the Half Line

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Abstract In this paper, the Fokas unified method is used to analyze the initial-boundary value for the Chen-Lee-Liu equation

$$i\partial_t u + \partial_{xx} u - i|u|^2\partial_x u = 0$$

on the half line $(-\infty, 0]$ with decaying initial value. Assuming that the solution $u(x,t)$ exists, we show that it can be represented in terms of the solution of a matrix Riemann-Hilbert problem formulated in the plane of the complex spectral parameter λ. The jump matrix has explicit (x,t) dependence and is given in terms of the spectral functions $\{a(\lambda), b(\lambda)\}$ and $\{A(\lambda), B(\lambda)\}$, which are obtained from the initial data $u_0(x) = u(x,0)$ and the boundary data $g_0(t) = u(0,t)$, $g_1(t) = u_x(0,t)$, respectively. The spectral functions are not independent, but satisfy a so-called global relation.

Keywords Chen-Lee-Liu equation; initial-value problem; Riemann-Hilbert problem; Fokas unified method; jump matrix
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1 Introduction

One of important integrable systems in mathematics and physics is the following Chen-Lee-Liu (C-L-L) equation[3]

$$i\partial_t u + \partial_{xx} u - i|u|^2\partial_x u = 0 \tag{1.1}$$

which has been derived as an integrable generalization of the nonlinear Schrödinger (NLS) equation by using bi-Hamiltonian methods[14]. The C-L-L equation is also called the derivative nonlinear Schrödinger II (DNLS II) equation[12]. Another two kinds of derivative type NLS equations are the famous KN or the so called DNLS I equation[20,21],

$$i\partial_t u + \partial_{xx} u + i\partial_x(|u|^2 u) = 0 \tag{1.2}$$

and the Gerdjikov-Ivanov equation or the DNLS III equation[18],

$$i\partial_t u + \partial_{xx} u - iu^2\partial_x \overline{u} + \frac{1}{2}|u|^4 u = 0. \tag{1.3}$$

It has been found that there exists gauge transformations among these three equations[2,9–11,23]. The DNLS equations have many applications in plasma physics and nonlinear optics fibers (see

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[8,13,19,29,32]). For example, it governs the evolution of small-amplitude Alfvén waves in a low-β plasma or the large-amplitude magnetohydrodynamic waves. The picosecond pulses in the single-mode nonlinear optical fibers are described by the DNLS equation.

A method to solving initial-boundary value problems for nonlinear integrable systems formulated on the half line and on a finite intervalis presented by Fokas in [15]. The Fokas method provides a generalization of the inverse scattering transformation formalism from initial value problem to initial-boundary value problems. In recent years, this method has been developed by several authors[3−6,22,26−28,31].

In this paper, we use the Fokas method for solving boundary value problems for (1.1) on the half line $(-\infty, 0]$. The paper is orginized as follows. In Section 2, we study the analytical properties of the eigenfunctions and spectral functions associated with the Lax pair of the C-L-L Equation (1.1). Then we change the initial value of the C-L-L Equation (1.1) into a matrix Riemann-Hilbert problem (RHP). The jump matrix has explicit (x, t) dependence and is given in terms of the spectral functions $\{a(\lambda), b(\lambda)\}$ and $\{A(\lambda), B(\lambda)\}$, which are obtained from the initial data $u_0(x) = u(x, 0)$ and the boundary data $g_0(t) = u(0, t), g_1(t) = u_x(0, t)$, respectively. In Section 3, we show that it can be represented in terms of the solution of a matrix RHP formulated in the plane of the complex spectral parameter λ. The problem has the jump across $\{\operatorname{Im} \lambda^4 = 0\}$.

2 Summary of Some Results and the Basic RHP

2.1 Lax Pair

We introduce some notation and definitions which are used throughout the paper.

- $\sigma_3 = \operatorname{diag}(1, -1)$ denotes the third Pauli's matrix, $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $\sigma_1 = \sigma_+ + \sigma_-$;
- A, B are two 2×2 matrixes, matrix commutator $[A, B] = AB - BA$;
- $\widehat{\sigma}_3$ denotes the matrix commutator with σ_3, $\widehat{\sigma}_3 A = [\sigma_3, A]$, then $e^{\widehat{\sigma}_3}$ can be easily computed: $e^{\widehat{\sigma}_3} A = e^{\sigma_3} A e^{-\sigma_3}$, where A is a 2×2 matrix;
- If $f(\bullet)$ is a function then $\overline{f(\bullet)}$ denotes the complex conjugate of $f(\bullet)$;
- D is an unbounded domain of $\mathbb{R} \cup i\mathbb{R}$, let $\mathcal{S}(D)$ denote the space of Schwartz class on D, i.e., the class of smooth scalar-valued functions $f(x)$ on D which together with all derivatives tend to zero faster than any positive power of $|x|^{-1}$ as $|x| \to \infty$;
- For
$$k = 1, 2, \qquad \mathcal{L}_k^{(2\times 2)}(D) \equiv \{F(\lambda) | \lambda \in D,\ F_{ij} \in \mathcal{L}^k(D),\ i, j = 1, 2\},$$
where
$$\mathcal{L}^k(D) \equiv \Big\{ f(\lambda) | \lambda \in D,\ \|f\|_{\mathcal{L}^k(D)} \equiv \Big(\int_D |f(\lambda)|^k |d\lambda| \Big)^{1/k} < \infty \Big\},$$
and
$$\mathcal{L}_\infty^{2\times 2}(D) \equiv \{G(\lambda) | \lambda \in D,\ \|G_{ij}\|_{\mathcal{L}^\infty(D)} \equiv \sup_{\lambda \in D} |G_{ij}(\lambda)| < \infty\ (i, j = 1, 2)\},$$
with the norms taking as follows
$$\|(\cdot)\|_{\mathcal{L}_n^{2\times 2}(D)} \equiv \max_{i,j=1,2} \|(\cdot)_{\{ij\}}\|_{\mathcal{L}^n(D)}, \qquad n = 1, 2, \cdots, \infty.$$

Definition 2.1.1. *Let the contour Γ be the union of a finite number of smooth and oriented curves on the Riemann sphere $\mathbb{C}$, such that $\mathbb{C}\setminus\Gamma$ has only a finite number of connected components. Let $J(\lambda)$ be a 2×2 matrix defined on the contour Γ. The RHP (Γ, J) is the problem of finding a 2×2 matrix-valued function $M(\lambda)$ that satisfies:*

(i) *$M(\lambda)$ is analytic for all $\lambda\in\mathbb{C}\setminus\Gamma$, and extends continuously to the contour Γ;*

(ii) *$M_+(\lambda) = M_-(\lambda)J(\lambda),\ \lambda\in\Gamma$;*

(iii) *$M(\lambda)\to I$, as $\lambda\to\infty$.*

Proposition 2.1.2. *The C-L-L Equation (1.1) admits the following Lax pairs*[11]*:*

$$\partial_x\Psi(x,t;\lambda) = \mathfrak{M}(x,t;\lambda)\Psi(x,t;\lambda), \qquad \partial_t\Psi(x,t;\lambda) = \mathfrak{N}(x,t;\lambda)\Psi(x,t;\lambda), \tag{2.1}$$

where

$$\begin{aligned}\mathfrak{M}(x,t;\lambda) =&\lambda(-i\lambda\sigma_3 + u\sigma_+ + v\sigma_-) - \frac{i}{4}uv\sigma_3,\\ \mathfrak{N}(x,t;\lambda) =&2\lambda^2\Big[-i\sigma_3\lambda^2 + (u\sigma_+ + v\sigma_-)\lambda - \frac{i}{2}uv\sigma_3\Big] + \lambda\Big[\frac{1}{2}uv(u\sigma_+ + v\sigma_-) + i(u_x\sigma_+ - v_x\sigma_-)\Big]\\ &- \Big[\frac{i}{8}u^2v^2 + \frac{1}{4}(uv_x - u_xv)\Big]\sigma_3,\end{aligned} \tag{2.2}$$

with $u(x,t) = \overline{v}(x,t)$. And $u(x,t)$, $v(x,t)$ satisfy the coupled C-L-L equations

$$i\partial_t u + \partial_{xx}u - i|u|^2\partial_x u = 0, \qquad -i\partial_t v + \partial_{xx}v + i|u|^2\partial_x v = 0. \tag{2.3}$$

Let $u(x,t), v(x,t)$ satisfy the two nonlinear (2.3) on the half line $-\infty < x < 0$, $0 < t < T$. Let $u(x,t)$ satisfy decaying initial conditions at $t = 0$, as well as appropriate boundary conditions at $x = 0$. We can prove that (2.3) are the Frobenius compatibility conditions for System (2.1).

Proposition 2.1.3. *Let $u(x,t)$(or $v(x,t)$) be a solution of (2.3). Then there exists a corresponding solution of System (2.1) such that $\Psi(x,t;0)$ is a diagonal matrix.*

Proof. For given $u(x,t)$, let $\widehat{\Psi}(x,t;\lambda)$ be a solution of System (2.1) which exists in accordance with Proposition 2.1.2. Then we obtain that $\widehat{\Psi}(x,t;0) = \exp\big(-\frac{i}{2}\sigma_3\int_{x_0}^{x}|u(\xi,t)|^2d\xi\big)\cdot\widehat{\mathcal{K}}_1$ or $\widehat{\Psi}(x,t;0) = \exp\big(-\frac{i}{4}\sigma_3\int_{t_0}^{t}\big[\frac{1}{2}u^2v^2 - i(uv_x - u_xv)\big](x,\eta)d\eta\big)\cdot\widehat{\mathcal{K}}_2$, for some $x_0, t_0\in\mathbb{R}$ and non-degenerate matrix $\widehat{\mathcal{K}}_1$, $\widehat{\mathcal{K}}_2$ which is independent of x, t, respectively. The function $\Psi(x,t;\lambda)\equiv\widehat{\Psi}(x,t;\lambda)\widehat{\mathcal{K}}_i^{-1}$ $(i = 1,2)$ is the solution of System (2.1) which is diagonal at $\lambda = 0$. □

2.2. Spectral Analysis

Extending the column vector ψ to a 2×2 matrix and letting

$$\psi = \Psi e^{i(\lambda^2x + 2\lambda^4t)\sigma_3}, \qquad -\infty < x < 0,\ \ 0 < t < T, \tag{2.4}$$

then we obtain the equivalent Lax pair

$$\begin{aligned}&\psi_x + i\lambda^2[\sigma_3,\psi] = \Big[\lambda Q - \frac{i}{4}Q^2\sigma_3\Big]\psi,\\ &\psi_t + 2i\lambda^4[\sigma_3,\psi] = \Big[2\lambda^3Q - i\lambda^2Q^2\sigma_3 + \lambda\Big(\frac{1}{2}Q^3 - iQ_x\sigma_3\Big) + P\Big]\psi,\end{aligned} \tag{2.5}$$

where

$$Q = u\sigma_+ + v\sigma_-, \qquad P = -\frac{i}{8}Q^4\sigma_3 - \frac{1}{4}[Q, Q_x]. \tag{2.6}$$

The Lax pair (2.5) can be written in full derivative form

$$d(e^{i(\lambda^2 x+2\lambda^4 t)\widehat{\sigma}_3)}\psi(x,t;\lambda)) = e^{i(\lambda^2 x+2\lambda^4 t)\widehat{\sigma}_3}U(x,t;\lambda)\psi, \qquad -\infty < x < 0, \quad 0 < t < T, \tag{2.7}$$

where

$$U(x,t;\lambda) = U_1(x,t;\lambda)dx + U_2(x,t;\lambda)dt,$$
$$U_1(x,t;\lambda) = \lambda Q - \frac{i}{4}Q^2\sigma_3, \qquad U_2(x,t;\lambda) = 2\lambda^3 Q - i\lambda^2 Q^2\sigma_3 + \lambda\Big(\frac{1}{2}Q^3 - iQ_x\sigma_3\Big) + P.$$

In order to formulate a Riemann-Hilbert problem for the solution of the inverse spectral problem, we seek the solutions of the spectral problem which approaches the 2×2 identity matrix as $\lambda \to \infty$. We use Lenell's method[22] to transform the solution $\psi(x,t;\lambda)$ of (2.7) into the desired asympotic behavior. Consider that a solution of (2.7) is of the form

$$\psi(x,t;\lambda) = D_0 + \frac{D_1}{\lambda} + \frac{D_2}{\lambda^2} + \frac{D_3}{\lambda^3} + \mathcal{O}\Big(\frac{1}{\lambda^4}\Big), \qquad \lambda \longrightarrow \infty,$$

where D_0, D_1, D_2, D_3 are independent of λ. Substituting the above expansion into the first equation of (2.6), and comparing the same order of frequency of λ, we have

$$O(\lambda^2) : i[\sigma_3, D_0] = 0,$$
$$O(\lambda) : i[\sigma_3, D_1] = QD_0,$$
$$O(1) : D_{0x} + i[\sigma_3, D_2] = QD_1 - \frac{i}{4}Q^2\sigma_3 D_0.$$

We know that D_0 is a diagonal matrix form $O(\lambda^2)$, and let $D_0 = \begin{pmatrix} D_0^{11} & 0 \\ 0 & D_0^{22} \end{pmatrix}$. From $O(\lambda)$ we have

$$D_1^{(o)} = \begin{pmatrix} 0 & -\frac{i}{2}uD_1^{22} \\ \frac{i}{2}vD_1^{11} & 0 \end{pmatrix},$$

where $D_1^{(o)}$ being the off-diagonal part of D_1. From $O(1)$, we have

$$D_{0x} = \frac{i}{4}uv\sigma_3 D_0. \tag{2.8}$$

On the other hand, substituting the above expansion into the second equation of (2.6), we have

$$O(\lambda^4) : 2i[\sigma_3, D_0] = 0,$$
$$O(\lambda^3) : 2i[\sigma_3, D_1] = 2QD_0,$$
$$O(\lambda^2) : 2i[\sigma_3, D_2] = 2QD_1 - iQ^2\sigma_3 D_0,$$
$$O(\lambda^1) : 2i[\sigma_3, D_3] = \Big(\frac{1}{2}Q^3 - iQ_x\sigma_3\Big)D_0 + 2QD_2 - iQ^2\sigma_3 D_1,$$
$$O(1) : D_{0t} = 2QD_3 - iQ^2\sigma_3 D_2 + \Big(\frac{1}{2}Q^3 - iQ_x\sigma_3\Big)D_1 - \Big(\frac{i}{8}Q^4\sigma_3 + \frac{1}{4}[Q, Q_x]\Big)D_0.$$

From $O(\lambda^1)$, we obtain the relation

$$2QD_3^{(o)} - iQ^2 D_2^{(d)}\sigma_3 = -\frac{1}{2}Q^3 D_1^{(o)} + \frac{i}{4}Q^4 D_0\sigma_3 + \frac{1}{2}QQ_x D_0, \tag{2.9}$$

where $D_3^{(o)}$ denotes the off-diagonal part of D_3, and $D_2^{(d)}$ denotes the diagonal part of D_2. By using (2.9) and from $O(1)$ we obtain

$$D_{0t} = \Big(\frac{i}{8}u^2v^2 + \frac{1}{4}(uv_x - u_xv)\Big)\sigma_3 D_0. \tag{2.10}$$

The (1.1) admits the conservation law $i(uv)_t = \big(\frac{i}{2}u^2v^2 + uv_x - u_xv\big)_x$. Then the two (2.8) and (2.10) for D_0 are consistent and are both satisfied if we define

$$D_0(x,t) = \exp\Big(i\int_{(x_0,t_0)}^{(x,t)} \Delta\sigma_3\Big), \tag{2.11}$$

where Δ is the closed real-valued one-form, and $\Delta(x,t) = \Delta_1(x,t)dx + \Delta_2(x,t)dt$, $\Delta_1(x,t) = \frac{1}{4}uv$, $\Delta_2(x,t) = \frac{1}{8}u^2v^2 - \frac{i}{4}(uv_x - u_xv)$, $(x_0,t_0) \in D$, simultaneity, for the convenience of calculation we denote $(x_0,t_0) = (0,0)$.

Noting that the integral in (2.11) is independent of the path of integration and the Δ is independent of λ, then we can introduce a new function $\mu(x,t;\lambda)$ as follows

$$\psi(x,t;\lambda) = e^{i\int_{(0,0)}^{(x,t)}\Delta\widehat{\sigma_3}}\mu(x,t;\lambda)D_0(x,t), \qquad -\infty < x < 0, \quad 0 < t < T. \tag{2.12}$$

Through direct calculation, the Lax pair of (2.7) becomes

$$d(e^{i(\lambda^2x+2\lambda^4t)\widehat{\sigma_3}}\mu(x,t;\lambda)) = W(x,t;\lambda), \qquad \lambda\in\mathbb{C}, \tag{2.13}$$

where

$$\begin{aligned} &W(x,t;\lambda) = e^{i(\lambda^2x+2\lambda^4t)\widehat{\sigma_3}}V(x,t;\lambda)\mu(x,t;\lambda),\\ &V(x,t;\lambda) = V_1(x,t;\lambda)dx + V_2(x,t;\lambda)dt = e^{-i\int_{(0,0)}^{(x,t)}\Delta\widehat{\sigma_3}}(U(x,t;\lambda) - i\Delta\sigma_3). \end{aligned}$$

Taking into account the definition of $U(x,t;\lambda)$ and Δ, we can get

$$\begin{aligned} V_1(x,t;\lambda) &= \begin{pmatrix} -\frac{i}{2}uv & \lambda u e^{-2i\int_{(0,0)}^{(x,t)}\Delta} \\ \lambda v e^{2i\int_{(0,0)}^{(x,t)}\Delta} & \frac{i}{2}uv \end{pmatrix},\\ V_2(x,t;\lambda) &= \begin{pmatrix} -i\lambda^2uv - \frac{i}{4}u^2v^2 - \frac{1}{2}(uv_x-u_xv) & \Big(2\lambda^3u + \lambda\Big(\frac{1}{2}u^2v + iu_x\Big)\Big)e^{-2i\int_{(0,0)}^{(x,t)}\Delta} \\ \Big(2\lambda^3v + \lambda\Big(\frac{1}{2}uv^2 - iv_x\Big)\Big)e^{2i\int_{(0,0)}^{(x,t)}\Delta} & i\lambda^2uv + \frac{i}{4}u^2v^2 + \frac{1}{2}(uv_x-u_xv) \end{pmatrix}. \end{aligned}$$

Then (2.13) for $\mu(x,t;\lambda)$ can be written as

$$\mu_x + i\lambda^2[\sigma_3,\mu] = V_1\mu, \qquad \mu_t + 2i\lambda^4[\sigma_3,\mu] = V_2\mu, \tag{2.14}$$

where $-\infty < x < 0,\ 0 < t < T,\ \lambda\in\mathbb{C}$.

2.3 Eigenfunctions and Their Relations

Assuming that $u(x,t)$ exists and is sufficiently smooth in $D = \{-\infty < x < 0,\ 0 < t < T\}$, $\mu_j(x,t,\lambda)$ $(j = 1,2,3)$ are the 2×2 matrix valued functions defined by

$$\mu_j(x,t;\lambda) = I + \int_{(x_j,t_j)}^{(x,t)} e^{-i(\lambda^2x+2\lambda^4t)\widehat{\sigma_3}}W(\xi,\tau,\lambda), \qquad -\infty < x < 0, \quad 0 < t < T. \tag{2.15}$$

The integral denotes a smooth curve from (x_j,t_j) to (x,t), and $(x_1,t_1) = (0,T)$, $(x_2,t_2) = (0,0)$, $(x_3,t_3) = (-\infty,t)$, see Figure 1.

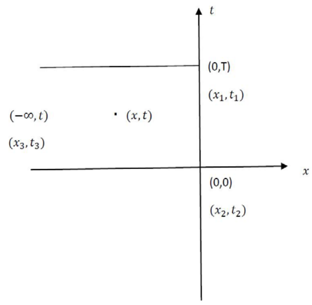

Figure 1. The three points in the (x,t)-domaint

The fundamental theorem of calculus implies that the functions $\mu_j(x,t;\lambda)$ $(j=1,2,3.)$ satisfy (2.13) and the one-form $W(x,t;\lambda)$ is exact, then $\mu_j(x,t;\lambda)(j=1,2,3.)$ are independent on the path of integration. The functions μ_1, μ_2 and μ_3 are defined from λ in some domain of the complex λ-plane. Following the idea in [16], we choose the specific contours depicted in Figure 2.

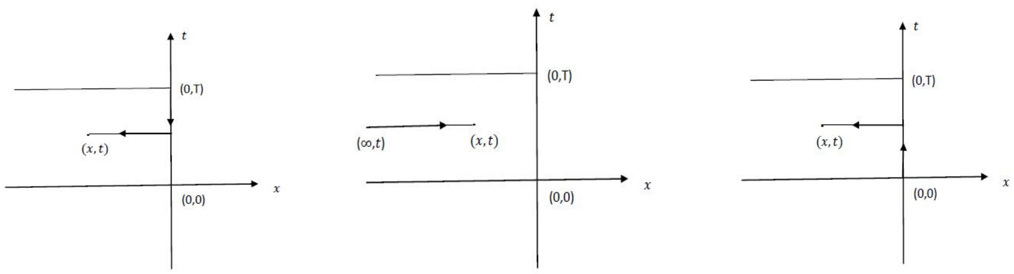

Figure 2. The Three Contours l_1, l_2, l_3 in the (x,t)-domaint

therefore we have

$$\begin{aligned}\mu_1(x,t;\lambda) =& I - \int_x^0 e^{i\lambda^2(\xi-x)\widehat{\sigma_3}}(V_1\mu_1)(\xi,t,\lambda)d\xi \\ &- e^{-i\lambda^2 x\widehat{\sigma_3}}\int_t^T e^{2i\lambda^4(\tau-t)\widehat{\sigma_3}}(V_2\mu_1)(0,\tau,\lambda)d\tau, \\ \mu_2(x,t;\lambda) =& I - \int_x^0 e^{i\lambda^2(\xi-x)\widehat{\sigma_3}}(V_1\mu_2)(\xi,t,\lambda)d\xi \\ &+ e^{-i\lambda^2 x\widehat{\sigma_3}}\int_0^t e^{2i\lambda^4(\tau-t)\widehat{\sigma_3}}(V_2\mu_2)(0,\tau,\lambda)d\tau, \\ \mu_3(x,t;\lambda) =& I + \int_{-\infty}^x e^{i\lambda^2(\xi-x)\widehat{\sigma_3}}(V_1\mu_3)(\xi,t,\lambda)d\xi.\end{aligned} \tag{2.16}$$

Assuming that the dependence of $V_1(x,t;\lambda)$, $V_2(x,t;\lambda)$ on λ is such that $\mu_j(x,t;\lambda) = I + \mathcal{O}(\frac{1}{\lambda})(j=1,2,3.)$ as $\lambda\to\infty$, it follows that the functions $\mu_j(x,t;\lambda)(j=1,2,3.)$ are the fundamental eigenfunctions needed for the formulation of a RHP in the complex λ-plane. And we note that this choice implies the following inequalities

$$(x_1,t_1)\to(x,t): x<\xi<0, \qquad t<\tau<T,$$

$$(x_2,t_2)\to(x,t): x<\xi<0, \qquad 0<\tau<t,$$
$$(x_3,t_3)\to(x,t): -\infty<\xi<x.$$

We find that the first column of the matrix (2.15) involves $e^{-2i(\lambda^2(\xi-x)+2\lambda^4(\tau-t))}$, and using the above inequalities implies that the exponential term of $\mu_j(x,t;\lambda)$ $(j=1,2,3.)$ is bounded in the following regions of the complex λ-plane,

$$(x_1,t_1)\to(x,t): \left\{\operatorname{Im}\lambda^2\le 0\right\}\cap\left\{\operatorname{Im}\lambda^4\le 0\right\},$$
$$(x_2,t_2)\to(x,t): \left\{\operatorname{Im}\lambda^2\le 0\right\}\cap\left\{\operatorname{Im}\lambda^4\ge 0\right\},$$
$$(x_3,t_3)\to(x,t): \left\{\operatorname{Im}\lambda^2\ge 0\right\}.$$

The second column of the matrix (2.15) involves the inverse of the above exponential, which is bounded in

$$\mu_1(x,t;\lambda),(x_1,t_1)\to(x,t): \{\mathrm{Im}\lambda^2\ge 0\}\cap\{Im\lambda^4\ge 0\},$$
$$\mu_2(x,t;\lambda),(x_2,t_2)\to(x,t): \{\mathrm{Im}\lambda^2\ge 0\}\cap\{Im\lambda^4\le 0\},$$
$$\mu_3(x,t;\lambda),(x_3,t_3)\to(x,t): \{\mathrm{Im}\lambda^2\le 0\}.$$

Then, we obtain

$$\begin{aligned}\mu_1(x,t;\lambda)&=(\mu_1^{D_4}(x,t;\lambda),\mu_1^{D_1}(x,t;\lambda)),\\ \mu_2(x,t;\lambda)&=(\mu_2^{D_3}(x,t;\lambda),\mu_2^{D_2}(x,t;\lambda)),\\ \mu_3(x,t;\lambda)&=(\mu_3^{D_1\cup D_2}(x,t;\lambda),\mu_3^{D_3\cup D_4}(x,t;\lambda)),\end{aligned}\tag{2.17}$$

where $\mu_j^{D_l}$ denotes μ_j which is bounded and analytic for $\lambda\in D_l$ and $D_l=\omega_l\cup(-\omega_l)$, $\omega_l=\left\{z\in\mathbb{C}|2k\pi+\frac{l-1}{4}\pi<\operatorname{Arg}z<2k\pi+\frac{l}{4}\pi\right\}$, $-\omega_l=\left\{z\in\mathbb{C}|2k\pi+\frac{l+3}{4}\pi<\operatorname{Arg}z<2k\pi+\frac{l+4}{4}\pi\right\}$, $j=1,2,3$, $l=1,2,3,4$, $k=0,\pm1,\pm2,\cdots$, $\operatorname{Arg}z$ denotes the argument of the complex z, see Figure 3.

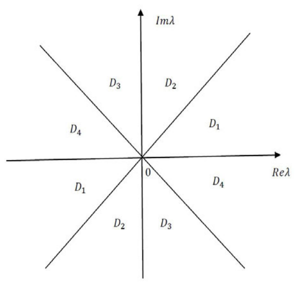

Figure 3. The Sets D_j, $j=1,2,3,4$, which Decompose the Complex λ-plane

More specifically,

$$\begin{aligned}\mu_1(0,t;\lambda)&=(\mu_1^{D_2\cup D_4}(0,t;\lambda),\mu_1^{D_1\cup D_3}(0,t;\lambda)),\\ \mu_2(0,t;\lambda)&=(\mu_2^{D_1\cup D_3}(0,t;\lambda),\mu_2^{D_2\cup D_4}(0,t;\lambda)),\\ \mu_1(x,T;\lambda)&=(\mu_1^{D_3\cup D_4}(x,T;\lambda),\mu_1^{D_1\cup D_2}(x,T;\lambda)),\\ \mu_2(x,0;\lambda)&=(\mu_2^{D_3\cup D_4}(x,0;\lambda),\mu_2^{D_1\cup D_2}(x,0;;\lambda)),\\ \mu_1(0,0;\lambda)&=(\mu_1^{D_2\cup D_4}(0,0;\lambda),\mu_1^{D_1\cup D_3}(0,0;\lambda)),\\ \mu_2(0,T;\lambda)&=(\mu_2^{D_1\cup D_3}(0,T;\lambda),\mu_2^{D_2\cup D_4}(0,T;\lambda)).\end{aligned}\tag{2.18}$$

For the purpose of deriving a RHP, we need to compute the jumps across the boundaries of the D_j's ($j=1,2,3,4.$). It turns out that the relevant jump matrices can be uniquely defined in terms of two 2×2 matrices valued spectral functions $s(\lambda)$ and $S(\lambda)$ defined as follows

$$\begin{aligned}\mu_3(x,t;\lambda)&=\mu_2(x,t;\lambda)e^{-i(\lambda^2x+2\lambda^4t)\widehat{\sigma}_3}s(\lambda),\\ \mu_1(x,t;\lambda)&=\mu_2(x,t;\lambda)e^{-i(\lambda^2x+2\lambda^4t)\widehat{\sigma}_3}S(\lambda).\end{aligned}\tag{2.19}$$

Evaluating the first equation of (2.19) at $(x,t)=(0,0)$ and the second equation of (2.19) at $(x,t)=(0,T)$, implies

$$s(\lambda)=\mu_3(0,0;\lambda),(S(\lambda))^{-1}=e^{2i\lambda^4T\widehat{\sigma}_3}\mu_2(0,T;\lambda).\tag{2.20}$$

From (2.18) and (2.19), we obtain

$$\mu_1(x,t;\lambda)=\mu_3(x,t;\lambda)e^{-i(\lambda^2x+2\lambda^4t)\widehat{\sigma}_3}(s(\lambda))^{-1}S(\lambda)\tag{2.21}$$

which will lead to the global relation.

Hence, the function $s(\lambda)$ can be obtained from the evaluations at $x=0$ of the function $\mu_3(x,0,\lambda)$ and $S(\lambda)$ can be obtained from the evaluations at $t=T$ of the function $\mu_2(0,t,\lambda)$. And these functions about $\mu_j(x,t;\lambda)$ ($j=1,2,3.$) satisfy the linear integral equations as follows

$$\begin{aligned}\mu_1(0,t;\lambda)&=I-\int_t^T e^{2i\lambda^4(\tau-t)\widehat{\sigma}_3}(V_2\mu_1)(0,\tau,\lambda)d\tau,\\ \mu_2(0,t;\lambda)&=I+\int_0^t e^{2i\lambda^4(\tau-t)\widehat{\sigma}_3}(V_2\mu_2)(0,\tau,\lambda)d\tau,\\ \mu_3(x,0;\lambda)&=I+\int_{-\infty}^x e^{i\lambda^2(\xi-x)\widehat{\sigma}_3}(V_1\mu_3)(\xi,0,\lambda)d\xi,\\ \mu_2(x,0;\lambda)&=I-\int_x^0 e^{i\lambda^2(\xi-x)\widehat{\sigma}_3}(V_1\mu_2)(\xi,0,\lambda)d\xi.\end{aligned}\tag{2.22}$$

Let $u_0(x)=u(x,0)$, $g_0(t)=u(0,t)$, and $g_1(t)=u_x(0,t)$ be the initial and boundary values of $u(x,t)$, then

$$V_1(x,0;\lambda)=\begin{pmatrix}-\frac{i}{2}|u_0|^2 & \lambda u_0e^{-\int_0^x\frac{i}{2}|u_0|^2d\xi}\\ \lambda\overline{u}_0e^{\int_0^x\frac{i}{2}|u_0|^2d\xi} & \frac{i}{2}|u_0|^2\end{pmatrix},$$

$$V_2(0,t;\lambda)=\begin{pmatrix}-i\lambda^2|g_0|^2-\frac{i}{4}|g_0|^4-\frac{1}{2}(g_0\overline{g}_1-g_1\overline{g}_0) & \left(2\lambda^3g_0+\lambda\left(\frac{1}{2}|g_0|^2g_0+ig_1\right)\right)e^{-2i\int_0^t\Delta_2(0,\tau)d\tau}\\ \left(2\lambda^3\overline{g}_0+\lambda\left(\frac{1}{2}|g_0|^2\overline{g}_0-i\overline{g}_1\right)\right)e^{2i\int_0^t\Delta_2(0,\tau)d\tau} & i\lambda^2|g_0|^2+\frac{i}{4}|g_0|^4+\frac{1}{2}(g_0\overline{g}_1-g_1\overline{g}_0)\end{pmatrix},$$

and $\Delta_2(0,\tau)=\frac{1}{8}|g_0|^4-\frac{i}{4}(g_0\overline{g}_1-g_1\overline{g}_0)$.

The analytic properties of (2×2) matrices $\mu_j(x,t;\lambda)$ ($j=1,2,3.$) that come from (2.15) are collected in the following propositions. We denote by $\mu_j^{(1)}(x,t;\lambda)$ and $\mu_j^{(2)}(x,t;\lambda)$ the first and second columns of $\mu_j(x,t;\lambda)$, respectively. Setting

$$\mu_j(x,t;\lambda)=(\mu_j^{(1)}(x,t;\lambda),\mu_j^{(2)}(x,t;\lambda))=\begin{pmatrix}\mu_j^{11} & \mu_j^{12}\\ \mu_j^{21} & \mu_j^{22}\end{pmatrix},\qquad j=1,2,3.$$

Proposition 2.3.1. *The matrices $\mu_j(x,t;\lambda)=(\mu_j^{(1)}(x,t;\lambda),\mu_j^{(2)}(x,t;\lambda))$ ($j=1,2,3.$) have the following properties*

- $det\mu_1(x,t;\lambda) = det\mu_2(x,t;\lambda) = det\mu_3(x,t;\lambda) = 1$;
- $\mu_1^{(1)}(x,t;\lambda)$ *is analytic, and* $\lim\limits_{\lambda\to\infty}\mu_1^{(1)}(x,t;\lambda) = (1,0)^T$, *where* $\lambda \in \{\operatorname{Im}\lambda^2 \le 0\} \cap \{\operatorname{Im}\lambda^4 \le 0\}$;
- $\mu_1^{(2)}(x,t;\lambda)$ *is analytic, and* $\lim\limits_{\lambda\to\infty}\mu_1^{(2)}(x,t;\lambda) = (0,1)^T$, *where* $\lambda \in \{\operatorname{Im}\lambda^2 \ge 0\} \cap \{\operatorname{Im}\lambda^4 \ge 0\}$;
- $\mu_2^{(1)}(x,t;\lambda)$ *is analytic, and* $\lim\limits_{\lambda\to\infty}\mu_2^{(1)}(x,t;\lambda) = (1,0)^T$, *where* $\lambda \in \{\operatorname{Im}\lambda^2 \le 0\} \cap \{\operatorname{Im}\lambda^4 \ge 0\}$;
- $\mu_2^{(2)}(x,t;\lambda)$ *is analytic, and* $\lim\limits_{\lambda\to\infty}\mu_2^{(2)}(x,t;\lambda) = (0,1)^T$, *where* $\lambda \in \{\operatorname{Im}\lambda^2 \ge 0\} \cap \{\operatorname{Im}\lambda^4 \le 0\}$;
- $\mu_3^{(1)}(x,t;\lambda)$ *is analytic, and* $\lim\limits_{\lambda\to\infty}\mu_3^{(1)}(x,t;\lambda) = (1,0)^T$, *where* $\lambda \in \{\operatorname{Im}\lambda^2 \ge 0\}$;
- $\mu_3^{(2)}(x,t;\lambda)$ *is analytic, and* $\lim\limits_{\lambda\to\infty}\mu_3^{(2)}(x,t;\lambda) = (0,1)^T$, *where* $\lambda \in \{\operatorname{Im}\lambda^2 \le 0\}$.

Proposition 2.3.2 (Symmetries). *The matrices*

$$\mu_j(x,t;\lambda) = \begin{pmatrix} \mu_j^{11}(x,t;\lambda) & \mu_j^{12}(x,t;\lambda) \\ \mu_j^{21}(x,t;\lambda) & \mu_j^{22}(x,t;\lambda) \end{pmatrix}, \qquad j = 1,2,3$$

have the following properties

- $\mu_j^{11}(x,t;\lambda) = \overline{\mu_j^{22}(x,t;\overline{\lambda})}$, $\mu_j^{12}(x,t;\lambda) = \overline{\mu_j^{21}(x,t;\overline{\lambda})}$;
- $\mu_j^{11}(x,t;-\lambda) = \mu_j^{11}(x,t;\lambda)$, $\mu_j^{12}(x,t;-\lambda) = -\mu_j^{12}(x,t;\lambda)$, $\mu_j^{21}(x,t;-\lambda) = -\mu_j^{21}(x,t;\lambda)$, $\mu_j^{22}(x,t;-\lambda) = \mu_j^{22}(x,t;\lambda)$.

Proposition 2.3.3. *The spectral function $s(\lambda)$ and $S(\lambda)$ are defined in (2.18) and (2.19) imply that*

$$\begin{aligned} s(\lambda) &= I + \int_{-\infty}^{0} e^{i\lambda^2(\xi-x)\widehat{\sigma_3}}(V_1\mu_3)(\xi,0;\lambda)d\xi, \\ S^{-1}(\lambda) &= I + \int_0^T e^{2i\lambda^4\tau\widehat{\sigma_3}}(V_2\mu_2)(0,\tau;\lambda)d\tau. \end{aligned} \tag{2.23}$$

According to Proposition 2.3.2, we can construct the following matrix functions $s(\lambda)$ and $S(\lambda)$,

$$s(\lambda) = \begin{pmatrix} \overline{a(\overline{\lambda})} & b(\lambda) \\ \overline{b(\overline{\lambda})} & a(\lambda) \end{pmatrix}, \qquad (\lambda) = \begin{pmatrix} \overline{A(\overline{\lambda})} & B(\lambda) \\ \overline{B(\overline{\lambda})} & A(\lambda) \end{pmatrix}. \tag{2.24}$$

By use of (2.19) and (2.23), we can obtain

- $\begin{pmatrix} b(\lambda) \\ a(\lambda) \end{pmatrix} = \mu_3^{(2)}(0,0;\lambda) = \begin{pmatrix} \mu_3^{12}(0,0;\lambda) \\ \mu_3^{22}(0,0;\lambda) \end{pmatrix}$
$\begin{pmatrix} e^{-4i\lambda^4 T}B(\lambda) \\ \overline{A(\overline{\lambda})} \end{pmatrix} = \mu_2^{(2)}(0,T;\lambda) = \begin{pmatrix} \mu_2^{12}(0,T;\lambda) \\ \mu_2^{22}(0,T;\lambda) \end{pmatrix}$.

- $\partial_x\mu_3^{(2)}(x,0;\lambda)+2i\lambda^2\sigma\mu_3^{(2)}(x,0;\lambda)=V_1(x,0;\lambda)\mu_3^{(2)}(x,0;\lambda),\ \lambda\in D_3\cup D_4,\ -\infty<x<0.$
  $\partial_t\mu_2^{(2)}(0,t;\lambda)+4i\lambda^4\sigma\mu_2^{(2)}(0,t;\lambda)=V_2(0,t;\lambda)\mu_2^{(2)}(x,0;\lambda),\ \lambda\in D_2\cup D_4,\ 0<t<T.$ *where* $\sigma=\begin{pmatrix}1&0\\0&0\end{pmatrix}$.
- $a(-\lambda)=a(\lambda),\ b(-\lambda)=-b(\lambda),\ A(-\lambda)=A(\lambda),\ B(-\lambda)=-B(\lambda).$
- $\det s(\lambda)=\det S(\lambda)=1.$
- $a(\lambda)=1+\mathcal{O}(\frac{1}{\lambda}),\ b(\lambda)=\mathcal{O}(\frac{1}{\lambda}),\ \lambda\to\infty,\ \operatorname{Im}\lambda^2\geq 0,$
  $A(\lambda)=1+\mathcal{O}(\frac{1}{\lambda}),\ B(\lambda)=\mathcal{O}(\frac{1}{\lambda}),\ \lambda\to\infty,\ \operatorname{Im}\lambda^4\geq 0.$

2.4 The Basic RHP

According to the paper [25], we can get that the Riemann-Hilbert problem of the C-L-L equation. (2.19) and (2.21), relating the various analytic eigenfunctions, can be rewritten in a form that determines the jump conditions of a (2×2) RHP, with unitary jump matrices on the real and imaginary axis. This involves tedious but straightforward algebraic manipulations.

Setting

$$\begin{aligned}
&\theta(\lambda)=\lambda^2x+2\lambda^4t;\\
&\alpha(\lambda)=\overline{a(\overline{\lambda})}A(\lambda)-\overline{b(\overline{\lambda})}B(\lambda);\\
&\beta(\lambda)=a(\lambda)B(\lambda)-b(\lambda)A(\lambda);\\
&\delta(\lambda)=\overline{a(\overline{\lambda})}\beta(\lambda)+b(\lambda)\alpha(\lambda).
\end{aligned}$$

Let $M(x,t;\lambda)$ be defined as below

$$\begin{aligned}
M_+(x,t;\lambda)&=(\frac{\mu_3^{D_1\cup D_2}(x,t;\lambda)}{\alpha(\lambda)},\mu_1^{D_1}(x,t;\lambda)),\lambda\in D_1;\\
M_-(x,t;\lambda)&=(\frac{\mu_3^{D_1\cup D_2}(x,t;\lambda)}{\overline{a(\overline{\lambda})}},\mu_2^{D_2}(x,t;\lambda)),\lambda\in D_2;\\
M_+(x,t;\lambda)&=(\mu_2^{D_3}(x,t;\lambda),\frac{\mu_3^{D_3\cup D_4}(x,t;\lambda)}{a(\lambda)}),\lambda\in D_3;\\
M_-(x,t;\lambda)&=(\mu_1^{D_4}(x,t;\lambda),\frac{\mu_3^{D_3\cup D_4}(x,t;\lambda)}{\overline{\alpha(\overline{\lambda})}}),\lambda\in D_4.
\end{aligned}\tag{2.25}$$

These definitions imply that

$$\det M(x,t;\lambda)=1,\qquad M(x,t;\lambda)=I+\mathcal{O}\Big(\frac{1}{\lambda}\Big),\qquad \lambda\to\infty.$$

Theorem 2.4.1. *Let $u(x,t;\lambda)$ is a smooth function, $\mu_1(x,t;\lambda),\mu_2(x,t;\lambda),\mu_3(x,t;\lambda)$ are defined by (2.16), and $M(x,t;\lambda)$ be defined by (2.25), then $M(x,t;\lambda)$ satisfies the jump condition*

$$M_+(x,t;\lambda)=M_-(x,t;\lambda)J(x,t;\lambda),\qquad \lambda^4\in\mathbb{R},\tag{2.26}$$

where

$$J(x,t,\lambda)=\begin{cases}J_1(x,t;\lambda), & \operatorname{Arg}\lambda^2=0;\\ J_2(x,t;\lambda), & \operatorname{Arg}\lambda^2=\frac{\pi}{2};\\ J_3(x,t;\lambda), & \operatorname{Arg}\lambda^2=\pi;\\ J_4(x,t;\lambda), & \operatorname{Arg}\lambda^2=\frac{3\pi}{2}.\end{cases}\tag{2.27}$$

and

$$J_1(x,t;\lambda)=\begin{pmatrix}\dfrac{1}{\alpha(\lambda)\overline{\alpha(\overline{\lambda})}} & \dfrac{\overline{\beta(\overline{\lambda})}}{\alpha(\lambda)}e^{-2i\theta(\lambda)}\\ -\dfrac{\beta(\lambda)}{\alpha(\lambda)}e^{2i\theta(\lambda)} & 1\end{pmatrix},$$

$$J_2(x,t;\lambda)=\begin{pmatrix}\dfrac{\overline{a(\overline{\lambda})}}{\alpha(\lambda)} & \delta(\lambda)e^{-2i\theta(\lambda)}\\ 0 & \dfrac{\alpha(\lambda)}{\overline{a(\overline{\lambda})}}\end{pmatrix},$$

$$J_3(x,t;\lambda)=\begin{pmatrix}1 & \dfrac{b(\lambda)}{a(\lambda)}e^{-2i\theta(\lambda)}\\ -\dfrac{\overline{b(\overline{\lambda})}}{\overline{a(\overline{\lambda})}}e^{2i\theta(\lambda)} & \dfrac{1}{a(\lambda)\overline{a(\overline{\lambda})}}\end{pmatrix},$$

$$J_4(x,t;\lambda)=\begin{pmatrix}\dfrac{a(\lambda)}{\overline{\alpha(\overline{\lambda})}} & 0\\ -\overline{\delta(\overline{\lambda})}e^{2i\theta(\lambda)} & \dfrac{\overline{\alpha(\overline{\lambda})}}{a(\lambda)}.\end{pmatrix}.$$

Proof. We can complete the proof as Proposition 2.2's idea in [17]. In order to derive the jump Condition (2.26) we write (2.19) and (2.21) in the following form

$$\begin{cases}\overline{a(\overline{\lambda})}\mu_2^{D_3}+\overline{b(\overline{\lambda})}e^{2i\theta(\lambda)}\mu_2^{D_2}=\mu_3^{D_1\cup D_2},\\ b(\lambda)e^{-2i\theta(\lambda)}\mu_2^{D_3}+a(\lambda)\mu_2^{D_2}=\mu_3^{D_3\cup D_4},\end{cases}\tag{2.28}$$

$$\begin{cases}\overline{A(\overline{\lambda})}\mu_2^{D_3}+\overline{B(\overline{\lambda})}e^{2i\theta(\lambda)}\mu_2^{D_2}=\mu_1^{D_4},\\ B(\lambda)e^{-2i\theta(\lambda)}\mu_2^{D_3}+A(\lambda)\mu_2^{D_2}=\mu_1^{D_1},\end{cases}\tag{2.29}$$

$$\begin{cases}\overline{\alpha(\overline{\lambda})}\mu_3^{D_1\cup D_2}+\overline{\beta(\overline{\lambda})}e^{2i\theta(\lambda)}\mu_2^{D_3\cup D_4}=\mu_1^{D_4},\\ \beta(\lambda)e^{-2i\theta(\lambda)}\mu_3^{D_1\cup D_2}+\alpha(\lambda)\mu_2^{D_3\cup D_4}=\mu_1^{D_1}.\end{cases}\tag{2.30}$$

Using (2.28), (2.29) and (2.30), we can derive that the jump matrices $J_i(x,t;\lambda)$ $(i=1,2,3,4.)$ satisfy

$$\begin{aligned}
&\Big(\frac{\mu_3^{D_1\cup D_2}(x,t;\lambda)}{\alpha(\lambda)},\mu_1^{D_1}(x,t;\lambda)\Big)=\Big(\mu_1^{D_4}(x,t;\lambda),\frac{\mu_3^{D_3\cup D_4}(x,t;\lambda)}{\overline{\alpha(\overline{\lambda})}}\Big)J_1(x,t;\lambda);\\
&\Big(\frac{\mu_3^{D_1\cup D_2}(x,t;\lambda)}{\alpha(\lambda)},\mu_1^{D_1}(x,t;\lambda)\Big)=\Big(\frac{\mu_3^{D_1\cup D_2}(x,t;\lambda)}{\overline{a(\overline{\lambda})}},\mu_2^{D_2}(x,t;\lambda)\Big)J_2(x,t;\lambda);\\
&\Big(\mu_2^{D_3}(x,t;\lambda),\frac{\mu_3^{D_3\cup D_4}(x,t;\lambda)}{a(\lambda)}\Big)=\Big(\frac{\mu_3^{D_1\cup D_2}(x,t;\lambda)}{\overline{a(\overline{\lambda})}},\mu_2^{D_2}(x,t,\lambda)\Big)J_3(x,t;\lambda);\\
&\Big(\mu_2^{D_3}(x,t;\lambda),\frac{\mu_3^{D_3\cup D_4}(x,t;\lambda)}{a(\lambda)}\Big)=\Big(\mu_1^{D_4}(x,t;\lambda),\frac{\mu_3^{D_3\cup D_4}(x,t;\lambda)}{\overline{\alpha(\overline{\lambda})}}\Big)J_4(x,t;\lambda).
\end{aligned}\tag{2.31}$$

The matrix $M(x,t;\lambda)$ of this RHP is a sectionally meromorphic function of λ in $\mathbb{C}\backslash\{\lambda^4\in\mathbb{R}\}$. The possible poles of $M(x,t;\lambda)$ are generated by the zeros of $a(\lambda)$, $\alpha(\lambda)$ and by the complex conjugates of these zeros. Since $a(\lambda)$, $\alpha(\lambda)$ are even functions, this means each zero λ_j of $a(\lambda)$

is accompanied by another zero at $-\lambda_j$. Similarly, each zero λ_j of $\alpha(\lambda)$ is accompanied by a zero at $-\lambda_j$. In particular, both $a(\lambda)$ and $\alpha(\lambda)$ have even number of zeros. □

Hypothesis 2.4.2. *We assume that*

- *$a(\lambda)$ has $2n$ simple zeros $\{\varepsilon_j\}_{j=1}^{2n}$, $2n = 2n_1 + 2n_2$, such that ε_j $(j = 1, 2, \cdots, 2n_1)$ lie in $D_3 \bigcup D_4$, and $\overline{\varepsilon}_j$ $(j = 2n_1 + 1, 2n_1 + 2, \cdots, 2n)$ lie in $D_1 \cup D_2$.*
- *$\alpha(\lambda)$ has $2N$ simple zeros $\{\gamma_j\}_{j=1}^{2N}$ $(2n = 2N_1 + 2N_2)$, such that γ_j $(j = 1, 2, \cdots, 2N_1)$, lie in $D_1 \bigcup D_2$, and $\overline{\gamma}_j$ $(j = 2N_1 + 1, 2N_1 + 2, \cdots, 2N)$, lie in $D_3 \cup D_4$.*
- *None of the zeros of $\alpha(\lambda)$ coincides with any of the zeros of $a(\lambda)$.*

The residues of the function $M(x, t; \lambda)$ at the corresponding poles can be computed using (2.19) and (2.21). Using the notation $[M(x, t; \lambda)]_1$ for the first column and $[M(x, t; \lambda)]_2$ for the second column of the solution $M(x, t; \lambda)$ of the RHP, and we write $\dot{a}(\lambda) = \frac{da}{d\lambda}$, then we get the following proposition.

Proposition 2.4.3.

(i) Res $\{[M(x,t;\lambda)]_1, \gamma_j\} = \frac{e^{2i\theta(\gamma_j)}}{\dot{\alpha}(\gamma_j)\beta(\gamma_j)}[M(x,t;\gamma_j)]_2$, $j = 1, 2, \cdots, 2N_1$.

(ii) Res $\{[M(x,t;\lambda)]_2, \overline{\gamma}_j\} = \frac{e^{-2i\theta(\overline{\gamma}_j)}}{\dot{\alpha}(\overline{\gamma}_j)\beta(\overline{\gamma_j})}[M(x,t;\overline{\gamma_j})]_1$, $j = 2N_1 + 1, 2N_1 + 2, \cdots, 2N$.

(iii) Res $\{[M(x,t;\lambda)]_1, \overline{\varepsilon}_j\} = \frac{e^{2i\theta(\overline{\varepsilon}_j)}\overline{b(\overline{\varepsilon}_j)}}{\dot{a}(\overline{\varepsilon}_j)}[M(x,t;\overline{\varepsilon}_j)]_2$, $j = 2n_1 + 1, 2n_1 + 2, \cdots, 2n$.

(iv) Res $\{[M(x,t;\lambda)]_2, \varepsilon_j\} = \frac{e^{-2i\theta(\varepsilon_j)}b(\varepsilon_j)}{\dot{a}(\varepsilon_j)}[M(x,t;\varepsilon_j)]_1$, $j = 1, 2, \cdots, 2n_1$.

Proof. According to the idea in [17], we only need to prove (i), and another three relations also have similar proof. Consider $M(x,t;\lambda) = (\frac{\mu_3^{D_1 \cup D_2}}{\alpha(\lambda)}, \mu_1^{D_1})$, the simple zeros γ_j $(j = 1, 2, \cdots, 2N_1.)$ of $\alpha(\lambda)$ are the simple poles of $\frac{\mu_3^{D_1 \cup D_2}}{\alpha(\lambda)}$. Then we have

$$\text{Res}\Big\{\frac{\mu_3^{D_1 \cup D_2}(x,t;\lambda)}{\alpha(\lambda)}, \gamma_j\Big\} = \lim_{\lambda \to \gamma_j} (\lambda - \gamma_j)\frac{\mu_3^{D_1 \cup D_2}(x,t;\lambda)}{\alpha(\lambda)} = \frac{\mu_3^{D_1 \cup D_2}(x,t;\gamma_j)}{\dot{\alpha}(\gamma_j)}.$$

Taking $\lambda = \gamma_j$ into the second equation of (2.30) we obtain

$$\mu_3^{D_1 \cup D_2}(x,t;\gamma_j) = \frac{\mu_1^{D_1}(x,t;\gamma_j)}{\beta(\gamma_j)}e^{2i\theta(\gamma_j)}.$$

Furthermore,

$$\text{Res}\Big\{\frac{\mu_3^{D_1 \cup D_2}(x,t;\lambda)}{\alpha(\lambda)}, \gamma_j\Big\} = \frac{e^{2i\theta(\gamma_j)}\mu_1^{D_1}(x,t;\gamma_j)}{\dot{\alpha}(\gamma_j)\beta(\gamma_j)}.$$

It is equivalent to Proposition 2.4.3(i). □

2.5 The Inverse Problem

Rewriting the jump condition

$$M_+(x,t;\lambda) - M_-(x,t;\lambda) = M_-(x,t;\lambda)J(x,t;\lambda) - M_-(x,t;\lambda),$$

then

$$M_+(x,t;\lambda) - M_-(x,t;\lambda) = M_-\widetilde{J}(x,t;\lambda), \tag{2.32}$$

where $\widetilde{J}(x,t;\lambda) = J(x,t;\lambda) - I$. The asymptotic conditions of (2.16) and the Proposition 2.3.1 imply

$$M(x,t;\lambda) = I + \frac{\overline{M}(x,t;\lambda)}{\lambda} + \mathcal{O}\Big(\frac{1}{\lambda}\Big), \qquad \lambda \to \infty, \quad \lambda \in \mathbb{C}\setminus\Gamma, \tag{2.33}$$

where $\Gamma = \{\lambda^4 = \mathbb{R}\}$. (2.32) and the condition (2.33) yield the following integral representation for the function $M(x,t;\lambda)$

$$M(x,t;\lambda) = I + \frac{1}{2\pi i}\int_\Gamma \frac{M_+(x,t;\lambda')\widetilde{J}(x,t;\lambda')}{\lambda - \lambda'}d\lambda', \qquad \lambda \in \mathbb{C}\setminus\Gamma, \tag{2.34}$$

then

$$\overline{M}(x,t;\lambda) = -\frac{1}{2\pi i}\int_\Gamma M_+(x,t;\lambda')\widetilde{J}(x,t;\lambda')d\lambda'. \tag{2.35}$$

Using (2.33) in the first ODE of the Lax pair (2.6), we find

$$-\frac{i}{4}[\sigma_3, \overline{M}(x,t;\lambda)] = i\frac{u_x(x,t) - iu_t(x,t)}{4}\sigma_1, \tag{2.36}$$

$$u_x(x,t) - iu_t(x,t) = 2(\overline{M(x,t;\lambda)})_{21} = 2\lim_{\lambda\to\infty}(\lambda M(x,t;\lambda))_{21}, \tag{2.37}$$

where σ_1, σ_3 denote the usual Pauli matrices.

The inverse problem involves reconstructing the potential $u(x,t)$ from the spectral functions $\mu_j, \ j = 1,2,3$. That means we will reconstruct the potential $u(x,t)$. We show in Section 2.2 that

$$D_1^{(o)} = \begin{pmatrix} 0 & -\frac{i}{2}uD_1^{22} \\ \frac{i}{2}vD_1^{11} & 0 \end{pmatrix},$$

when $\psi(x,t;\lambda) = D_0 + \frac{D_1}{\lambda} + \frac{D_2}{\lambda^2} + \frac{D_3}{\lambda^3} + \mathcal{O}(\frac{1}{\lambda^4})$ $(\lambda \to \infty)$ is a solution of (2.7). This implies that

$$u(x,t) = 2im(x,t)e^{2i\int_{(0,0)}^{x,t}\Delta}, \tag{2.38}$$

where

$$\mu(x,t;\lambda) = I + \frac{m^{(1)}(x,t;\lambda)}{\lambda} + \frac{m^{(2)}(x,t;\lambda)}{\lambda^2} + \frac{m^{(3)}(x,t;\lambda)}{\lambda^3} + \mathcal{O}\Big(\frac{1}{\lambda^4}\Big)(\lambda \to \infty)$$

is the corresponding solution of (2.13) related to $\psi(x,t;\lambda)$ via (2.12), and we write $m(x,t)$ for $m_{12}^{(1)}(x,t)$. From (2.38) and its complex conjugate, we obtain

$$uv = 4|m|^2, \qquad uv_x - u_xv = 4(\overline{m}_x m - m_x\overline{m}) - 32i|m|^4.$$

Thus, we are able to express the one-form Δ defined in (2.10) in terms of $m(x,t;\lambda)$ as

$$\Delta = |m|^2dx - (6|m|^4 + i(\overline{m}_x m - m_x\overline{m}))dt. \tag{2.39}$$

Then we can solve the inverse problem as follows

(i) Use any one of the three spectral functions μ_j $(j = 1,2,3.)$ to compute $m(x,t)$ according to

$$m(x,t) = \lim_{\lambda\to\infty}(\lambda\mu_j(x,t;\lambda))_{12}.$$

(ii) Determine $\Delta(x,t)$ from (2.39).

(iii) Finally, $u(x,t)$ is given by (2.38).

3 The Spectral Functions and the Principal RHP

3.1 The Definition of Spectral Functions

The analysis of Section 2 motivates the following definitions for the spectral functions.

Definition 3.1.1 (The spectral functions $a(\lambda)$ and $b(\lambda)$). *Given the smooth function $u_0(x) = u(x,0)$, we define the map*

$$\mathbb{S} : \{u_0(x)\} \longrightarrow \{a(\lambda), b(\lambda)\}$$

by

$$\begin{pmatrix} b(\lambda) \\ a(\lambda) \end{pmatrix} = \mu_3^{(2)}(x,0;\lambda) = \begin{pmatrix} \mu_3^{12}(x,0;\lambda) \\ \mu_3^{22}(x,0;\lambda) \end{pmatrix}, \qquad \mathrm{Im}\,\lambda^2 \le 0.$$

where $\mu_3(x,0;\lambda)$ is the unique solution of the Volterra linear integral equation

$$\mu_3(x,0;\lambda) = I + \int_{-\infty}^{x} e^{i\lambda^2(\xi - x)\widehat{\sigma_3}}(V_1\mu_3)(\xi,0;\lambda)d\xi$$

and $V_1(x,0;\lambda)$ is given in terms of $u(x,0;\lambda)$ by

$$V_1(x,0;\lambda) = \begin{pmatrix} -\frac{i}{2}|u_0|^2 & \lambda u_0 e^{-\int_x^0 \frac{i}{2}|u_0|^2 d\xi} \\ \lambda \overline{u}_0 e^{\int_x^0 \frac{i}{2}|u_0|^2 d\xi} & \frac{i}{2}|u_0|^2 \end{pmatrix}.$$

Proposition 3.1.2. *The spectral functions $a(\lambda)$ and $b(\lambda)$ have the following properties*

(i) *$a(\lambda)$ and $b(\lambda)$ are analytic for $Im\lambda^2 < 0$, continuous and bounded for $Im\,\lambda^2 \le 0$.*

(ii) *$a(\lambda) = 1 + \mathcal{O}(\frac{1}{\lambda}), b(\lambda) = \mathcal{O}(\frac{1}{\lambda})$ as $\lambda \to \infty$, $Im\,\lambda^2 \le 0$;*

(iii) *$a(\lambda)\overline{a(\overline{\lambda})} - b(\lambda)\overline{b(\overline{\lambda})} = 1$, $\lambda^2 \in \mathbb{R}$;*

(iv) *$a(-\lambda) = a(\lambda)$, $b(-\lambda) = -b(\lambda)$, $Im\,\lambda^2 \le 0$;*

(v) *The map $\mathbb{Q} : \{a(\lambda), b(\lambda)\} \to \{u_0(x)\}$, the inverse map $\mathbb{S}$ of $\mathbb{Q}$ is defined by*

$$u_0(x) = 2im(x)e^{4i\int_x^0 |m(x')|dx'}, \qquad m(x) = \lim_{\lambda\to\infty}(\lambda M^{(x)}(x,\lambda))_{12}$$

where, $M^{(x)}(x,\lambda)$ is the unique solution of the following RHP (see Remark 3.1.3);

(vi) *$\mathbb{S}^{-1} = \mathbb{Q}$.*

Remark 3.1.3. The Definition 3.1.1 gives rise to the map

$$\mathbb{S} : \{u_0(x)\} \to \{a(\lambda), b(\lambda)\}.$$

The inverse of this map

$$\mathbb{Q} : \{a(\lambda), b(\lambda)\} \to \{u_0(x)\}$$

can be defined as follows

$$u_0(x) = 2im(x)e^{4i\int_x^0 |m(x')|dx'}, \qquad m(x) = \lim_{\lambda\to\infty} (\lambda M^{(x)}(x,\lambda))_{12},$$

where $M^{(x)}(x,\lambda)$ is the unique solution of the following RHP

- $M^{(x)}(x,\lambda) = \begin{cases} M_-^{(x)}(x,\lambda), & \operatorname{Im}\lambda^2 \le 0 \\ M_+^{(x)}(x,\lambda), & \operatorname{Im}\lambda^2 \ge 0 \end{cases}$ is a sectionally meromorphic function.
- $M_+^{(x)}(x,\lambda) = M_-^{(x)}(x,\lambda)(J^{(x)}(x,\lambda))^{-1}$, $\lambda^2 \in \mathbb{R}$, and

$$J^{(x)}(x,\lambda) = \begin{pmatrix} 1 & \dfrac{b(\lambda)}{a(\lambda)}e^{-2i\lambda^2 x} \\ -\dfrac{\overline{b(\overline{\lambda})}}{\overline{a(\overline{\lambda})}}e^{2i\lambda^2 x} & 1 \end{pmatrix}, \qquad \lambda^2 \in \mathbb{R}. \tag{3.1}$$

- $M^{(x)}(x,\lambda) = I + \mathcal{O}\big(\frac{1}{\lambda}\big)$, $\lambda \to \infty$.
- $a(\lambda)$ has $2n$ simple zeros $\{\varepsilon_j\}_1^{2n}$, $2n = 2n_1 + 2n_2$, such that, ε_j $(j = 1, 2, \cdots, 2n_1)$ lie in $D_3 \cup D_4$, $\overline{\varepsilon}_j$ $(j = 2n_1 + 1, 2n_1 + 2, \cdots, 2n)$ lie in $D_1 \cup D_2$.
- The first column of $M_-^{(x)}(x,\lambda)$ has simple poles at $\lambda = \overline{\varepsilon}_j$ $(j = 1, 2, \cdots, 2n)$, the second column of $M_+^{(x)}(x,\lambda)$ has simple poles at $\lambda = \varepsilon_j$ $(j = 1, 2, \cdots, 2n)$.

  The associated residues are given by

$$\operatorname{Res}\{[M^{(x)}(x,\lambda)]_1, \overline{\varepsilon}_j\} = \frac{e^{2i\overline{\varepsilon_j}^2 x}\overline{b(\overline{\varepsilon}_j)}}{\overline{\dot a(\overline{\varepsilon}_j)}}\big[M^{(x)}(x,\overline{\varepsilon}_j)\big]_2, \qquad j = 1, 2, \cdots, 2n, \tag{3.2}$$

$$\operatorname{Res}\{[M^{(x)}(x,\lambda)]_2, \varepsilon_j\} = \frac{e^{-2i\varepsilon_j^2 x}b(\varepsilon_j)}{\dot a(\varepsilon_j)}\big[M^{(x)}(x,\varepsilon_j)\big]_1, \qquad j = 1, 2, \cdots, 2n. \tag{3.3}$$

Definition 3.1.4. (The spectral functions $A(\lambda)$ and $B(\lambda)$). *Let $g_0(t)$, $g_1(t)$ be smooth functions, we define the map*

$$\widetilde{S} : \{g_0(t), g_1(t)\} \to \{A(\lambda), B(\lambda)\}$$

by

$$\begin{pmatrix} B(\lambda) \\ A(\lambda) \end{pmatrix} = \mu_1^{(2)}(0,\lambda) = \begin{pmatrix} \mu_1^{12}(0,\lambda) \\ \mu_1^{22}(0,\lambda) \end{pmatrix} \operatorname{Im}\lambda^2 \le 0,$$

where $\mu_1(0,\lambda)$ is the unique solution of the Volterra linear integral equation

$$\mu_1(0,\lambda) = I - \int_t^T e^{2i\lambda^4(\tau - T)\widehat{\sigma}_3}(V_2\mu_1)(\tau,\lambda)d\tau$$

and $V_2(0,T;\lambda)$ is given by

$$V_2(0,t;\lambda) = \begin{pmatrix} -i\lambda^2|g_0|^2 - \frac{i}{4}|g_0|^4 - \frac{1}{2}(g_0\overline{g}_1 - g_1\overline{g}_0) & (2\lambda^3 g_0 + \lambda(\frac{1}{2}|g_0|^2 g_0 + ig_1))e^{-2i\int_0^t \Delta_2(0,\tau)d\tau} \\ (2\lambda^3\overline{g}_0 + \lambda(\frac{1}{2}|g_0|^2\overline{g}_0 - i\overline{g}_1))e^{2i\int_0^t \Delta_2(0,\tau)d\tau} & i\lambda^2|g_0|^2 + \frac{i}{4}|g_0|^4 + \frac{1}{2}(g_0\overline{g}_1 - g_1\overline{g}_0) \end{pmatrix},$$

where $\Delta_2(0,\tau) = \frac{1}{8}|g_0|^4 - \frac{i}{4}(g_0\overline{g}_1 - g_1\overline{g}_0)$.

Proposition 3.1.5. *The spectral functions $A(\lambda)$ and $B(\lambda)$ have the following properties*

(i) *$A(\lambda)$ and $B(\lambda)$ are analytic for $Im\lambda^4 > 0$ and continuous and bounded for $Im\,\lambda^4 \geq 0$;*

(ii) $A(\lambda) = 1 + \mathcal{O}(\frac{1}{\lambda}), B(\lambda) = \mathcal{O}(\frac{1}{\lambda})$ *as* $\lambda \to \infty,\ \ Im\,\lambda^4 \geq 0$*;*

(iii) $A(\lambda)\overline{A(\overline{\lambda})} - B(\lambda)\overline{B(\overline{\lambda})} = 1,\ \ \lambda^4 \in \mathbb{R}$*;*

(iv) $A(-\lambda) = A(\lambda), B(-\lambda) = -B(\lambda),\ \ \mathrm{Im}\,\lambda^4 \geq 0$*;*

(v) *The Map $\widetilde{Q} : \{A(\lambda), B(\lambda)\} \to \{g_0(t), g_1(t)\}$, the inverse map $\widetilde{S}$ of $\widetilde{Q}$ is defined by*

$$g_0(t) = 2im_{12}^{(1)}(t)e^{2i\int_0^t \Delta_2(\tau)d\tau}, \tag{3.4}$$

$$g_1(t) = (4m_{12}^{(3)}(t) + |g_0(t)|^2 m_{12}^{(1)}(t))e^{2i\int_0^t \Delta_2(\tau)d\tau} + ig_0(t)(2m_{22}^{(2)}(t) + |g_0(t)|^2), \tag{3.5}$$

where $\Delta_2(t) = 4|m_{12}^{(1)}|^4 + 8(\mathrm{Re}\,[m_{12}^{(1)}\overline{m}_{12}^{(3)}] - |m_{12}^{(1)}|^2\mathrm{Re}\,[m_{22}^{(2)}])$, and the functions $m^{(1)}(t)$, $m^{(2)}(t)$, $m^{(3)}(t)$ are determined by the asymptotic expansion $M^{(t)}(t,\lambda) = I + \frac{m^{(1)}(t,\lambda)}{\lambda} + \frac{m^{(2)}(t,\lambda)}{\lambda^2} + \frac{m^{(3)}(t,\lambda)}{\lambda^3} + \mathcal{O}(\frac{1}{\lambda^4})$ $(\lambda \to \infty)$, where $M^{(t)}(t,\lambda)$ is the unique solution of the following RHP (see Remark 3.1.6);

(vi) $\widetilde{S}^{-1} = \widetilde{Q}$.

Remark 3.1.6. Let

$$M_+^{(t)}(t,\lambda) = \Big(\frac{\mu_2^{D_1\cup D_3}(t,\lambda)}{A(\lambda)}, \mu_1^{D_1\cup D_3}(t,\lambda)\Big), \qquad \mathrm{Im}\,\lambda^4 \geq 0,$$

$$M_-^{(t)}(t,\lambda) = \Big(\mu_1^{D_2\cup D_4}(t,\lambda), \frac{\mu_2^{D_2\cup D_4}(t,\lambda)}{\overline{A(\overline{\lambda})}}\Big), \qquad \mathrm{Im}\,\lambda^4 \leq 0. \tag{3.6}$$

$M^{(t)}(t,\lambda)$ is the unique solution of the following RHP

- $M^{(t)}(t,\lambda) = \begin{cases} M_+^{(t)}(t,\lambda), & \mathrm{Im}\,\lambda^4 \geq 0 \\ M_+^{(t)}(t,\lambda), & \mathrm{Im}\,\lambda^4 \leq 0 \end{cases}$ is a sectionally meromorphic function.

- $M_+^{(t)}(t,\lambda) = M_-^{(t)}(t,\lambda)J^{(t)}(t,\lambda),\ \ \lambda^4 \in \mathbb{R}$, and

$$J^{(t)}(t,\lambda) = \begin{pmatrix} \frac{1}{A(\lambda)\overline{A(\overline{\lambda})}} & \frac{B(\lambda)}{\overline{A(\overline{\lambda})}}e^{-4i\lambda^4 t} \\ -\frac{\overline{B(\overline{\lambda})}}{A(\lambda)}e^{4i\lambda^4 t} & 1 \end{pmatrix} \lambda^4 \in \mathbb{R}. \tag{3.7}$$

- $M^{(t)}(T,\lambda) = I + \mathcal{O}(\frac{1}{\lambda})$ $(\lambda \to \infty)$.

- $A(\lambda)$ has $2k$ simple zeros $\{\zeta_j\}_1^{2k}$, $2k = 2k_1 + 2k_2$, such that, ζ_j $(j = 1, 2, \cdots, 2k_1)$ lie in $D_1 \cup D_3$, $\overline{\zeta}_j$ $(j = 2k_1+1, 2k_1+2, \cdots, 2k)$ lie in $D_2 \cup D_4$.

- The first column of $M_+^{(t)}(t,\lambda)$ has simple poles at $\lambda = \zeta_j$ $(j = 1, 2, \cdots, 2k)$, the second column of $M_-^{(t)}(t,\lambda)$ has simple poles at $\lambda = \overline{\zeta}_j$, $j = 1, 2, \cdots, 2k$.

The associated residues are given by

$$\operatorname{Res}\{[M^{(t)}(t,\lambda)]_1,\zeta_j\} = \frac{e^{4i\zeta_j^4 t}}{\dot{A}(\zeta_j)B(\zeta_j)}[M^{(t)}(t,\zeta_j)]_2, \qquad j=1,2,\cdots,2k, \tag{3.8}$$

$$\operatorname{Res}\{[M^{(t)}(t,\lambda)]_2,\overline{\zeta}_j\} = \frac{e^{-4i\zeta_j^4 t}}{\dot{A}(\overline{\zeta}_j)B(\overline{\zeta}_j)}[M^{(t)}(t,\overline{\zeta}_j)]_1, \qquad j=1,2,\cdots,2k. \tag{3.9}$$

Definition 3.1.7 (The spectral functions $\alpha(\lambda)$ and $\beta(\lambda)$). *Given the spectral functions*

$$\alpha(\lambda) = \overline{a(\overline{\lambda})}A(\lambda) - \overline{b(\overline{\lambda})}B(\lambda), \qquad \beta(\lambda) = a(\lambda)B(\lambda) - b(\lambda)A(\lambda)$$

and the smooth functions $h_T(x) = u(x,T)$. We define the map

$$\tilde{\tilde{S}} : \{h_T(x)\} \to \{\alpha(\lambda), \beta(\lambda)\}$$

by

$$\begin{pmatrix} \beta(\lambda) \\ \alpha(\lambda) \end{pmatrix} = \mu_1^{(2)}(0,\lambda) = \begin{pmatrix} \mu_1^{12}(0,\lambda) \\ \mu_1^{22}(0,\lambda) \end{pmatrix}, \qquad \operatorname{Im}\lambda^2 \geq 0,$$

where $\mu_1(x,T;\lambda)$ is the unique solution of the Volterra linear integral equation

$$\mu_1(x,T;\lambda) = I - \int_x^0 e^{i\lambda^2(\xi-x)\widehat{\sigma}_3}(V_1\mu_1)(\xi,T;\lambda)d\xi$$

and $V_2(x,T;\lambda)$ is given by

$$V_2(x,t;\lambda) = \begin{pmatrix} -\frac{i}{2}|h_T|^2 & \lambda h_T e^{-\int_x^0 \frac{i}{2}|h_T|^2 d\xi} \\ \lambda \overline{h}_T e^{\int_x^0 \frac{i}{2}|h_T|^2 d\xi} & \frac{i}{2}|h_T|^2 \end{pmatrix}.$$

Proposition 3.1.8. *The spectral functions $\alpha(\lambda)$ and $\beta(\lambda)$ have the following properties*

(i) *$\alpha(\lambda)$ and $\beta(\lambda)$ are analytic for $\operatorname{Im}\lambda^2 > 0$ and continuous and bounded for $\operatorname{Im}\lambda^2 \geq 0$;*

(ii) *$\alpha(\lambda) = 1 + \mathcal{O}(\frac{1}{\lambda}), \beta(\lambda) = \mathcal{O}(\frac{1}{\lambda})$ as $\lambda \to \infty$, $\operatorname{Im}\lambda^2 \geq 0$;*

(iii) *$\alpha(\lambda)\overline{\alpha(\overline{\lambda})} - \beta(\lambda)\overline{\beta(\overline{\lambda})} = 1$, $\lambda^2 \in \mathbb{R}$;*

(iv) *$\alpha(-\lambda) = \alpha(\lambda)$, $\beta(-\lambda) = -\beta(\lambda)$, $\operatorname{Im}\lambda^2 \geq 0$;*

(v) *The Map $\tilde{\tilde{Q}} : \{\alpha(\lambda), \beta(\lambda)\} \to \{h_T(x)\}$, the inverse Map $\tilde{\tilde{S}}$ of $\tilde{\tilde{Q}}$ is defined by*

$$h_T(x) = 2im_t(x)e^{4i\int_x^0 |m_T(x')|dx'}, \tag{3.10}$$

$$m_t(x) = \lim_{\lambda\to\infty}(\lambda M^{(T)}(x,\lambda))_{12}, \tag{3.11}$$

where $M^{(T)}(x,\lambda)$ is the unique solution of the following RHP;

(vi) $\tilde{\tilde{S}}^{-1} = \tilde{\tilde{Q}}$.

Remark 3.1.9. Let

$$M_+^{(T)}(x,\lambda)=\Big(\frac{\mu_3^{D_1\cup D_2}(x,\lambda)}{\alpha(\lambda)},\mu_1^{D_1\cup D_2}(x,\lambda)\Big),\qquad \operatorname{Im}\lambda^2\geq 0,$$
$$M_-^{(T)}(x,\lambda)=\Big(\mu_1^{D_3\cup D_4}(x,\lambda),\frac{\mu_3^{D_3\cup D_4}(x,\lambda)}{\overline{\alpha(\overline{\lambda})}}\Big),\qquad \operatorname{Im}\lambda^2\leq 0. \tag{3.12}$$

$M^{(T)}(x,\lambda)$ is the unique solution of the following RHP

- $M^{(T)}(t,\lambda)=\begin{cases} M_+^{(T)}(x,\lambda), & \operatorname{Im}\lambda^2\geq 0\\ M_+^{(T)}(x,\lambda), & \operatorname{Im}\lambda^2\leq 0\end{cases}$ is a sectionally meromorphic function.
- $M_+^{(T)}(x,\lambda)=M_-^{(T)}(x,\lambda)J^{(T)}(x,\lambda)$, $\lambda^2\in\mathbb{R}$, and

$$J^{(T)}(x,\lambda)=\begin{pmatrix}\frac{1}{\alpha(\lambda)\overline{\alpha(\overline{\lambda})}} & \frac{\overline{\beta(\lambda)}}{\overline{\alpha(\overline{\lambda})}}e^{-2i(\lambda^2x+2\lambda^4T)}\\ -\frac{\overline{\beta(\overline{\lambda})}}{\alpha(\lambda)}e^{2i(\lambda^2x+2\lambda^4T)} & 1\end{pmatrix},\qquad \lambda^2\in\mathbb{R}. \tag{3.13}$$

- $M^{(T)}(x,\lambda)=I+\mathcal{O}(\frac{1}{\lambda}),\ \lambda\to\infty.$
- $\alpha(\lambda)$ has $2N$ simple zeros $\{\gamma_j\}_1^{2N}$, $2N=2N_1+2N_2$, such that, $\gamma_j\ \ (j=1,2,\cdots,2N_1)$ lie in $D_1\cup D_2$), $\overline{\gamma}_j\ \ (j=2N_1+1,2N_1+2,\cdots,2N)$ lie in $D_3\cup D_4$.
- The first column of $M_+^{(T)}(x,\lambda)$ has simple poles at $\lambda=\gamma_j\ \ (j=1,2,\cdots,2N)$, the second column of $M_-^{(T)}(x,\lambda)$ has simple poles at $\lambda=\overline{\gamma}_j\ \ (j=1,2,\cdots,2N)$. The associated residues are given by

$$\operatorname{Res}\{[M^{(T)}(x,\lambda)]_1,\gamma_j\}=\frac{e^{2i(\gamma_j^2x+2\gamma_j^4t)}}{\dot{\alpha}(\gamma_j)\beta(\gamma_j)}[M^{(T)}(x,\gamma_j)]_2,\qquad j=1,2,\cdots,2N, \tag{3.14}$$
$$\operatorname{Res}\{[M^{(T)}(x,\lambda)]_2,\overline{\gamma}_j\}=\frac{e^{-2i(\gamma_j^2x+2\gamma_j^4t)}}{\overline{\dot{\alpha}(\overline{\gamma}_j)\beta(\overline{\gamma}_j)}}[M^{(T)}(x,\overline{\gamma}_j)]_1,\qquad j=1,2,\cdots,2N. \tag{3.15}$$

3.2 The Principal RHP

Theorem 3.2.1. *Let $u_0(x)\in\mathcal{S}(\mathbb{R}^-)$ a smooth function. Suppose that the function $g_0(t),g_1(t)$ are compatible with the function $u_0(t)$. Define the spectral function $a(\lambda)$, $b(\lambda)$, $A(\lambda)$ and $B(\lambda)$, in terms of $u_0(x)$, $g_0(t)$, and $g_1(t)$ of Definition 3.1.1 and Definition 3.1.4. Suppose that $a(\lambda)$, $b(\lambda)$, $A(\lambda)$ and $B(\lambda)$ satisfy the global relation*

$$a(\lambda)B(\lambda)-b(\lambda)A(\lambda)=e^{4i\lambda^4T}c^+(\lambda),\qquad \operatorname{Im}\lambda^2\geq 0,$$

where $s(\lambda)=\mu_3(0,0;\lambda),S(\lambda)=S(T,\lambda)=(e^{2i\lambda^4T}\mu_2(0,T;\lambda))^{-1}$, if $\lambda\to\infty$ the global relation is replaced by $a(\lambda)B(\lambda)-b(\lambda)A(\lambda)=0$. Assume that the possible zeros of $\{\varepsilon_j\}_{j=1}^{2n}$ are $a(\lambda)$ and $\{\gamma_j\}_{j=1}^{2N}$ of $\alpha(\lambda)$, then define the $M(x,t,\lambda)$ as the solution of the following RHP

- *$M(x,t;\lambda)$ is sectionally meromorphic in $\mathbb{C}\setminus\{\lambda^4\in\mathbb{R}\}$.*
- *The first column of $M(x,t;\lambda)$ has simple poles at $\lambda=\varepsilon_j$, $j=1,2,\cdots,2n$, and $\lambda=\gamma_j$, $j=1,2,\cdots,2N$. The second column of $M(x,t;\lambda)$ has simple poles at $\lambda=\overline{\varepsilon}_j$, $j=1,2,\cdots,2n$ and $\lambda=\overline{\gamma}_j$, $j=1,2,\cdots,2N$.*
- *$M(x,t;\lambda)$ satisfies the jump condition*

$$M_+(x,t;\lambda)=M_-(x,t;\lambda)J(x,t;\lambda),\qquad \lambda^4\in\mathbb{R}. \tag{3.16}$$

- $M(x,t;\lambda)=I+\mathcal{O}\big(\frac{1}{\lambda}\big),\ \lambda\to\infty.$
- *$M(x,t;\lambda)$ satisfies the residue conditions of Proposition 2.4.3.*

Then $M(x,t;\lambda)$ exists and is unique, we define $u(x,t)$ in terms of $M(x,t;\lambda)$ by

$$\begin{aligned}&u(x,t)=2im(x,t)e^{2i\int_{(0,0)}^{(x,t)}\Delta},\\&m(x,t)=\lim_{\lambda\to\infty}(\lambda M(x,t;\lambda))_{12},\\&\Delta=|m|^2dx-(6|m|^4+i(\overline{m}_xm-m_x\overline{m}))dt.\end{aligned}\tag{3.17}$$

Furthermore $u(x,t)$ is the solution of the C-L-L Equation (1.1), and $u(x,0)=f_0(x)$, $u(0,t)=g_0(t)$, $q_x(0,t)=g_1(t)$.

Proof. In fact, if we assume that $a(\lambda)$ and $\alpha(\lambda)$ have no zeroes, then the (2×2) function $M(x,t;\lambda)$ satisfies a non-sigular RHP. Using the fact that the jump matrix $J(x,t;\lambda)$ matches with the symmetry conditions, we can show that this problem has a unique global solution[1]. The case that $a(\lambda)$ and $\alpha(\lambda)$ have a finite number of zeros can be mapped to the case of no zeros supplemented by an algebraic system of equations which is always uniquely solvable. □

Theorem 3.2.2. *The RHP in Theorem 3.2.1 with the vanishing boundary condition $M(x,t;\lambda)\to0(\lambda\to\infty)$, has only the zero solution.*

Proof. Assume that $M(x,t;\lambda)$ is a solution of the RHP in Theorem 3.2.1 such that $M_\pm(x,t;\lambda)\to\infty(\lambda\to\infty)$. A is a (2×2) matrix, $A^\dagger$ denotes the complex conjugate transpose of A.

Define

$$\begin{aligned}&H_+(\lambda)=M_+(\lambda)M_-^\dagger(-\overline{\lambda}),\qquad \operatorname{Im}\lambda^4\ge0,\\&H_-(\lambda)=M_-(\lambda)M_+^\dagger(-\overline{\lambda}),\qquad \operatorname{Im}\lambda^4\le0,\end{aligned}\tag{3.18}$$

where the x and t are dependence. $H_+(\lambda)$ and $H_+(\lambda)$ are analytic in $\{\lambda\in\mathbb{C}\setminus\operatorname{Im}\lambda^4>0\}$ and $\{\lambda\in\mathbb{C}\setminus\operatorname{Im}\lambda^4<0\}$ respectively. By the symmetry relations $a(-\lambda)=a(\lambda),b(-\lambda)=-b(\lambda)$ and $A(-\lambda)=A(\lambda),B(-\lambda)=-B(\lambda)$, we infer that

$$J_1^\dagger(-\overline{\lambda})=J_1(\lambda),\qquad J_3^\dagger(-\overline{\lambda})=J_3(\lambda),\qquad J_2^\dagger(-\overline{\lambda})=J_4(\lambda). \tag{3.19}$$

Then

$$\begin{aligned}&H_+(\lambda)=M_-(\lambda)J(\lambda)M_-^\dagger(-\overline{\lambda}),\qquad \operatorname{Im}\lambda^4\in\mathbb{R},\\&H_-(\lambda)=M_-(\lambda)J^\dagger(-\overline{\lambda})M_-^\dagger(-\overline{\lambda}),\qquad \operatorname{Im}\lambda^4\in\mathbb{R}.\end{aligned}\tag{3.20}$$

(3.19) and (3.20) mean that $H_+(\lambda)=H_-(\lambda)$ for $\operatorname{Im}\lambda^4\in\mathbb{R}$. Therefore, $H_+(\lambda)$ and $H_-(\lambda)$ define an entire function vanishing at infinity, so $H_+(\lambda)$ and $H_-(\lambda)$ are identically zero. Noting $J_3(i\kappa)(\kappa\in\mathbb{R})$ is a Hermitian matrix with unit determinant and $(2,2)$ entry 1 for any $\kappa\in\mathbb{R}$. Therefore, $J_3(i\kappa)(\kappa\in\mathbb{R})$ is a positive definite matrix. Since $H_-(\kappa)$ vanishes identically for $\kappa\in i\mathbb{R}$, i.e.,

$$M_+(i\kappa)J_3(i\kappa)M_+^\dagger(i\kappa)=0,\qquad \kappa\in\mathbb{R}. \tag{3.21}$$

We can deduce that $M_+(i\kappa) = 0$ as $\kappa \in \mathbb{R}$. It follows that $M_+(\lambda)$ and $M_-(\lambda)$ vanish identically. □

Proposition 3.2.3. *$u(x,t)$ satisfies the C-L-L equation.*

Proof. Using arguments of the dressing method[26], it can be verified directly that if $M(x,t;\lambda)$ is defined as the unique solution of the above RHP, and if $u(x,t)$ is defined in terms of $M(x,t;\lambda)$ by (3.17), then $u(x,t)$ and $M(x,t;\lambda)$ satisfy two parts of the Lax pair, hence $u(x,t)$ is solvable on C-L-L equation. □

Proposition 3.2.4. *$u(x,0) = u_0(x)$.*

Proof. Noting the (2.27) at $t = 0$ we can divide the jump matrix into product of (2×2) matrix

$$\begin{aligned}
J_1(x,0;\lambda) &= \begin{pmatrix} \dfrac{1}{\alpha(\lambda)\overline{\alpha(\overline{\lambda})}} & \dfrac{\beta(\lambda)}{\overline{\alpha(\overline{\lambda})}}e^{-2i\lambda^2 x} \\ -\dfrac{\overline{\beta(\overline{\lambda})}}{\alpha(\lambda)}e^{2i\lambda^2 x} & 1 \end{pmatrix}, \\
J_2(x,0;\lambda) &= \begin{pmatrix} \dfrac{\overline{a(\overline{\lambda})}}{\alpha(\lambda)} & \delta(\lambda)e^{-2i\lambda^2 x} \\ 0 & \dfrac{\alpha(\lambda)}{\overline{a(\overline{\lambda})}} \end{pmatrix}, \\
J_3(x,0;\lambda) &= \begin{pmatrix} 1 & \dfrac{b(\lambda)}{a(\lambda)}e^{-2i\lambda^2 x} \\ -\dfrac{\overline{b(\overline{\lambda})}}{\overline{a(\overline{\lambda})}}e^{\lambda^2 x} & \dfrac{1}{a(\lambda)\overline{a(\overline{\lambda})}} \end{pmatrix}, \\
J_4(x,0;\lambda) &= \begin{pmatrix} \dfrac{a(\lambda)}{\overline{\alpha(\overline{\lambda})}} & 0 \\ -\overline{\delta(\overline{\lambda})}e^{2i\lambda^2 x} & \dfrac{\overline{\alpha(\overline{\lambda})}}{a(\lambda)} \end{pmatrix}.
\end{aligned} \tag{3.22}$$

Define

$$\begin{aligned}
M^{(x)}(x,\lambda) &= M(x,0;\lambda), \qquad \lambda \in D_1 \cup D_4, \\
M^{(x)}(x,\lambda) &= M(x,0;\lambda)(J_2(x,0;\lambda))^{-1}, \qquad \lambda \in D_2, \\
M^{(x)}(x,\lambda) &= M(x,0;\lambda)J_4(x,0;\lambda), \qquad \lambda \in D_3,
\end{aligned} \tag{3.23}$$

then we set

$$\begin{aligned}
M_+^{(x)}(x,\lambda) &= \Big(\frac{\mu_3^{D_1\cup D_2}(x,\lambda)}{\alpha(\lambda)}, \mu_1^{D_1\cup D_2}(x,\lambda)\Big), \qquad \lambda \in D_1 \cup D_2, \\
M_-^{(x)}(x,\lambda) &= \Big(\mu_1^{D_3\cup D_4}(x,\lambda), \frac{\mu_3^{D_3\cup D_4}(x,\lambda)}{\overline{\alpha(\overline{\lambda})}}), \qquad \lambda \in D_3 \cup D_4, \\
M_-^{(x)}(x,\lambda) &= \Big(\mu_2^{D_3\cup D_4}(x,\lambda), \frac{\mu_3^{D_3\cup D_4}(x,\lambda)}{a(\lambda)}\Big), \qquad \lambda \in D_3 \cup D_4, \\
M_+^{(x)}(x,\lambda) &= \Big(\frac{\mu_3^{D_1\cup D_2}(x,\lambda)}{\overline{a(\overline{\lambda})}}, \mu_2^{D_1\cup D_2}(x,\lambda)\Big), \qquad \lambda \in D_1 \cup D_2,
\end{aligned} \tag{3.24}$$

where $M^{(x)}(x,\lambda)$ satisfies

$$\begin{aligned} M_+^{(x)}(x,\lambda) &= M_-^{(x)}(x,\lambda)J_1^{(x)}(x,\lambda), \qquad \lambda\in\mathbb{R}; \\ M_+^{(x)}(x,\lambda) &= M_-^{(x)}(x,\lambda)(J_3^{(x)}(x,\lambda))^{-1}, \qquad \lambda^2\in\mathbb{R}. \end{aligned} \tag{3.25}$$

$M^{(x)}(x,\lambda)=I+\mathcal{O}\big(\frac{1}{\lambda}\big)$ $(\lambda\to\infty)$. According to Proposition 3.1.2,

$$u_0(x)=2im(x)e^{4i\int_x^0 |m(x')|dx'}, \qquad m(x)=\lim_{\lambda\to\infty}(\lambda M^{(x)}(x,\lambda))_{12},$$

comparing this with (3.17) evaluated at $t=0$, we conclude that $u_0=u(x,0)$. □

Proposition 3.2.5. *The sets* $\{\varepsilon_j\}_{j=1}^{2n}$ *and* $\{\gamma_j\}_{j=1}^{2N}$ *are not empty.*

Proof. The first column of $M(x,t;\lambda)$ has poles at $\{\overline{\varepsilon}_j\}_{2n_1+1}^{2n}$ for $\lambda\in D_2$ and has poles $\{\gamma_j\}_1^{2N_1}$ for $\lambda\in D_1$. On the other hand, the first column of $M^{(x)}(x,\lambda)$ should have poles at $\{\gamma_j\}_{j=1}^{2N}$ or have poles at $\{\varepsilon_j\}_{j=1}^{2n}$. We will now show that the transformation defined by (3.23) maps the former poles to the latter ones.

Setting $M(x,0;\lambda)=(M^{(1)}(x,0;\lambda),M^{(2)}(x,0;\lambda))$, (3.23) can be written as

$$M^{(x)}(x,\lambda)=\Big(\frac{a(\lambda)}{\overline{\alpha(\overline{\lambda})}}M^{(1)}-\overline{\delta(\overline{\lambda})}e^{-2i\lambda^2x}M^{(2)},\frac{\overline{\alpha(\overline{\lambda})}}{a(\lambda)}M^{(2)}\Big), \qquad \lambda\in D_3. \tag{3.26}$$

The residue condition of Proposition 2.4.3, (iii) at $\overline{\varepsilon}_j$ implies that $M^{(x)}(x,\lambda)$ has no poles at $\overline{\varepsilon}_j$ on the other hand, (3.26) shows that $M^{(x)}(x,\lambda)$ has poles at $\{\overline{\varepsilon}_j\}_{2N_1+1}^{2N}$ with residues given by

$$\begin{aligned} &\operatorname{Res}\{[M^{(x)}(x,\lambda)]_1,\overline{\gamma_j}\}=-\operatorname{Res}\{\overline{\delta(\overline{\lambda})},\overline{\gamma_j}\}e^{-2i\overline{\gamma}_j^2x}M^{(x)}(x,\overline{\gamma}_j), \\ &\qquad j=2N_1+1,2N_1+2,\cdots,2N. \end{aligned} \tag{3.27}$$

Similar considerations apply to ε_j and γ_j. □

Proposition 3.2.6. $u(0,t)=g_0(t)$, $u_x(0,t)=g_1(t)$.

Proof. Define

$$M^{(t)}(t,\lambda)=M(0,t;\lambda)G(t,\lambda), \tag{3.28}$$

where $G(t,\lambda)$ is given by $G^{(j)}(t,\lambda)$ for $\lambda\in D_j$, $j=1,2,3,4$. Noting that $M(0,t;\lambda)$ satisfies Theorem 2.4.1 on the respective parts of the boundary separating the $D_j's$, then $M^{(t)}(t,\lambda)$ satisfies the RHP defined in Remark 3.1.6. Suppose we can find matrices $G^{(1)}$ and $G^{(2)}$ holomorphic for Im $\lambda^2>0$ (and continuous for Im$\lambda^2>0$), matrices $G^{(3)}$ and $G^{(4)}$ holomorphic for Im$\lambda^2<0$ (continuous for Im$\lambda^2<0$), which tend to I as $\lambda\to\infty$, and which satisfy

$$\begin{aligned} J_2(0,t;\lambda)G^{(1)}(t,\lambda)&=G^{(2)}(t,\lambda)J^{(t)}(t,\lambda), \\ J_1(0,t;\lambda)G^{(1)}(t,\lambda)&=G^{(4)}(t,\lambda)J^{(t)}(t,\lambda), \\ J_3(0,t;\lambda)G^{(3)}(t,\lambda)&=G^{(2)}(t,\lambda)J^{(t)}(t,\lambda), \\ J_4(0,t;\lambda)G^{(3)}(t,\lambda)&=G^{(4)}(t,\lambda)J^{(t)}(t,\lambda), \end{aligned} \tag{3.29}$$

where $J^{(t)}(t,\lambda)$ is the jump matrix defined in (3.7).

We can obtain that such $G^{(j)}(t,\lambda)$ $(j=1,2,3,4.)$ matrices are

$$\begin{aligned}
G^{(1)}(t,\lambda) &= \begin{pmatrix} \dfrac{\alpha(\lambda)}{A(\lambda)} & c^{+}(\lambda)e^{4i\lambda^4(T-t)} \\ 0 & \dfrac{A(\lambda)}{\alpha(\lambda)} \end{pmatrix}, \\
G^{(2)}(t,\lambda) &= \begin{pmatrix} \delta(\lambda) & -\dfrac{b(\lambda)}{\overline{A(\bar{\lambda})}}e^{-4i\lambda^4 t} \\ 0 & \dfrac{1}{\delta(\lambda)} \end{pmatrix}, \\
G^{(3)}(t,\lambda) &= \begin{pmatrix} \dfrac{1}{\overline{\delta(\bar{\lambda})}} & 0 \\ -\dfrac{\overline{b(\bar{\lambda})}}{A(\lambda)}e^{4i\lambda^4 t} & \overline{\delta(\bar{\lambda})} \end{pmatrix}, \\
G^{(4)}(t,\lambda) &= \begin{pmatrix} \dfrac{\overline{A(\bar{\lambda})}}{\overline{\alpha(\bar{\lambda})}} & 0 \\ \overline{c^{+}(\bar{\lambda})}e^{-4i\lambda^4(T-t)} & \dfrac{\overline{\alpha(\bar{\lambda})}}{\overline{A(\bar{\lambda})}} \end{pmatrix}.
\end{aligned} \tag{3.30}$$

By using directly calculation, we can verify these $G^{(j)}(t,\lambda)$ $(j=1,2,3,4.)$ matrices satisfy the conditions (3.29). As for the proof of the equation $q(x,0)=q_0(x)$, it can be verified that the transformation (3.28) replaces the residue conditions of Proposition 2.4.3 by the residue conditions of Remark 3.1.6.

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