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A Note about Multi-Hilbert Transform on $\mathfrak{D}(\mathbb{R}^n)$

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Abstract In this paper, we first prove that, for a non-zero function $f\in\mathfrak{D}(\mathbb{R}^n)$, its multi-Hilbert transform H_nf is bounded and does not have compact support. In addition, a new distribution space $\mathfrak{D}'_H(\mathbb{R}^n)$ is constructed and the definition of the multi-Hilbert transform is extended to it. It is shown that $\mathfrak{D}'_H(\mathbb{R}^n)$ is the biggest subspace of $\mathfrak{D}'(\mathbb{R}^n)$ on which the extended multi-Hilbert transform is a homeomorphism.

Keywords Multi-Hilbert transform; $\mathfrak{D}(\mathbb{R}^n)$; $\mathfrak{D}'_H(\mathbb{R}^n)$
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1 Introduction

For $f\in L^p(\mathbb{R}),\ 1\le p<\infty$, the classical Hilbert transform is defined as

$$Hf(x):=\frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}}\frac{f(t)}{x-t}dt, \tag{1.1}$$

where, p.v. is the *Cauchy principal value* (see [5]), that is

$$\text{p.v.}\int_{\mathbb{R}}\frac{f(t)}{x-t}dt=\lim_{\varepsilon\to0^+}\int_{|x-t|>\varepsilon}\frac{f(t)}{x-t}dt.$$

In [13], Lihua Yang proved that $\mathfrak{D}(\mathbb{R})\cap H(\mathfrak{D}(\mathbb{R}))=\{0\}$, where,

$$\mathfrak{D}(\mathbb{R}^n)=\Big\{\phi:\phi\in C_c^\infty(\mathbb{R}^n),\ \forall\,\alpha\in\mathbb{N}_0^n,\ \rho_\alpha(\phi)=\sup_{x\in\mathbb{R}^n}|D^\alpha\phi(x)|<\infty\Big\}$$

for $n\in\mathbb{N}$. That is to say the function Hf does not have compact support for all non-zero function $f\in\mathfrak{D}(\mathbb{R})$.

Consider n-dimensional Euclidean space $\mathbb{R}^n$ $(n\in\mathbb{N})$. For $f\in L^p(\mathbb{R}^n),\ 1\le p<\infty$, $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n,\ t=(t_1,t_2,\cdots,t_n)\in\mathbb{R}^n$, the n-dimensional Hilbert transform H_n is defined as

$$H_nf(x):=\frac{1}{\pi^n}\text{p.v.}\int_{\mathbb{R}^n}\frac{f(t)}{(x_1-t_1)(x_2-t_2)\cdots(x_n-t_n)}dt. \tag{1.2}$$

For $n=2$, Xiaona Cui[3] obtained the following result:

$$\mathfrak{D}(\mathbb{R}^2)\cap H_2(\mathfrak{D}(\mathbb{R}^2))=\{0\}.$$

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It is natural for us to raise the following question: When $n > 2$, whether there holds

$$\mathfrak{D}(\mathbb{R}^n) \cap H_n(\mathfrak{D}(\mathbb{R}^n)) = \{0\}. \tag{1.3}$$

In Section 3, we will give an affirmative answer for the above question.

Up till now, many achievements have been made to extend the classical Hilbert transform to some generalized function spaces[1−9]. Most of them on this topic[1−6] is to extend the Hilbert transform to a preexistent distribution space by using the analytic representation of distributions. The notable one among them is [6] by Orton, in which, Hilbert transform is extended to $\mathfrak{D}'(\mathbb{R})$. Her extension depends on the analytic representation, which is unique up to an entire analytic function, namely, the Hilbert transform of $f \in \mathfrak{D}'(\mathbb{R})$ is essentially an equivalent class. In [7], Hilbert transform is extended to $\mathfrak{D}'(\mathbb{R})$ directly with conjugate operator by introducing the topology on $H(\mathfrak{D}(\mathbb{R}))$. With the extension, for any $f \in \mathfrak{D}'(\mathbb{R})$, its Hilbert transform Hf is in $H'(\mathfrak{D}(\mathbb{R}))$, which is called a space of ultradistributions[7]. It can be verified that $H'(\mathfrak{D}(\mathbb{R}))$ is not a subspace of $\mathfrak{D}'(\mathbb{R})$ since $H\phi \notin \mathfrak{D}(\mathbb{R})$ for $\phi \in \mathfrak{D}(\mathbb{R})$ unless $\phi = 0$. Let us recall that, the similar case occurs for Fourier transform since the Fourier transform $\hat{\phi}$ of $\phi \in \mathfrak{D}(\mathbb{R})$ is not in $\mathfrak{D}(\mathbb{R})$ unless $\phi = 0$. To extend Fourier transform to distributions, the Schwartz space $\mathfrak{S}(\mathbb{R})$ of rapidly decreasing functions is considered. It is well known that $\mathfrak{D}(\mathbb{R}) \hookrightarrow \mathfrak{S}(\mathbb{R})$ (see Section 2 or [10] for the exact meaning of embedding $'\hookrightarrow'$) and the Fourier transform is a homeomorphism on $\mathfrak{S}(\mathbb{R})$. Therefore, the dual space of $\mathfrak{S}(\mathbb{R})$ satisfies $\mathfrak{S}'(\mathbb{R}) \hookrightarrow \mathfrak{D}'(\mathbb{R})$ and the Fourier transform is extended to $\mathfrak{S}'(\mathbb{R})$ successfully. Following this idea, this paper will establish a new space of distributions and extend the n-dimensional Hilbert transform to it such that n-dimensional Hilbert transform is a homeomorphism.

In our paper, a distribution space $\mathfrak{D}_H(\mathbb{R}^n)$ is constructed and some characterizations are given in Section 3. It is also shown in this section that $\mathfrak{D}_H(\mathbb{R}^n)$ is the smallest space with our desired properties and correspondingly its dual space $\mathfrak{D}'_H(\mathbb{R}^n)$ is the biggest distribution space such that $\mathfrak{D}'_H(\mathbb{R}^n) \hookrightarrow \mathfrak{D}'(\mathbb{R}^n)$ and the extended multi-Hilbert transform is a homeomorphism on $\mathfrak{D}'_H(\mathbb{R}^n)$. In Section 4, the classical Hilbert transform is extended to $\mathfrak{D}'_H(\mathbb{R}^n)$. The case $n = 1$ was proved by Lihua Yang in [13] and Xiaona Cui[3] got the similar results for $n = 2$. So we will mainly consider the case $n \geq 3$.

2 Preliminaies

Before we formulate the main results, some useful lemmas and the definitions will be given first.

For clarification, let us denote some commonly used notations as follows: For a Lebesgue measurable set $E \in \mathbb{R}^n$, let $L^p(E)$ be the space of p-power Lebesgue integrable functions with the well-known $L^p(E)$ norm for $1 \leq p \leq \infty$, $L_{\text{loc}}(\mathbb{R}^n)$ be the space of all the locally integrable functions on $\mathbb{R}^n$, $C^k(\mathbb{R}^n)$ $(k \in \mathbb{N})$ be that of all the k-times differentiable functions on $\mathbb{R}^n$, $C(\mathbb{R}^n) := C^0(\mathbb{R}^n)$, $C^\infty(\mathbb{R}^n) := \bigcap\limits_{k\in\mathbb{N}} C^k(\mathbb{R}^n)$.

Definition 2.1 (Space $X(\mathbb{R}^n)$). *The space* $X(\mathbb{R}^n)$ *is defined by*

$$X(\mathbb{R}^n) = \Big\{\varphi : \varphi(x) = \sum_{|\mu|=0}^{k} \varphi_{\mu_1}(x_1)\cdots\varphi_{\mu_n}(x_n),\ \varphi_{\mu_j}(x_j) \in \mathfrak{D}(\mathbb{R}),\ \forall\, k \in \mathbb{N}\Big\}, \tag{2.1}$$

where $\mu = (\mu_1, \cdots, \mu_n) \in \mathbb{N}^n$ *and* $x \in \mathbb{R}^n$ *and the topology of* $X(\mathbb{R}^n)$ *is induced by the topology of the space* $\mathfrak{D}(\mathbb{R})$.

Definition 2.2 (Space $\mathfrak{D}_{L^p}(\mathbb{R}^n)$). *For* $1 < p < \infty$, *the space* $\mathfrak{D}_{L^p}(\mathbb{R}^n)$ *is defined by*

$$\mathfrak{D}_{L^p}(\mathbb{R}^n) = \Big\{\varphi \in C^\infty(\mathbb{R}^n) : \forall\, \alpha \in \mathbb{N}^n,\ \gamma_\alpha(\varphi) = \Big(\int_{\mathbb{R}^n} |D^\alpha\varphi(x)|^p\, dx\Big)^{\frac{1}{p}} < \infty\Big\}. \tag{2.2}$$

From the study of Hilbert transform on $\mathfrak{D}(\mathbb{R})$, J.N. Pandey[4] extended the test space of n-dimensional Hilbert transform to $\mathfrak{D}_{L^p}(\mathbb{R}^n)$ by the space $X(\mathbb{R}^n)$.

Lemma 2.1. *The test space $X(\mathbb{R}^n)$ is dense in $\mathfrak{D}(\mathbb{R}^n)$.*

Lemma 2.2. *The space $\mathfrak{D}(\mathbb{R}^n)$ is dense in $\mathfrak{D}_{L^p}(\mathbb{R}^n)$.*

From Lemma 2.1 and Lemma 2.2, we immediately derive:

Lemma 2.3. *The test space $X(\mathbb{R}^n)$ is dense in $\mathfrak{D}_{L^p}(\mathbb{R}^n)$.*

Lemma 2.4. *The n-dimensional Hilbert transform H_n is a linear continuous map from $X(\mathbb{R}^n)$ to $\mathfrak{D}_{L^p}(\mathbb{R}^n)$ and we have $H_n^2 = (-1)^n I$.*

Lemma 2.5. *For all $1 < p < \infty$, the n-dimensional Hilbert transform H_n is a linear homeomorphism map from $\mathfrak{D}_{L^p}(\mathbb{R}^n)$ to itself and there holds $H_n^2 = (-1)^n I$.*

For the sake of convenience, we adopt the multi-index notation. A multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a n-tuple of nonnegative integers, i.e. $\alpha_i \in \mathbb{N},\ i = 1, 2, \cdots, n$. $\mathbb{N}_0^n$ denotes the set of multi-indices. For any multi-index $\alpha \in \mathbb{N}_0^n$ and any $t \in \mathbb{R}^n$, we set $t^\alpha = {t_1}^{\alpha_1} {t_2}^{\alpha_2} \cdots {t_n}^{\alpha_n}$. The length of α is defined as $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Other related notations are as follows:

$$\begin{aligned}
&\mathbf{1} = (1, \cdots, 1), \qquad \alpha + \mathbf{1} = (\alpha_1 + 1, \cdots, \alpha_n + 1), \qquad \alpha^{\mathbf{1}} = \alpha_1 \cdots \alpha_n,\\
&\alpha \geq \beta \ \text{ i.e. } \ \alpha_i \geq \beta_i, \qquad \text{for } \ i = 1, 2, \cdots, n,\\
&\alpha > \beta \ \text{ i.e. } \ \alpha_i > \beta_i, \qquad \text{for } \ i = 1, 2, \cdots, n,\\
&D^\alpha = \Big(\frac{\partial}{\partial \xi}\Big)^\alpha = \frac{\partial^{|\alpha|}}{(\partial \xi_1)^{\alpha_1} (\partial \xi_2)^{\alpha_2} \cdots (\partial \xi_n)^{\alpha_n}}.
\end{aligned}$$

We sort the elements of $\mathbb{N}_0^n$ according to Graded lexicographic order[12]. “Graded” refers to the fact that the total degree $|\alpha|$ is the main criterion. Graded lexicographic ordering means that the multi-indices are arranged as

$$\begin{aligned}
&(0, 0, \cdots, 0), (0, 0, \cdots, 0, 1), (0, 0, \cdots, 1, 0), \cdots, (1, 0, \cdots, 0),\\
&(0, 0, \cdots, 2), (0, 0, \cdots, 1, 1), (0, 0, \cdots, 2, 0), \cdots, (2, 0, \cdots, 0), \cdots
\end{aligned}$$

Definition 2.3[12] (Graded lexicographic order). *Graded lexicographic order $<_{\mathrm{tdeg}}$: for any $\alpha = (\alpha_1, \cdots, \alpha_n),\ \beta = (\beta_1, \cdots, \beta_n) \in \mathbb{N}_0^n$,*

(a) Suppose $|\alpha| < |\beta|$. Then $\alpha <_{\mathrm{tdeg}} \beta$;

(b) Suppose $|\alpha| = |\beta|$ and there exists a integer i $(1 \leq i \leq n)$ such that $\alpha_i < \beta_i$ and $\alpha_j = \beta_j$ $(1 \leq j < i)$. Then $\alpha <_{\mathrm{tdeg}} \beta$.

At the same time, we endow every $\alpha \in \mathbb{N}_0^n$ an order number according to the “Graded lexicographic order”.

Definition 2.4. *For the multi-index $\alpha \in \mathbb{N}_0^n$, the order number $N(\alpha)$ is a nonnegative integer satisfy:*

(1) For $\mathbf{0} = (0, \cdots, 0)$, $N(\mathbf{0}) = 0$;

(2) Suppose $\alpha, \beta \in \mathbb{N}_0^n$ and $\alpha <_{tdeg} \beta$. If there are no $\gamma \in \mathbb{N}_0^n$ such that $\alpha <_{\mathrm{tdeg}} \gamma <_{\mathrm{tdeg}} \beta$, then $N(\beta) = N(\alpha) + 1$.

Note 1. It is easy to check $\{N(\alpha) : \alpha \in \mathbb{N}_0^n\}$ is an one-to-one correspondence with $\mathbb{N}$, and furthermore, the expression of $N(\alpha)$ is appropriately as follow:

$$N(\alpha) = \sum_{j=0}^{|\alpha|-1} \binom{n + j - 1}{n - 1} + \sum_{j=1}^{n} \sum_{k=0}^{\alpha_j - 1} \binom{n - j + \sum_{i=j}^{n} \alpha_i - k - 1}{n - j - 1},$$

where, for $a \geq b \in \mathbb{N}$, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ is the binomial coefficient.

Note 2. Let $n = 3$, we sort $\mathbb{N}_0^3$ as follow:

$N(\alpha)$	α				
.	.	24	$(0,4,0)$	32	$(3,0,1)$
.	.	25	$(1,0,3)$	33	$(3,1,0)$
.	.	26	$(1,1,2)$	34	$(4,0,0)$
.	.	27	$(1,2,1)$	35	$(0,0,5)$
20	$(0,0,4)$	28	$(1,3,0)$	36	$(0,1,4)$
21	$(0,1,3)$	29	$(2,0,2)$	.	.
22	$(0,2,2)$	30	$(2,1,1)$	.	.
23	$(0,3,1)$	31	$(2,2,0)$	.	.

Definition 2.5 (Embedding). *Assume $\mathfrak{X}$ and $\mathfrak{Y}$ are two topological vector spaces and $\mathfrak{X} \subset \mathfrak{Y}$. We say $\mathfrak{X}$ can be embedded into $\mathfrak{Y}$ consistently if we can deduce $x_i \to 0$ in $\mathfrak{Y}$ from $x_i \to 0$ in $\mathfrak{X}$ for all sequence $\{x_i\}_{i=1}^{\infty} \subset \mathfrak{X}$.*

In order to prove our results, we need the following two lemmas:

Lemma 2.6 (Decomposition Theorem). *Let $k, \gamma \in \mathbb{N}_0^n$, $\gamma = (|k|+1, |k|+1, \cdots, |k|+1)$. Denoting $\Lambda = \{\beta : \beta \in \mathbb{N}_0^n, |\beta| = |k|+1, \ \exists\, i \in \{1,2,\cdots,n\}, \ s.t.\, \beta_i = |k|+1\}$, we have*

$$\sum_{N(\alpha)>N(k)} S_\alpha \leq \sum_{\beta\in\Lambda}\sum_{\alpha\geq\beta} S_\alpha + \sum_{\alpha<\gamma, N(\alpha)>N(k)} S_\alpha,$$

where $S_\alpha \geq 0, \ \alpha \in \mathbb{N}_0^n$.

Proof. It is natural that

$$\sum_{N(\alpha)>N(k)} S_\alpha = \sum_{\alpha<\gamma, N(\alpha)>N(k)} S_\alpha + \sum_{\alpha\geq\gamma, N(\alpha)>N(k)} S_\alpha. \tag{2.3}$$

When $\alpha \geq \gamma$, there exists an integer i such satisfies $\alpha_i \geq |k|+1$, which means $\alpha \geq \beta$, where $\beta = (0, \cdots, |k|+1, \cdots, 0) \in \Lambda$, that is to say

$$\sum_{\alpha\geq\gamma, N(\alpha)>N(k)} S_\alpha \leq \sum_{\beta\in\Lambda}\sum_{\alpha\geq\beta} S_\alpha. \tag{2.4}$$

Then (2.3) and (2.4) implies

$$\sum_{N(\alpha)>N(k)} S_\alpha \leq \sum_{\beta\in\Lambda}\sum_{\alpha\geq\beta} S_\alpha + \sum_{\alpha<\gamma, N(\alpha)>N(k)} S_\alpha.$$

□

Lemma 2.7 (Weight Theorem). *Suppose $\delta, \gamma \in \mathbb{N}_0^n$ and $N(\delta)+1 < N(\gamma)$, then for all $\beta \in \mathbb{N}_0^n$ which satisfy $N(\delta) \leq N(\beta) < N(\alpha) \leq N(\gamma)$, there exists a vector $Q \in \mathbb{R}_+^n$ which satisfies $Q \cdot \beta < Q \cdot \alpha$.*

Proof. We only need to find Q satisfy $Q \cdot \beta < Q \cdot \alpha$ for all pairs (β, α) such that $N(\delta) \leq N(\beta) < N(\gamma)$ and $N(\alpha) = N(\beta)+1$.

For convenience, let $Q_n = 1$. Denote $\{Q_{i-1} - Q_i\}$ with $\Delta = \{\Delta_1, \cdots, \Delta_{n-1}\}$ as follows:

$$\begin{cases} Q_{n-1} = 1 + \Delta_1, \\ Q_{n-2} = 1 + \Delta_1 + \Delta_2, \\ \quad\vdots \\ Q_1 = 1 + \Delta_1 + \cdots + \Delta_{n-1}. \end{cases} \tag{2.5}$$

Firstly we consider all the elements $\beta \in \mathbb{N}_0^n$ such that $|\beta| = k$, where k is a positive integer between $|\delta|$ and $|\gamma|$, i.e. $|\delta| \le k \le |\gamma|$. We divide these β into n parts according to the first β_i that is not equal to 0:

$$\begin{cases} I_n = \{(0, \cdots, 0, k)\}, \\ I_{n-1} = \{(0, \cdots, 0, 1, k-1), \cdots, (0, \cdots, 0, k, 0)\}, \\ \quad \vdots \\ I_1 = \{(1, 0, \cdots, 0, k-1), \cdots, (k, 0, \cdots, 0)\}. \end{cases}$$

Noting the last element of I_n and all the elements of I_{n-1}, we obtain $k < (1 + \Delta_1) + (k-1)$, namely $\Delta_1 > 0$. Successively considering the last element of I_i and all the elements of I_{i-1}, we get

$$\begin{cases} k(1 + \Delta_1) < (1 + \Delta_1 + \Delta_2) + (k-1), \\ k(1 + \Delta_1 + \Delta_2) < (1 + \Delta_1 + \Delta_2 + \Delta_3) + (k-1), \\ \quad \vdots \\ k\Big(1 + \sum_{i=1}^{n-2} \Delta_i\Big) < \Big(1 + \sum_{i=1}^{n-1} \Delta_i\Big) + (k-1). \end{cases} \tag{2.6}$$

It follows from (2.6) that

$$\begin{cases} \Delta_2 > (k-1)\Delta_1, \\ \Delta_3 > (k-1)(\Delta_1 + \Delta_2), \\ \quad \vdots \\ \Delta_{n-1} > (k-1) \sum_{i=1}^{n-2} \Delta_i. \end{cases}$$

In addition, $\alpha = (k, 0, \cdots, 0)$ and $\beta = (0, \cdots, 0, k+1)$ should also satisfy $Q \cdot \alpha < Q \cdot \beta$. So

$$k\Big(1 + \sum_{i=1}^{n-1} \Delta_i\Big) < k + 1. \tag{2.7}$$

In order to cover all the elements between δ and γ, we let $k = |\gamma|$ and set

$$\begin{cases} \Delta_2 = k\Delta_1 = |\gamma|\Delta_1, \\ \Delta_3 = k(\Delta_1 + \Delta_2) = |\gamma|(\Delta_1 + \Delta_2) = |\gamma|(|\gamma| + 1)\Delta_1, \\ \quad \vdots \\ \Delta_{n-1} = k \sum_{i=1}^{n-2} \Delta_i = |\gamma| \sum_{i=1}^{n-2} \Delta_i = |\gamma|(|\gamma| + 1)^{n-2}\Delta_1. \end{cases} \tag{2.8}$$

Placing $k = |\gamma|$ in (2.7) and noting the equations (2.8), we get

$$\Delta_1 < \frac{|\gamma| + 1}{|\gamma|((|\gamma| + 1)^{n-1}|\gamma| + 1)}.$$

Let

$$\Delta_1 = \frac{1}{2} \cdot \frac{|\gamma| + 1}{|\gamma|((|\gamma| + 1)^{n-1}|\gamma| + 1)},$$

and then we find Q denoted by (2.5) and (2.8). □

Definition 2.6 (High-dimension vanishing moments). *Let $f \in C(\mathbb{R}^n)$. If there exists $k \in \mathbb{N}_0^n$ such that, for any $\alpha \in \{\alpha \in \mathbb{N}_0^n\colon N(\alpha) \le N(k)\}$,*

$$\int_{\mathbb{R}^n} t^\alpha f(t)dt = 0,$$

we call that f has k vanishing moments.

Note 1. A function f is said to have exactly k vanishing moments if it has k vanishing moments and for all $\alpha \in \mathbb{N}^n$ with $N(\alpha) = N(k) + 1$, there holds $\int_{\mathbb{R}^n} t^\alpha f(t)dt \neq 0$.

Note 2. In particular, let $n = 2$, the two-dimension vanishing moments defined here are more general than that defined in [3] (see [3, Definition 1]). A function has α vanishing moments in [3] implies that it has α vanishing moment here, however, the contrary is not true. That is to say the two-dimension vanishing moments defined here need less conditions.

3 The Multi-Hilbert Transform on $\mathfrak{D}(\mathbb{R}^n)$

Lemma 3.1. *Let $x, t \in \mathbb{R}^n$, $\alpha \in \mathbb{N}_0^n$, and $|x_i| > |t_i|$ for all $i \in \{1, 2, \cdots, n\}$. Then*

$$\sum_{N(\alpha)=0}^{\infty} \frac{t^\alpha}{x^{\alpha+\mathbf{1}}} = \frac{1}{(x_1 - t_1)(x_2 - t_2)\cdots(x_n - t_n)}. \tag{3.1}$$

Proof. For all $i \in \{1, 2, \cdots, n\}$, since $|x_i| > |t_i|$, we have

$$\sum_{\alpha_i=0}^{\infty} \frac{{t_i}^{\alpha_i}}{{x_i}^{\alpha_i}} = \frac{x_i}{x_i - t_i}. \tag{3.2}$$

Then we derive from (3.2) that

$$\begin{aligned}\sum_{N(\alpha)=0}^{\infty} \frac{t^\alpha}{x^{\alpha+\mathbf{1}}} =& \frac{1}{x^{\mathbf{1}}}\Big(\sum_{\alpha_1=0}^{\infty}\cdots\sum_{\alpha_n=0}^{\infty} \frac{t_1^{\alpha_1}\cdots t_n^{\alpha_n}}{x_1^{\alpha_1}\cdots x_n^{\alpha_n}}\Big)\\ =& \frac{1}{x^{\mathbf{1}}}\frac{x_1\cdots x_n}{(x_1 - t_1)(x_2 - t_2)\cdots(x_n - t_n)}\\ =& \frac{1}{(x_1 - t_1)(x_2 - t_2)\cdots(x_n - t_n)}.\end{aligned}$$

□

Theorem 3.2. *Suppose that $f \in \mathfrak{D}(\mathbb{R}^n)$. Then the n-dimensional Hilbert transform of f is bounded on $\mathbb{R}^n$.*

Proof. We denote the support of the function f by $D = [-a_1, a_1] \times \cdots \times [-a_n, a_n]$, where $a_i > 0,\ \ i = 1, \cdots, n$. Then the integral region in (1.2) is contained in D.

Considering the integral

$$\text{p.v.}\int_{\mathbb{R}} \frac{1}{x_i - t_i} = 0, \qquad i = 1, \cdots, n, \tag{3.3}$$

for all $x \in \mathbb{R}^n$, the Hilbert transform can be rewritten as

$$\begin{aligned}H_n f(x) =&\frac{1}{\pi^n}\text{p.v.}\int_{\mathbb{R}^n} \frac{f(t)}{(x_1-t_1)\cdots(x_n-t_n)}\,dt\\ =&\frac{1}{\pi^n}\text{p.v.}\int_{D} \frac{f(t)}{(x_1-t_1)\cdots(x_n-t_n)}\,dt\\ =&\frac{1}{\pi^n}\text{p.v.}\int_{D} \frac{f(t_1,t_2,\cdots,t_n)-f(x_1,t_2,\cdots,t_n)}{(x_1-t_1)\cdots(x_n-t_n)}\,dt,\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}&\text{p.v.}\int_{D} \frac{f(t_1,t_2,\cdots,t_n)-f(x_1,t_2,\cdots,t_n)}{(x_1-t_1)\cdots(x_n-t_n)}\,dt\\ =&\lim_{\varepsilon_i\to 0^+}\int_{|x_i-t_i|>\varepsilon_i>0, i=1,\cdots,n; t\in D} \frac{f(t_1,t_2,\cdots,t_n)-f(x_1,t_2,\cdots,t_n)}{(x_1-t_1)\cdots(x_n-t_n)}\,dt.\end{aligned}$$

Obviously, there holds

$$f(t_1,t_2,\cdots,t_n)-f(x_1,t_2,\cdots,t_n)=(t_1-x_1)\int_0^1 \partial_1 f(x_1+\theta_1(t_1-x_1),t_2,\cdots,t_n)d\theta_1. \tag{3.5}$$

Then we get from (3.4) and (3.5) that

$$\begin{aligned}H_n f(x) =&\frac{1}{\pi^n}\text{p.v.}\int_{D} \frac{F_1(x_1,t_1,\cdots,t_n)}{-(x_2-t_2)\cdots(x_n-t_n)}dt\\ =&\frac{1}{\pi^n}\text{p.v.}\int_{D} \frac{F_1(x_1,t_1,\cdots,t_n)-F_1(x_1,t_1,x_2,t_3,\cdots,t_n)}{-(x_2-t_2)\cdots(x_n-t_n)}dt,\end{aligned} \tag{3.6}$$

where

$$F_1(x_1,t_1,\cdots,t_n)=\int_0^1 \partial_1 f(x_1+\theta_1(t_1-x_1),t_2,\cdots,t_n)d\theta_1. \tag{3.7}$$

Similar to (3.5), we have

$$\begin{aligned}&F_1(x_1,t_1,\cdots,t_n)-F_1(x_1,t_1,x_2,t_3,\cdots,t_n)\\ =&(t_2-x_2)\int_0^1 \partial_2 F_1(x_1,t_1,x_2+\theta_2(t_2-x_2),t_3,\cdots,t_n)d\theta_2.\end{aligned} \tag{3.8}$$

Combining (3.7) with (3.8), it yields

$$\begin{aligned}H_n f(x) =&\frac{1}{\pi^n}\text{p.v.}\int_{D} \frac{F_1(x_1,t_1,\cdots,t_n)}{-(x_2-t_2)\cdots(x_n-t_n)}dt\\ =&\frac{1}{\pi^n}\text{p.v.}\int_{D} \frac{F_2(x_1,t_1,x_2,t_2,t_3,\cdots,t_n)}{(-1)^2(x_3-t_3)\cdots(x_n-t_n)}dt,\end{aligned} \tag{3.9}$$

where

$$F_2(x_1,t_1,x_2,t_2,t_3,\cdots,t_n)=\int_0^1 \partial_2 F_1(x_1,t_1,x_2+\theta_2(t_2-x_2),t_3,\cdots,t_n)d\theta_2. \tag{3.10}$$

Repeating the process, we deduce

$$\begin{aligned}H_n f(x) =&\frac{1}{\pi^n}\text{p.v.}\int_{D} \frac{F_{n-1}(x_1,t_1,x_2,t_2,t_3,\cdots,t_n)}{(-1)^{n-1}(x_n-t_n)}dt\\ =&\frac{1}{\pi^n}\text{p.v.}\int_{D} (-1)^n \partial_n F_{n-1}(x_1,t_1,x_2,t_2,\cdots,x_{n-1},t_{n-1},x_n\\ &+\theta_n(t_n-x_n))d\theta_n dt.\end{aligned} \tag{3.11}$$

Since $f \in \mathfrak{D}(\mathbb{R}^n)$ and the integral region is contained in the bounded region D, there exists a constant M which satisfies

$$|\partial_n \partial_{n-1} \cdots \partial_1 f(x)| \leq M, \tag{3.12}$$

for all $x \in \mathbb{R}^n$. So we have

$$|H_n f(x)| \leq \frac{1}{\pi^n} \int_{-a_1}^{a_1} \cdots \int_{-a_1}^{a_1} M dt \leq \frac{2^n a_1 \cdots a_n}{\pi^n} M, \tag{3.13}$$

which implies our conclusion. □

Theorem 3.2 tells us that the multi-Hilbert transform of functions in $\mathfrak{D}(\mathbb{R}^n)$ are bounded. In fact, they also tend to 0 at the infinity.

Theorem 3.3. *Suppose* $f \in \mathfrak{D}(\mathbb{R}^n)$ *and* supp $f \subset \boldsymbol{B}_a := [-a_1, a_1] \times \cdots \times [-a_i, a_i]$, $a_i > 0$, $i = 1, \cdots, n$. *Then for all* $x \in \mathbb{R}^n$ *with* $|x_i| > a_i$, $i = 1, \cdots, n$, *we have*

$$\left| x^{\mathbf{1}} H_n f(x) - \frac{1}{\pi^n} \int_{\mathbb{R}^n} f(t) dt \right| \leq I(|x_1|, \cdots, |x_n|) \|f\|_{L^1(\mathbb{R}^n)}, \tag{3.14}$$

where,

$$\lim_{|x_i| \to \infty, i=1,\cdots,n} I(|x_1|, \cdots, |x_n|) = 0. \tag{3.15}$$

Proof. From the definition of multi-Hilbert transform in (1.2) and that the support of f is contained in D, we have

$$H_n f(x) = \frac{1}{\pi^n} \text{p.v.} \int_D \frac{f(t)}{(x_1 - t_1) \cdots (x_n - t_n)} dt. \tag{3.16}$$

For all $x \in \mathbb{R}^n$ with $|x_i| > a_i$, $i = 1, \cdots, n$, we get from (3.16) and Lemma 3.1 that

$$H_n f(x) = \frac{1}{\pi^n} \text{p.v.} \int_D \sum_{N(\alpha)=0}^{\infty} \frac{t^\alpha}{x^{\alpha + \mathbf{1}}} f(t) dt. \tag{3.17}$$

Evidently,

$$\begin{aligned} \sum_{N(\alpha)=0}^{\infty} \left| \frac{t^\alpha}{x^{\alpha+\mathbf{1}}} f(t) \right| &= \frac{|f(t)|}{(|x_1| - |t_1|) \cdots (|x_n| - |t_n|)} \\ &\leq \frac{|f(t)|}{(|x_1| - |a_1|) \cdots (|x_n| - |a_n|)} \in L^1(D). \end{aligned} \tag{3.18}$$

Using Fubini-Tonelli's Theorem (see [13]) and combining (3.17) with (3.18), we get

$$H_n f(x) = \frac{1}{\pi^n} \text{p.v.} \sum_{N(\alpha)=0}^{\infty} \frac{1}{x^{\alpha+\mathbf{1}}} \int_D t^\alpha f(t) dt. \tag{3.19}$$

From the identity (3.19), we deduce

$$x^{\mathbf{1}} H_n f(x) - \frac{1}{\pi^n} \int_{\mathbb{R}^n} f(t) dt = \frac{1}{\pi^n} \sum_{\alpha \geq \mathbf{0}, \alpha \neq \mathbf{0}} \frac{1}{x^\alpha} \int_D t^\alpha f(t) dt. \tag{3.20}$$

Utilizing (3.20), we get for all $x\in\mathbb{R}^n$ with $|x_i|>a_i,\ \ i=1,\cdots,n$, there holds

$$\begin{aligned}
&\Big|x^{\mathbf{1}}H_nf(x)-\frac{1}{\pi^n}\int_{\mathbb{R}^n}f(t)dt\Big|\\
\le&\frac{1}{\pi^n}\sum_{\alpha\ge\mathbf{0},\alpha\ne\mathbf{0}}\Big(\frac{a_1}{|x_1|}\Big)^{\alpha_1}\cdots\Big(\frac{a_n}{|x_n|}\Big)^{\alpha_n}\int_D|f(t)|dt\\
=&\frac{1}{\pi^n}\int_D|f(t)|dt\Big\{\sum_{\alpha_1=1}^{\infty}\Big(\frac{a_1}{|x_1|}\Big)^{\alpha_1}\sum_{\alpha_2=0}^{\infty}\Big(\frac{a_2}{|x_2|}\Big)^{\alpha_2}\cdots\sum_{\alpha_n=0}^{\infty}\Big(\frac{a_n}{|x_n|}\Big)^{\alpha_n}\\
&+\sum_{\alpha_2=1}^{\infty}\Big(\frac{a_2}{|x_2|}\Big)^{\alpha_2}\sum_{\alpha_3=0}^{\infty}\Big(\frac{a_3}{|x_3|}\Big)^{\alpha_3}\cdots\sum_{\alpha_n=0}^{\infty}\Big(\frac{a_n}{|x_n|}\Big)^{\alpha_n}+\cdots+\sum_{\alpha_n=1}^{\infty}\Big(\frac{a_n}{|x_n|}\Big)^{\alpha_n}\Big\}\\
=&\frac{1}{\pi^n}\int_D|f(t)|dt\Big\{\frac{a_1}{|x_1|-a_1}\frac{x_2}{|x_2|-a_2}\cdots\frac{x_n}{|x_n|-a_n}\\
&+\frac{a_2}{|x_2|-a_2}\frac{x_3}{|x_3|-a_3}\cdots\frac{x_n}{|x_n|-a_n}+\cdots+\frac{a_n}{|x_n|-a_n}\Big\}\\
=&I(|x_1|,\cdots,|x_n|)\|f\|_{L^1(\mathbb{R}^n)},
\end{aligned}\tag{3.21}$$

where

$$\begin{aligned}
I(|x_1|,\cdots,|x_n|)=&\frac{1}{\pi^n}\Big\{\frac{a_1}{|x_1|-a_1}\frac{x_2}{|x_2|-a_2}\cdots\frac{x_n}{|x_n|-a_n}\\
&+\frac{a_2}{|x_2|-a_2}\frac{x_3}{|x_3|-a_3}\cdots\frac{x_n}{|x_n|-a_n}+\cdots+\frac{a_n}{|x_n|-a_n}\Big\}.
\end{aligned}\tag{3.22}$$

Since the summation in (3.22) is finite, we have

$$\lim_{|x_i|\to\infty,i=1,\cdots,n}I(|x_1|,\cdots,|x_n|)=0.$$

□

Note 3. It is notable that the limitation in (3.14) is taken from $|x_i|\to\infty$ for all $i=1,\cdots,n$, but not $|x|\to\infty$. Taking $|x_i|\to\infty$ for all $i=1,\cdots,n$ on the left side of (3.14), it is obvious that

$$\lim_{|x_i|\to\infty,\,i=1,\cdots,n}Hf(x)=0.$$

Using the same method as in the proof of Theorem 3.3, we can get the following corollary.

Corollary 3.4. *Suppose* $f\in\mathfrak{D}(\mathbb{R}^n)$, $\operatorname{supp}f\subset\boldsymbol{B}_a:=[-a_1,a_1]\times\cdots\times[-a_i,a_i]$, $a_i>0$, $i=1,\cdots,n$. *Then for all* $x\in\mathbb{R}^n$ *with* $|x_i|>a_i$, $i=1,\cdots,n$, *we have*

$$\Big|\frac{x^{\mathbf{1}}}{x_1}H_nf(x)-\frac{1}{\pi^n}\frac{1}{x_1}\int_{\mathbb{R}^n}f(t)dt\Big|\le I_1(|x_1|,\cdots,|x_n|)\|f\|_{L^1(\mathbb{R}^n)}.\tag{3.24}$$

Moreover,

$$\lim_{|x_1|\to\infty}I_1(|x_1|,\cdots,|x_n|)=0.\tag{3.25}$$

Proof. We just need to choose

$$I_1(|x_1|,\cdots,|x_n|)=\frac{1}{|x_1|}I(|x_1|,\cdots,|x_n|).$$

□

Note 4. Similarly, for fixed $x_2, \cdots, x_n$, taking $|x_1| \to \infty$ in (3.24), we get $\lim_{|x_1|\to\infty} H_n f(x) = 0$, which means $H_n f(x)$ tends to 0 when $|x| \to \infty$ and x parallel to the x_1-axe.

Theorem 3.5. *Let $f \in \mathfrak{D}(\mathbb{R}^n)$, $\operatorname{supp} f \subset \boldsymbol{B}_a := [-a, a]^n$, $a > 0$ and suppose f has k ($k \in \mathbb{N}_0^n$) vanishing moments with $k' \in \mathbb{N}_0^n$ and $N(k') = N(k) + 1$. Then there exists a unbounded route $\Gamma \subseteq \{x \in \mathbb{R}^n : |x_i| > a,\ i = 1, 2, \cdots, n\}$ which satisfies the following property:*

$$\lim_{x\to\infty, x\in\Gamma} \left| x^{k'+\mathbf{1}} H_n f(x) - \frac{1}{\pi^n} \int_{\mathbb{R}^n} t^{k'} f(t) dt \right| = 0. \tag{3.26}$$

Proof. We just consider $a > 1$ for convenience.

Using the definition of n-dimensional Hilbert transform (1,2) and the assumption $\operatorname{supp} f \subset \mathbf{B}_a$, we have

$$H_n f(x) = \frac{1}{\pi^n} \text{p.v.} \int_{-a}^{a} \cdots \int_{-a}^{a} \frac{f(t)}{(x_1 - t_1) \cdots (x_n - t_n)} dt. \tag{3.27}$$

For $x \in \mathbb{R}^n$ and $|x_i| > a$ $(i = 1, 2, \cdots, n)$, the Cauchy principle value in (3.27) can be wiped off.

Using (3.1), the identity (3.27) can be rewritten as

$$H_n f(x) = \frac{1}{\pi^n} \int_{-a}^{a} \cdots \int_{-a}^{a} \sum_{N(\alpha)=0}^{\infty} \frac{t^\alpha}{x^{\alpha+\mathbf{1}}} f(t) dt. \tag{3.28}$$

Since

$$\begin{aligned}\sum_{N(\alpha)=0}^{\infty} \left| \frac{t^\alpha}{x^{\alpha+\mathbf{1}}} f(t) \right| \leq & \frac{|f(t)|}{(|x_1| - |t_1|)(|x_2| - |t_2|) \cdots (|x_n| - |t_n|)} \\ \leq & \frac{|f(t)|}{(|x_1| - a)(|x_2| - a) \cdots (|x_n| - a)} \in L^1(\mathbf{B}_a),\end{aligned}$$

we conclude form Fubini-Tonelli's theorem (see [4]) that (3.28) can be written as

$$H_n f(x) = \frac{1}{\pi^n} \sum_{N(\alpha)=0}^{\infty} \frac{1}{x^{\alpha+\mathbf{1}}} \int_{-a}^{a} \cdots \int_{-a}^{a} t^\alpha f(t) dt.$$

By the assumption that f has k vanishing moments, we have

$$H_n f(x) = \frac{1}{\pi^n} \sum_{N(\alpha)=N(k')}^{\infty} \frac{1}{x^{\alpha+\mathbf{1}}} \int_{-a}^{a} \cdots \int_{-a}^{a} t^\alpha f(t) dt.$$

Consequently,

$$x^{k'+\mathbf{1}} H_n f(x) - \frac{1}{\pi^n} \int_{-a}^{a} \cdots \int_{-a}^{a} t^{k'} f(t) dt = \frac{1}{\pi^n} \sum_{N(\alpha)=N(k')+1}^{\infty} \frac{x^{k'}}{x^\alpha} \int_{-a}^{a} \cdots \int_{-a}^{a} t^\alpha f(t) dt,$$

and then

$$\left| x^{k'+\mathbf{1}} H_n f(x) - \frac{1}{\pi^n} \int_{-a}^{a} \cdots \int_{-a}^{a} t^{k'} f(t) dt \right| \leq \frac{1}{\pi^n} \sum_{N(\alpha)=N(k')+1}^{\infty} \frac{|x^{k'}|}{|x^\alpha|} \int_{-a}^{a} \cdots \int_{-a}^{a} |t^\alpha| |f(t)| dt.$$

Denoting a set Λ as $\Lambda=\{\beta:\ \beta\in\mathbb{N}_0^n,\ |\beta|=|k'|+1,\ \exists\, i\in\{1,2,\cdots,n\},\ \text{s.t.}\ \beta_i=|k'|+1\}$, where $\gamma=(|k'|+1,|k'|+1,\cdots,|k'|+1)$. Using Lemma 2.6, we get

$$\begin{aligned}&\Big|x^{k'+\mathbf{1}}H_nf(x)-\frac{1}{\pi^n}\int_{-a}^{a}\cdots\int_{-a}^{a}t^{k'}f(t)dt\Big|\\ \le&\frac{1}{\pi^n}\Big(\sum_{\beta\in\Lambda}\sum_{\alpha\ge\beta}+\sum_{\alpha<\gamma,\ N(\alpha)>N(k')}\Big)\frac{|x^{k'}|}{|x^\alpha|}\int_{-a}^{a}\cdots\int_{-a}^{a}|t^\alpha||f(t)|dt\\ =&S_1+S_2,\end{aligned}$$

where

$$S_1=\frac{1}{\pi^n}\sum_{\beta\in\Lambda}\sum_{\alpha\ge\beta}\frac{|x^{k'}|}{|x^\alpha|}\int_{-a}^{a}\cdots\int_{-a}^{a}|t^\alpha||f(t)|dt,$$

and

$$S_2=\frac{1}{\pi^n}\sum_{\alpha<\gamma,N(\alpha)>N(k')}\frac{|x^{k'}|}{|x^\alpha|}\int_{-a}^{a}\cdots\int_{-a}^{a}|t^\alpha||f(t)|dt.$$

For k' and γ, using Lemma 2.7, there exists a vector Q which satisfies $Q\cdot\beta<Q\cdot\alpha$ if $N(k')\le N(\beta)<N(\alpha)<N(\gamma)$.

Taking a special route $\Gamma=\{x\in\mathbb{R}^n: x_n>a,\ x_i={x_n}^{Q_i}\}$, we will prove

$$\lim_{x_n\to\infty,\, x\in\Gamma}(S_1+S_2)=0. \tag{3.29}$$

Let $x\in\Gamma$ and denote $Z^i=(0,\cdots,|k'|+1,0,\cdots,0)$, where $Z^i_i=|k'|+1$.

For S_1, we get the following estimates:

$$\begin{aligned}S_1\le&\frac{|x^{k'}|}{\pi^n}\sum_{\beta\in\Lambda}\sum_{\alpha\ge\beta}\frac{a^{\alpha_1}\cdot a^{\alpha_2}\cdots a^{\alpha_n}}{|x^\alpha|}\|f\|_{L^1}\\ =&\frac{|x^{k'}|}{\pi^n}\sum_{i=1}^{n}\Big(\frac{a^{|k'|+1}}{(|x_i|-a)|x_i|^{|k'|}}\cdot\frac{|x_1|}{|x_1|-a}\cdots\frac{|x_n|}{|x_n|-a}\cdot\frac{|x_i|-a}{|x_i|}\Big)\|f\|_{L^1}\\ =&\frac{1}{\pi^n}\sum_{i=1}^{n}\Big(\frac{{x_n}^{Q\cdot k'}a^{|k'|+1}}{{x_n}^{Q\cdot Z^i}(1-\frac{a}{|x_i|})}\cdot\frac{|x_1|}{|x_1|-a}\cdots\frac{|x_n|}{|x_n|-a}\cdot\frac{|x_i|-a}{|x_i|}\Big)\|f\|_{L^1}\\ =&\frac{a^{|k'|+1}}{\pi^n}\sum_{i=1}^{n}\Big(\frac{{x_n}^{Q\cdot k'}}{{x_n}^{Q\cdot Z^i}}\cdot\frac{|x_1|}{|x_1|-a}\cdots\frac{|x_n|}{|x_n|-a}\Big)\|f\|_{L^1}.\end{aligned}$$

Since $N(k')<N(Z^i)$, we conclude from Lemma 2.7 that $Q\cdot k'<Q\cdot Z^i$, which deduces that

$$\lim_{x_n\to\infty,\, x\in\Gamma}S_1=0. \tag{3.30}$$

Similarly, for S_2, we have

$$S_2\le\frac{a^{n(|k'|+1)}}{\pi^n}\sum_{\alpha<\gamma,\, N(\alpha)>N(k')}\frac{{x_n}^{Q\cdot k'}}{{x_n}^{Q\cdot\alpha}}\|f(t)\|_{L^1},$$

since $a>1$.

Hence

$$\lim_{x_n\to\infty,\, x\in\Gamma}S_2=0. \tag{3.31}$$

Combining (3.30) with (3.31), it yields that (3.29) and (3.26) hold. □

The following theorem shows that a function f in $H_n(\mathfrak{D}(\mathbb{R}^n))$ doesn't have a compact support unless $f = 0$.

Theorem 3.6. $\mathfrak{D}(\mathbb{R}^n) \cap H_n(\mathfrak{D}(\mathbb{R}^n)) = \{0\}$.

Proof. If $\mathfrak{D}(\mathbb{R}^n) \cap H_n(\mathfrak{D}(\mathbb{R}^n)) \neq \{0\}$, there exists $\mu \in \mathfrak{D}(\mathbb{R}^n) \cap H_n(\mathfrak{D}(\mathbb{R}^n))$ such that $\mu \not\equiv 0$. Let $\nu \in \mathfrak{D}(\mathbb{R}^n)$, $\nu \not\equiv 0$ satisfy $\mu = H_n\nu$.

We assert that there exists $k \in \mathbb{N}_0^n$ such that ν has exactly k vanishing moments. Otherwise, ν has arbitrary vanishing moments.

Indeed, let $\operatorname{supp}\nu \subset \mathbf{B}_a$ for some $a > 0$, then for any multivariate polynomial $p(t)$, we have

$$\int_{-a}^{a} \cdots \int_{-a}^{a} p(t)\nu(t)dt = 0.$$

Due to the density of multivariate polynomials in $C(\mathbf{B}_a)$, we obtain $\nu \equiv 0$, which leads to a contradiction.

For k, using Theorem 3.5, we have

$$\lim_{x_n\to\infty,\, x\in\Gamma} x^{k'+\mathbf{1}}\mu(x) = \lim_{x_n\to\infty,\, x\in\Gamma} x^{k'+\mathbf{1}} H_n\nu(x) = \frac{1}{\pi^n}\int_{\mathbb{R}^n} t^{k'}\nu(t)dt \neq 0,$$

which contradicts with $\mu \in \mathfrak{D}(\mathbb{R}^n)$. Thus we finish our proof. □

4 The Spaces $\mathfrak{D}_H(\mathbb{R}^n)$ and $\mathfrak{D}'_H(\mathbb{R}^n)$

The space $\mathfrak{D}_H(\mathbb{R}^n)$ is defined as

$$\mathfrak{D}_H(\mathbb{R}^n) := \mathfrak{D}(\mathbb{R}^n) + H_n(\mathfrak{D}(\mathbb{R}^n)). \tag{4.1}$$

Since $\mathfrak{D}(\mathbb{R}^n) \cap H_n(\mathfrak{D}(\mathbb{R}^n)) = \{0\}$ in Theorem 3.6, we get $\mathfrak{D}_H(\mathbb{R}^n) := \mathfrak{D}(\mathbb{R}^n) \dot{+} H(\mathfrak{D}(\mathbb{R}^n))$. This is a direct sum. We define the convergence in $\mathfrak{D}_H(\mathbb{R}^n)$ as follows.

For all $\phi_n + H_n\psi_n \subset \mathfrak{D}_H(\mathbb{R}^n)$, we say $\phi_n + H_n\psi_n \to 0$ (in $\mathfrak{D}_H(\mathbb{R}^n)$) if $\phi_n, \psi_n \to 0$ (in $\mathfrak{D}(\mathbb{R}^n)$). Endowed with this topology, $\mathfrak{D}_H(\mathbb{R}^n)$ becomes a topological vector space.

From Lemma 2.5, the operator H_n is a homeomorphism from $\mathfrak{D}_{L^p}(\mathbb{R}^n)$ to itself. Since $\mathfrak{D}(\mathbb{R}^n) \subset \mathfrak{D}_{L^p}(\mathbb{R}^n)$, we have $H_n(\mathfrak{D}(\mathbb{R}^n)) \subset \mathfrak{D}_{L^p}(\mathbb{R}^n)$ and thus the direct sum space $\mathfrak{D}_H(\mathbb{R}^n) \subset \mathfrak{D}_{L^p}(\mathbb{R}^n)$ obviously.

Since there holds $H_n^2 = (-1)^n I$ on $\mathfrak{D}_H(\mathbb{R}^n)$ by Lemma 2.5, we immediately obtain the following theorem.

Theorem 4.1. *The n-dimensional Hilbert transform H_n is a homeomorphism from $\mathfrak{D}_H(\mathbb{R}^n)$ to itself.*

With the topology $\mathfrak{D}_H(\mathbb{R}^n)$ defined above, it is easy to see that $\mathfrak{D}(\mathbb{R}^n) \hookrightarrow \mathfrak{D}_H(\mathbb{R}^n)$. Therefore $\mathfrak{D}'_H(\mathbb{R}^n) \hookrightarrow \mathfrak{D}'(\mathbb{R}^n)$. Moreover, the following theorem shows that $\mathfrak{D}_H(\mathbb{R}^n)$ is the smallest space that $\mathfrak{D}(\mathbb{R}^n) \hookrightarrow \mathfrak{D}_H(\mathbb{R}^n)$ and H_n is a homeomorphism from $\mathfrak{D}_H(\mathbb{R}^n)$ to itself.

Theorem 4.2. *Let $\mathfrak{X}(\mathbb{R}^n)$ be a topological vector space such that $\mathfrak{D}(\mathbb{R}^n) \hookrightarrow \mathfrak{X}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, and the n-dimensional Hilbert transform $H_n : \mathfrak{X}(\mathbb{R}^n) \to \mathfrak{X}(\mathbb{R}^n)$ be a continuous linear operator. Then $\mathfrak{D}_H(\mathbb{R}^n) \hookrightarrow \mathfrak{X}(\mathbb{R}^n)$, and consequently $\mathfrak{X}'(\mathbb{R}^n) \hookrightarrow \mathfrak{D}'(\mathbb{R}^n)$.*

Proof. Embedding $\mathfrak{D}(\mathbb{R}^n) \hookrightarrow \mathfrak{X}(\mathbb{R}^n)$ implies $H_n(\mathfrak{D}(\mathbb{R}^n)) \subset H_n(\mathfrak{X}(\mathbb{R}^n)) \subset \mathfrak{X}$. Therefore $\mathfrak{D}(\mathbb{R}^n) + H_n(\mathfrak{D}(\mathbb{R}^n)) \subset \mathfrak{X}$, i.e., $\mathfrak{D}_H(\mathbb{R}^n) \subset \mathfrak{X}$.

For any $\phi_n + H_n\psi_n \subset \mathfrak{D}_H(\mathbb{R}^n)$, $\phi_n + H_n\psi_n \to 0$ (in $\mathfrak{D}_H(\mathbb{R}^n)$), we have $\phi_n, \psi_n \to 0$ (in $\mathfrak{D}(\mathbb{R}^n)$). Then $\phi_n, \psi_n \to 0$ (in $\mathfrak{X}(\mathbb{R}^n)$) and consequently $H_n\psi_n \to 0$ (in $\mathfrak{X}(\mathbb{R}^n)$). Hence $\phi_n + H_n\psi_n \to 0$ (in $\mathfrak{X}(\mathbb{R}^n)$), which shows that $\mathfrak{D}_H(\mathbb{R}^n) \hookrightarrow \mathfrak{X}$. □

The classical multi-Hilbert transform H_n can be extended to $\mathfrak{D}'_H(\mathbb{R}^n)$ by using the conjugate operator. Before doing this, let us recall the following equality[11]:

$$\int_{\mathbb{R}}(Hf)(x)\phi(x)dx = -\int_{\mathbb{R}} f(x)(H\phi)(x)dx, \qquad \forall\, f, \phi \in L^2(\mathbb{R}). \tag{4.2}$$

Repeating (4.2), it is not hard for us to get

$$\int_{\mathbb{R}^n}(H_n f)(x)\phi(x)dx = (-1)^n \int_{\mathbb{R}^n} f(x)(H_n\phi)(x)dx, \qquad \forall\, f,\ \phi \in L^2(\mathbb{R}^n). \tag{4.3}$$

Considering the constraint of H_n on $\mathfrak{D}_H(\mathbb{R}^n)$, we know that: $H_n : \mathfrak{D}_H(\mathbb{R}^n) \to \mathfrak{D}_H(\mathbb{R}^n)$ is a continuous and linear operator, which implies that its conjugate operator $H_n^* : \mathfrak{D}'_H(\mathbb{R}^n) \to \mathfrak{D}'_H(\mathbb{R}^n)$, which is defined as $\langle H_n^* f, \phi\rangle := \langle f,\ H_n\phi\rangle$ $\ (\forall\ f \in \mathfrak{D}'_H(\mathbb{R}^n),\ \phi \in \mathfrak{D}_H(\mathbb{R}^n))$, is a continuous and linear operator. For any $f \in \mathfrak{D}_H(\mathbb{R}^n)$, by (4.3), we have

$$\begin{aligned}\langle H_n f, \phi\rangle &:= \int_{\mathbb{R}^n}(H_n f)(x)\phi(x)dx = (-1)^n \int_{\mathbb{R}^n} f(x)(H_n\phi)(x)dx \\ &=(-1)^n \langle f, H_n\phi\rangle = \langle (-1)^n H_n^* f, \phi\rangle, \qquad \forall\, f \in \mathfrak{D}_H(\mathbb{R}^n)).\end{aligned} \tag{4.4}$$

The identity (4.4) yields that

$$H_n f = (-1)^n H^* f, \quad \text{in} \ \ \mathfrak{D}'_H(\mathbb{R}^n). \tag{4.5}$$

Moreover, if $S : \mathfrak{D}_H(\mathbb{R}^n) \to \mathfrak{D}_H(\mathbb{R}^n)$ is also a continuous and linear operator satisfying $H_n f = S^* f$ $\ (\forall\, f \in \mathfrak{D}_H(\mathbb{R}^n))$, then

$$\int_{\mathbb{R}^n} f(x)(H_n\phi)(x)dx = (-1)^n \int_{\mathbb{R}^n} f(x)(S\phi)(x)dx, \qquad \forall\, f,\quad \phi \in \mathfrak{D}_H(\mathbb{R}^n),$$

which yields that $S\phi = (-1)^n H_n\phi$. Therefore, $(-1)^n H_n^*$ can be defined as the extension of H_n to the distribution space $\mathfrak{D}'_H(\mathbb{R}^n)$, namely, we have

Definition 4.1. *Let $H^* : \mathfrak{D}'_H(\mathbb{R}^n) \to \mathfrak{D}'_H(\mathbb{R}^n)$ be the conjugate operator of the n-dimensional Hilbert transform $H_n : \mathfrak{D}_H(\mathbb{R}^n) \to \mathfrak{D}_H(\mathbb{R}^n)$. Then $(-1)^n H_n^* : \mathfrak{D}'_H(\mathbb{R}^n) \to \mathfrak{D}'_H(\mathbb{R}^n)$ is defined as the extension of H_n to the distribution space $\mathfrak{D}'_H(\mathbb{R}^n)$, and denoted as H_n still if no confusion occurs.*

It is easy to see that the extended n-dimensional Hilbert transform is a homeomorphism from $\mathfrak{D}'_H(\mathbb{R}^n)$ to itself.

Theorem 4.3. *$H_n : \mathfrak{D}'_H(\mathbb{R}^n) \to \mathfrak{D}'_H(\mathbb{R}^n)$ satisfies $H_n^2 = (-1)^n I$.*

Proof. For any $f \in \mathfrak{D}'_H(\mathbb{R}^n)$, we have

$$\left\langle H_n^2 f, \phi\right\rangle = (-1)^n \langle H_n f, H_n\phi\rangle = \left\langle f, H_n^2\phi\right\rangle = (-1)^n \langle f, \phi\rangle, \qquad \forall\, \phi \in \mathfrak{D}_H(\mathbb{R}^n)$$

which concludes that $H_n^2 = (-1)^n I$. □

From Theorem 4.2 and Theorem 4.3, we deduce that $\mathfrak{D}'_H(\mathbb{R}^n)$ is the biggest subspace of $\mathfrak{D}'(\mathbb{R}^n)$ on which the extended multi-Hilbert transform is a homeomorphism.

Theorem 4.4. *Let $f \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$. Then, $H_n f$, as the n-dimensional Hilbert transform, coincides with the extended one defined in (4.3).*

Proof. The n-dimensional Hilbert transform $H_n f$ can be regarded as a distribution on $\mathfrak{D}_H(\mathbb{R}^n)$, whose function on $\phi \in \mathfrak{D}_H(\mathbb{R}^n)$ is

$$\langle H_n f, \phi\rangle = \int_{\mathbb{R}^n}(H_n f)(x)\phi(x)dx. \tag{4.6}$$

On the other hand, as the extended n-dimensional Hilbert transform, $H_n f$ is also a distribution on $\mathfrak{D}_H(\mathbb{R}^n)$, whose function on $\phi \in \mathfrak{D}_H(\mathbb{R}^n)$ is

$$\langle H_n f, \phi \rangle = (-1)^n \langle H^* f, \phi \rangle = (-1)^n \langle f, H_n \phi \rangle = (-1)^n \int_{\mathbb{R}^n} f(x)(H_n \phi)(x) dx. \tag{4.7}$$

It can be verified from (4.6) and (4.7) that

$$\begin{aligned} &\int_{\mathbb{R}^n} (H_n f)(x)\phi(x) dx \\ =&(-1)^n \langle f, H_n \phi \rangle = (-1)^n \int_{\mathbb{R}^n} f(x)(H_n \phi)(x) dx, \qquad \forall f \in L^p(\mathbb{R}^n), \quad \phi \in \mathfrak{D}_H(\mathbb{R}^n), \end{aligned}$$

which is our desired conclusion. □

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