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Generalized Competition Index of Primitive Digraphs

Li-hua YOU[1,†], Fang CHEN[2,a], Jian SHEN[2,b], Bo ZHOU[1,c]

[1]School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China
(E-mail: [†]ylhua@scnu.edu.cn, [c]zhoubo@scnu.edu.cn)
[2]Department of Mathematics, Texas State University, San Marcos, TX 78666, USA
(E-mail: [a]358109416@qq.com, [b]js48@txstate.edu)

Abstract For any positive integers k and m, the k-step m-competition graph $C_m^k(D)$ of a digraph D has the same set of vertices as D and there is an edge between vertices x and y if and only if there are distinct m vertices $v_1, v_2, \cdots, v_m$ in D such that there are directed walks of length k from x to v_i and from y to v_i for all $1 \le i \le m$. The m-competition index of a primitive digraph D is the smallest positive integer k such that $C_m^k(D)$ is a complete graph. In this paper, we obtained some sharp upper bounds for the m-competition indices of various classes of primitive digraphs.

Keywords competition index; m-competition index; scrambling index; primitive digraph
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1 Introduction

For terminology and notation used here we follow[4,5,9,21]. Let $D = (V, E)$ denote a *digraph* (directed graph) with vertex set $V = V(D)$ and arc set $E = E(D)$ on n vertices. Loops are permitted but multiple arcs are not. *A $u \to v$ walk* in D is a sequence of vertices $u, u_1, \cdots, u_t, v \in V(D)$ and a sequence of arcs $(u, u_1), (u_1, u_2), \cdots, (u_t, v) \in E(D)$, where the vertices and arcs are not necessarily distinct. A *closed walk* is a $u \to v$ walk where $u = v$. A *cycle* is a closed $u \to v$ walk with distinct vertices except for $u = v$. The *length of a walk* W, denoted by $|W|$, is the number of arcs in W. The notation $u \xrightarrow{k} v$ is used to indicate that there is a $u \to v$ walk of length k. The *distance*, $d(u, v)$, from a vertex u to a vertex v in D is the length of the shortest walk from u to v. An *l-cycle*, denoted by C_l, is a cycle of length l. The length of the shortest cycle in D is called the *girth* of D.

A diagraph D is *primitive* if there exists some positive integer k such that $u \xrightarrow{k} v$ for every pair $u, v \in V(D)$. The smallest such k is called the *exponent* of D, denoted by $\exp(D)$. It is well known that D is primitive if and only if D is strongly connected and the greatest common divisor of lengths of its cycles is 1. The competition graph of D was introduced by [9] when he studied a problem in ecology.

Definition 1.1[9]**.** *The competition graph of a digraph D, denoted by $C(D)$, has the same set of vertices as D and there is an edge between vertices x and y if and only if there is a vertex z such that (x, z) and (y, z) are arcs of D.*

Since the notion of competition graphs was introduced, there has been numerous literature on competition graphs. We refer to [19] for a surveys on competition graphs. In addition to ecology, their various applications include applications to channel assignments, coding, and

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modeling of complex economic and energy systems. Cho et al.[6] generalized competition graph to k-step competition graph.

Definition 1.2[6]. *Let D be a digraph and k be a positive integer. A vertex z of D is a k-step common prey for x and y if $x \xrightarrow{k} z$ and $y \xrightarrow{k} z$.*

Definition 1.3[6]. *The k-step competition graph of D, denoted by $C^k(D)$, has the same vertex set as D and there is an edge between distinct vertices x and y if and only if x and y have a k-step common prey in D.*

Definition 1.4[6]. *The k-step digraph of D, denoted by D^k, has the same vertex set as D and there is an arc (x, y) in D^k if and only if $x \xrightarrow{k} y$ in D.*

Clearly, the k-step competition graph of D is the competition graph of D^k, i.e. $C^k(D) = C(D^k)$ (see [11]).

Definition 1.5[12]. *For a positive integer m, the m-competition graph of a digraph D, denoted by $C_m(D)$, has the same vertex set as D and there is an edge between x and y if and only if there are at least m distinct vertices $v_1, v_2, \cdots, v_m$ and arcs (x, v_i) and (y, v_i) for $1 \le i \le m$. That is, there is an edge between x and y in $C_m(D)$ if and only if x and y have at least m common preys in D.*

Definition 1.6[12]. *The k-step m-competition graph $C_m^k(D)$ has the same vertex set as D and there is an edge between x and y if and only if there are at least m distinct vertices $v_1, v_2, \cdots, v_m$ such that each v_i $(1 \le i \le m)$ is a k-step common prey for vertices x and y; i.e., $x \xrightarrow{k} v_i$ and $y \xrightarrow{k} v_i$ for all i $(1 \le i \le m)$.*

Proposition 1.1[12]. *For any digraph D and positive integers m and k, we have $C_m^k(D) = C_m(D^k)$.*

Proposition 1.2[12]. *For any primitive digraph D on n vertices and for each positive integer m with $1 \le m \le n$, there is a positive integer k such that $C_m^k(D) = K_n$, where K_n denotes the complete graph on n vertices. Also $C_m^{k+1}(D) = K_n$ whenever $C_m^k(D) = K_n$.*

Definition 1.7[12]. *Let D be a primitive digraph on n vertices and m be a positive integer with $1 \le m \le n$. The m-competition index of D is the smallest positive integer k such that for every pair of vertices x and y, there exist distinct vertices $v_1, v_2, \cdots, v_m$ such that $x \xrightarrow{k} v_i$ and $y \xrightarrow{k} v_i$ for $1 \le i \le m$ in D, denoted by $k_m(D)$.*

Clearly, the m-competition index of D on n vertices is the smallest positive integer k such that every pair of vertices x and y has at least m common preys in D^k. From Proposition 1.2, $k_m(D)$ is the smallest positive integer k such that $C_m^k(D) = C_m^{k+i}(D) = K_n$ for every positive integer i. An analogous definition can be given for nonnegative matrices.

Definition 1.8[12]. *The m-competition index of a primitive matrix A, denoted by $k_m(A)$, is the smallest positive integer k such that any two rows of A^k have positive elements in at least m identical columns.*

Remark 1.1. In [1,2], Akelbek and Kirkland introduced the *scrambling index* of a primitive digraph D, denoted by $k(D)$, and in [7,11], Cho and Kim introduced the *competition index* of a digraph. In fact, the definitions of scrambling index and competition index are the same for primitive digraphs, they are also called common consequent index in [31,32]. Furthermore, these definitions are the same as the definition of the m-competition index of a primitive digraph when $m = 1$, and the exponent of a primitive digraph is the same as the definition of the m-competition index when $m = n$. So the m-competition index of a primitive digraph is a generalization of the competition index and the exponent of a primitive digraph. For this reason, we call $k_m(D)$ the generalized competition index. It is easy to see that $k(D) = k_1(D) \le k_2(D) \le \cdots \le k_n(D) = \exp(D)$. We refer to [1,2,3,8,10,17,22,25,26,30] for research on (generalized) scrambling index and (generalized) competition index, respectively.

Remark 1.2. In [23], Moon and Moser showed almost all $(0,1)$-matrices are primitive. In other words, let $\mathbb{B}_n$ denote the set of all $(0,1)$ square matrices of order n, $\mathbb{P}_n$ denote the set of all primitive matrices in $\mathbb{B}_n$, then $\lim\limits_{n\to\infty} \frac{|\mathbb{P}_n|}{|\mathbb{B}_n|} = 1$. Based on the one-to-one correspondence relation between the simple digraphs on n vertices and all $(0,1)$-matrices of order n, we know that almost all simple digraphs on n vertices are primitive. Let D be a primitive digraph. Then $k_m(D)$ is the minimum number of steps that information at any two vertices of D can be sent to at least m vertices in D. When $m = 1$ and $m = n$, these reduce to the classical cases: $k(D) = k_1(D)$ is the minimum number of steps that information at any two vertices of D can be sent to some vertex in D, and $k_n(D) = \exp(D)$ is the minimum number of steps that information at any two vertices of D can be sent to all vertices in D.

In this paper, we investigate the m-competition index of primitive digraphs. We obtain some sharp upper bounds for the m-competition indices of various classes of primitive digraphs including primitive digraphs with d loops, primitive symmetric digraphs, r-indecomposable digraphs, primitive Cayley digraphs and primitive digraphs with girth s.

2 m-Competition Index of Primitive Digraphs

In this section, we assume that n and m are integers with $1 \le m \le n$. Let P_n denote the set of all primitive digraphs on n vertices. The m-competition indices of various classes of primitive digraphs on n vertices are investigated.

Let $R_t^D(X)$ denote the set of all those vertices which can be reached by a walk of length t in D starting from some vertex in $X \subseteq V(D)$. Clearly, $R_0^D(X) = X$.

Lemma 2.1[21]**.** *Let D be a strongly connected digraph on n vertices, s be an integer and $W = \{i_1, \cdots, i_s\}$ be the set of loop vertices in D. Then for each integer t,*

$$|R_t^D(W)| \ge \min\{s + t, n\}.$$

2.1 Primitive Digraphs with d Loops

Let d be an integer with $1 \le d \le n$, and $P_n(d)$ denote the set of primitive digraphs with n vertices and d loops.

Let $L_{n,d} \in P_n(d)$ be the digraph with vertex set $V = \{1, 2, \cdots, n\}$ and arc set $E = \{(i, i+1) | 1 \le i \le n-1\} \cup \{(n, 1)\} \cup \{(i, i) | n-d+1 \le i \le n\}$. It is well known that $\exp(L_{n,d}) = 2n-d-1$ (see [21]).

Theorem 2.1. *Let $D \in P_n(d)$ and $1 \le m \le n$. Then $k_m(D) \le n - d + \lfloor \frac{m+n-1}{2} \rfloor$.*

Proof. Let W be the set of loop vertices in D and x, y be any two different vertices in D. There exist vertices u_1 and u_2 of W such that $x \xrightarrow{n-d} u_1$ and $y \xrightarrow{n-d} u_2$.

Case 1. $u_1 = u_2$.

Then, by Lemma 2.1,

$$\left| R^D_{\lfloor \frac{m+n-1}{2} \rfloor}(\{u_1\}) \right| \ge \min\left\{ \left\lfloor \frac{m+n-1}{2} \right\rfloor + 1, n \right\} = \left\lfloor \frac{m+n-1}{2} \right\rfloor + 1 \ge m$$

and thus $k_m(D) \le n - d + \lfloor \frac{m+n-1}{2} \rfloor$.

Case 2. $u_1 \neq u_2$.

Let $t = \lfloor \frac{m+n-1}{2} \rfloor$. Then, by Lemma 2.1, $|R_t^D(\{u_i\})| \geq t+1$ for $i = 1, 2$. Hence

$$\Big|\bigcap_{i=1}^{2} R_t^D(\{u_i\})\Big| \geq 2(t+1) - n \geq 2\Big(\frac{m+n-1-1}{2} + 1\Big) - n = m$$

and thus $k_m(D) \leq n - d + \lfloor \frac{m+n-1}{2} \rfloor$.

Combining the above two cases, we have $k_m(D) \leq n - d + \lfloor \frac{m+n-1}{2} \rfloor$. □

By Theorem 2.1, and taking $m = n$, we can obtain the following result.

Corollary 2.1[21]**.** *Let $D \in P_n(d)$. Then $\exp(D) = k_n(D) \leq 2n - d - 1$, and equality holds when $D \cong L_{n,d}$.*

Lemma 2.2[20]**.** *Let $D \in P_n$ and $\emptyset \neq X \subseteq V(D)$. Then for any nonnegative integers i, j, t, k with $i \geq j$, we have*

$$R_i^D(X) = R_{i-j}^D(R_j^D(X)), \qquad \Big|\bigcup_{t=0}^{k} R_t^D(X)\Big| \geq \min\{|X| + k, n\}.$$

For convenience, we let $|a|_n$ denote the least positive integer t with $t \equiv a \pmod n$, and let a set $\{a_1, \cdots, a_s\} \pmod n$ denote the set $\{|a_1|_n, \cdots, |a_s|_n\}$.

Theorem 2.2. *Let $D \in P_n(d)$ and $1 \leq m \leq d \leq n$. Then $k_m(D) \leq n - \lceil \frac{d-m+1}{2} \rceil$, and equality holds when $D \cong L_{n,d}$.*

Proof. For any two vertices $u_1, u_2 \in V(D)$, by Lemma 2.2, for $i = 1, 2$, we have

$$\Big|\bigcup_{t=0}^{n-\lceil \frac{d-m+1}{2} \rceil} R_t^D(\{u_i\})\Big| \geq \min\Big\{1 + \Big(n - \lceil \frac{d-m+1}{2} \rceil\Big), n\Big\} = n - \Big\lceil \frac{d-m+1}{2} \Big\rceil + 1.$$

Hence

$$\Big|\bigcap_{i=1}^{2}\Big[\bigcup_{t=0}^{n-\lceil \frac{d-m+1}{2} \rceil} R_t^D(\{u_i\})\Big]\Big| \geq 2\Big(n - \frac{d-m+2}{2} + 1\Big) - n \geq n - d + m.$$

Note that $D \in P_n(d)$ contains exactly d loops, thus there exist at least m loop vertices

$$w_1, w_2, \cdots, w_m \in \bigcap_{i=1}^{2}\Big[\bigcup_{t=0}^{n-\lceil \frac{d-m+1}{2} \rceil} R_t^D(\{u_i\})\Big],$$

such that $u_i \xrightarrow{n-\lceil \frac{d-m+1}{2} \rceil} w_1, w_2, \cdots, w_m$. Therefore, $k_m(D) \leq n - \lceil \frac{d-m+1}{2} \rceil$.

Now we show that the bound can be attained by $L_{n,d}$. Let $l = n - \lceil \frac{d-m+1}{2} \rceil - 1$. Clearly, $k_m(L_{n,d}) \leq l + 1$ by $L_{n,d} \in P_n(d)$. We note that

$$\begin{aligned} &R_l^{L_{n,d}}(\{1\}) = \Big\{n-d+1, n-d+2, \cdots, n - \Big\lceil \frac{d-m+1}{2} \Big\rceil\Big\}, \\ &R_l^{L_{n,d}}\Big(\Big\{n - \Big\lceil \frac{d}{2} \Big\rceil + 1\Big\}\Big) \\ =&\Big\{n - \Big\lceil \frac{d}{2} \Big\rceil + 1, n - \Big\lceil \frac{d}{2} \Big\rceil + 2, \cdots, 2n - \Big\lceil \frac{d-m+1}{2} \Big\rceil - \Big\lceil \frac{d}{2} \Big\rceil\Big\} \pmod n, \end{aligned}$$

therefore vertex 1 and vertex $n - \lceil \frac{d}{2} \rceil + 1$ only have s common preys in $L_{n,d}^{l}$, where

$$s = \left(n - \left\lceil \frac{d-m+1}{2} \right\rceil\right) - \left(n - \left\lceil \frac{d}{2} \right\rceil + 1\right) + 1 = \left\lceil \frac{d}{2} \right\rceil - \left\lceil \frac{d-m+1}{2} \right\rceil < m$$

if

$$\left|2n - \left\lceil \frac{d-m+1}{2} \right\rceil - \left\lceil \frac{d}{2} \right\rceil\right|_n < n - d + 1$$

and

$$\begin{aligned} s =& \left[\left(n - \left\lceil \frac{d-m+1}{2} \right\rceil\right) - \left(n - \left\lceil \frac{d}{2} \right\rceil + 1\right) + 1\right] \\ &+ \left[\left|2n - \left\lceil \frac{d-m+1}{2} \right\rceil - \left\lceil \frac{d}{2} \right\rceil\right|_n - (n-d+1) + 1\right] \\ =& d - 2\left\lceil \frac{d-m+1}{2} \right\rceil \leq m - 1 < m \end{aligned}$$

if

$$\left|2n - \left\lceil \frac{d-m+1}{2} \right\rceil - \left\lceil \frac{d}{2} \right\rceil\right|_n \geq n - d + 1.$$

Hence, $k_m(L_{n,d}) > l$, which implies that

$$k_m(L_{n,d}) = l + 1 = n - \left\lceil \frac{d-m+1}{2} \right\rceil.$$

□

By Theorem 2.2, and taking $m = 1$, we can obtain the following result.

Corollary 2.2[22]**.** *Let $D \in P_n(d)$. Then $k(D) = k_1(D) \leq n - \lceil \frac{d}{2} \rceil$, and the equality holds when $D \cong L_{n,d}$.*

2.2 Primitive Symmetric Digraphs

A symmetric digraph D is a digraph such that for any vertices u and v, (u, v) is an arc if and only if (v, u) is an arc. Let S_n denote the set of primitive symmetric digraphs on n vertices[24]. Clearly, if $D \in S_n$, there must be an odd cycle in D since D is primitive and symmetric.

Lemma 2.3. *Let $D \in S_n$ and C_r be an odd cycle in D with length r. Then for any vertex $j \in V(C(r))$ and $1 \leq t \leq r$, we have $|R_t^D(\{j\})| \geq t + 1$.*

Proof. Let $V(C_r) = \{1, 2, \cdots, r\}$ and $E(C_r) = \{(i, i+1) | 1 \leq i \leq r - 1\} \cup \{(r, 1)\}$. Then we complete the proof by the following two cases.

Case 1. $t < r$.

Then

$$V(C_r) \cap R_t^D(\{j\}) \supseteq \begin{cases} \{j, j+2, \cdots, j+t, j-2, j-4, \cdots, j-t\} \pmod r, & \text{if } t \text{ is even;} \\ \{j+1, j+3, \cdots, j+t, j-1, j-3, \cdots, j-t\} \pmod r, & \text{if } t \text{ is odd.} \end{cases}$$

Thus we always have $|R_t^D(\{j\})| \geq |V(C_r) \cap R_t^D(\{j\})| \geq t + 1$.

Case 2. $t = r$.

Then $V(C_r) \subseteq R_{r-1}^D(\{j\})$ by the same argument in Case 1. Since C_r is an odd cycle, it is easy to see that j can reach all vertices of C_r by a walk with length r, say, $V(C_r) \subseteq R_r^D(\{j\})$.

On the other hand, since $D \in S_n \subseteq P_n$ is strongly connected, there must be a vertex $x \in V(C_r)$ such that $d^+(x) \geq 3$, where $d^+(x)$ denotes the outdegree of x, i.e., there exists a vertex

$w \notin V(C_r)$ but $(x, w) \in E(D)$. Since $x \in V(C_r) \subseteq R^D_{r-1}(\{j\})$, we have $\{w\} \cup V(C_r) \subseteq R^D_r(\{j\})$ and thus $|R^D_r(\{j\})| \geq r + 1 = t + 1$. □

Let G_n be the digraph with the vertex set $V = \{1, 2, \cdots, n\}$ and the arc set $E = \{(i, i+1), (i+1, i) | 1 \leq i \leq n-1\} \cup \{(1, 1)\}$. Then $G_n \in S_n$.

Lemma 2.4. *Let G_n be defined as above and $1 \leq m \leq n$. Then $k_m(G_n) \geq m + n - 2$.*

Proof. Let $t = m + n - 3$. We only need to show $k_m(G_n) > t$.

Case 1. n is odd.

Subcase 1.1. m is odd. Then $R^{G_n}_t(\{n\}) = \{1, 2, \cdots, m-1, m+1, m+3 \cdots, n-3, n-1\}$ and $R^{G_n}_t(\{n-1\}) = \{1, 2, \cdots, m-1, m, m+2, \cdots, n-2, n\}$. So vertex $n-1$ and vertex n have only $m-1$ common preys in G^t_n.

Subcase 1.2. m is even. Then $R^{G_n}_t(\{n\}) = \{1, 2, \cdots, m-1, m+1, m+3, \cdots, n-2, n\}$ and $R^{G_n}_t(\{n-1\}) = \{1, 2, \cdots, m-1, m, m+2, \cdots, n-3, n-1\}$. So vertex $n-1$ and vertex n also have only $m-1$ common preys in G^t_n.

Combining the above two subcases, we have $k_m(G_n) > t$ if n is odd.

Case 2. n is even.

By an argument similar to that in Case 1, we can also obtain $k_m(G_n) > t$. □

Theorem 2.3. *Let $D \in S_n$, C_r be an odd cycle in D with length r. Then*

(1)

$$k_m(D) \leq \begin{cases} n - r + \lfloor \dfrac{m+r-1}{2} \rfloor, & \text{if } 1 \leq m \leq r; \\ n - r + m - 1, & \text{if } r+1 \leq m \leq n. \end{cases}$$

(2) *If $r = 1$, we have $k_m(D) \leq m + n - 2$, and the equality holds when $D \cong G_n$.*

Proof. For any two vertices $x, y \in V(D)$, there exist two vertices $u_1, u_2 \in C_r$, such that $x \stackrel{n-r}{\longrightarrow} u_1$, and $y \stackrel{n-r}{\longrightarrow} u_2$. Let

$$t = \begin{cases} \left\lfloor \dfrac{m+r-1}{2} \right\rfloor, & \text{if } 1 \leq m \leq r; \\ m - 1, & \text{if } r+1 \leq m \leq n. \end{cases}$$

Now we show there exist at least m distinct vertices $v_1, v_2, \cdots, v_m$ such that $u_i \stackrel{t}{\longrightarrow} v_j$ in D for $i = 1, 2$ and $1 \leq j \leq m$.

Case 1. $1 \leq m \leq r$.

Then $t \leq r - 1$. By Lemma 2.3, $|R^D_t(\{u_i\})| \geq |R^D_t(\{u_i\}) \cap V(C_r)| \geq t + 1$ with $i = 1, 2$ and then

$$\left| \bigcap_{i=1}^{2} R^D_t(\{u_i\}) \right| \geq \left| \left[\bigcap_{i=1}^{2} R^D_t(\{u_i\}) \right] \bigcap V(C_r) \right| \geq 2(t+1) - r \geq m.$$

Thus $k_m(D) \leq n - r + \lfloor \frac{m+r-1}{2} \rfloor$ when $1 \leq m \leq r$.

Case 2. $r + 1 \leq m \leq n$.

Firstly, we know $V(C_r) \subseteq R^D_s(\{u_i\})$ for $i = 1, 2$ and $s \geq r - 1$. So

$$\left| \left[\bigcap_{i=1}^{2} R^D_s(\{u_i\}) \right] \bigcap V(C_r) \right| = r.$$

We only need show there exist at least $m - r$ distinct vertices $v_1, v_2, \cdots, v_{m-r} \in V(D) \setminus V(C_r)$ such that $u_i \stackrel{m-1}{\longrightarrow} v_j$ in D for $i = 1, 2$ and $1 \leq j \leq m - r$.

Since $D \in S_n \subseteq P_n$ is strongly connected, there must be a vertex $x \in V(C_r)$ such that $d^+(x) \geq 3$, say, there exists a vertex $w \notin V(C_r)$ but $(x,w) \in E(D)$. Let $D(V_1)$ denote the induced graph of $V_1 \subseteq V(D)$ in D. Let l be the length of the longest path of $D \setminus E(D(V(C_r))))$.

Subcase 2.1. $l \geq m-r$.

Then there exists some vertex in $V(C_r)$, say, $u \in V(C_r)$, and at least $m-r$ distinct vertices $v_1, v_2, \cdots, v_{m-r} \in V(D) \setminus V(C_r)$ such that $u \overset{m-r}{\longrightarrow} v_j$ or $u \overset{m-r-1}{\longrightarrow} v_j$ in D for $1 \leq j \leq m-r$. Let $v \in N(u) \cap V(C_r)$ where $N(u)$ be the neighbors of u in D. Thus there exists a walk of length $m-1 = (r-1)+(m-r)$ from u_i to v_j by $u_i \overset{r-1}{\longrightarrow} u \overset{m-r}{\longrightarrow} v_j$ or $u_i \overset{r-1}{\longrightarrow} v \overset{1}{\longrightarrow} u \overset{m-r-1}{\longrightarrow} v_j$ for $i = 1,2$ and $j \in \{1,2,\cdots,m-r\}$.

Subcase 2.2. $l < m-r$.

We can show $V(D) \setminus V(C_r) \subseteq R^D_{m-1}(\{u_i\}))$ by the similar proof of Subcase 2.1, so we omit it.

Combining the above arguments, we know $k_m(D) \leq n-r+m-1$ when $r+1 \leq m \leq n$.

When $r = 1$, we have $k_m(D) \leq m+n-2$ by (1) and the equality holds when $D \cong G_n$ by Lemma 2.4. □

By Theorem 2.3, we have

Corollary 2.3[8,22,24]**.** *Let $D \in S_n$ and C_r be an odd cycle in D with length r. Then $k(D) = k_1(D) \leq n - \frac{r+1}{2}$ and $\exp(D) = k_n(D) \leq 2n-2$.*

2.3 r-Indecomposable Digraphs

Let r, n be integers with $1 \leq r < n$. An $n \times n$ Boolean matrix A is called r-indecomposable (shortly, r-inde) if it contains no $k \times l$ zero submatrix with $k+l = n-r+1$. If A is r-inde, $D(A)$ is said to be r-inde. Let $B_{n,r}$ be the set of all r-inde digraphs on n vertices. It is known that $B_{n,r+1} \subseteq B_{n,r}$ $(1 \leq r \leq n-1)$, and every r-inde digraph $(r \geq 1)$ is primitive[29]. $B_{n,1} = F_n$ is also called the set of fully indecomposable digraphs on n vertices. A nearly decomposable digraph D is a digraph such that $D \in F_n$, and $D - e \notin F_n$ for any $e \in E(D)$. Let NF_n be the set of all nearly decomposable digraphs on n vertices.

Lemma 2.5[27]**.** *Let r be an integer with $1 \leq r < n$. Then $D \in B_{n,r}$ if and only if $|R^D_1(X)| \geq |X| + r$ for any $X \subseteq V(D)$ with $1 \leq |X| \leq n-1$.*

Let H_n be the digraph with vertex set $V = \{1,2,\cdots,n\}$ and the arc set $E = \{(i,i+1),(i+1,i) | 1 \leq i \leq n-1\} \cup \{(1,1),(n,n)\}$. Clearly, $H_n \in NF_n \subseteq F_n$.

Theorem 2.4. *Let $1 \leq m \leq n$, $r \geq 1$ and $D \in B_{n,r}$. Then $k_m(D) \leq \lceil \frac{1}{r} \lfloor \frac{m+n-1}{2} \rfloor \rceil$.*

Proof. Let $t = \lceil \frac{1}{r} \lfloor \frac{m+n-1}{2} \rfloor \rceil$. By Lemma 2.5, for any two vertices $u_1, u_2 \in V(D)$,

$$|R^D_t(\{u_i\})| \geq |R^D_{t-1}(\{u_i\})| + r \geq \cdots \geq |\{u_i\}| + tr \geq 1 + \left\lfloor \frac{m+n-1}{2} \right\rfloor \quad \text{with} \quad i = 1,2.$$

Hence,

$$\Big|\bigcap_{i=1}^{2} R^D_t(\{u_i\})\Big| \geq 2\Big(1 + \Big\lfloor \frac{m+n-1}{2} \Big\rfloor\Big) - n \geq m.$$

So there exist $w_1, w_2, \cdots, w_m \in V(D)$ such that $u_i \overset{t}{\longrightarrow} w_1, w_2, \cdots, w_m$. Therefore, $k_m(D) \leq \lceil \frac{1}{r} \lfloor \frac{m+n-1}{2} \rfloor \rceil$. □

Corollary 2.4[24,29]**.** *Let $r \geq 1$ and $D \in B_{n,r}$. Then*

$$k(D) = k_1(D) \leq \left\lceil \frac{1}{r} \left\lfloor \frac{n}{2} \right\rfloor \right\rceil, \qquad \exp(D) = k_n(D) \leq \left\lceil \frac{n-1}{r} \right\rceil.$$

It is not difficult to verify that $k_m(H_n) = \lfloor \frac{m+n-1}{2} \rfloor$. Then by Theorem 2.4, we have

Theorem 2.5. *Let $D \in F_n$ or $D \in NF_n$ and $1 \le m \le n$. Then $k_m(D) \le \lfloor \frac{m+n-1}{2} \rfloor$, and equality holds when $D \cong H_n$.*

2.4 Primitive Cayley Digraphs

Let G be a multiplicative group of order n with identity element e, and let $A = \{a_1, a_2, \cdots a_p\}$ be a subset of G. The (right) Cayley digraph[29] is the digraph Cay $(G, A) = (V, E)$, where $V = G$ and $E = \{(x, y)|x^{-1}y \in A\}$.

Lemma 2.6[29]**.** *Let $A = \{a_1, a_2, \cdots, a_p\}$ $(1 \le p \le n)$ be a subset of an Abelian group G. If Cay (G, A) is primitive, then Cay (G, A) is $\lceil \frac{p}{2} \rceil$-inde.*

By Theorem 2.4 and Lemma 2.6, we have

Corollary 2.5. *Let $A = \{a_1, a_2, \cdots a_p\}$ $(1 \le p \le n)$ be a subset of an Abelian group G and $1 \le m \le n$. If Cay (G, A) is primitive, then*

$$k_m(Cay\,(G, A)) \le \left\lceil \frac{1}{\lceil \frac{p}{2} \rceil} \left\lfloor \frac{m+n-1}{2} \right\rfloor \right\rceil \le \left\lfloor \frac{m+n-1}{2} \right\rfloor.$$

Corollary 2.6[22,29]**.** *Let $A = \{a_1, a_2, \cdots, a_p\}$ $(1 \le p \le n)$ be a subset of an Abelian group G. If Cay (G, A) is primitive, then*

$$k(Cay(G, A)) = k_1(Cay(G, A)) \le \left\lceil \frac{\lfloor \frac{n}{2} \rfloor}{\lceil \frac{p}{2} \rceil} \right\rceil,$$
$$\exp(Cay(G, A)) = k_n(Cay(G, A)) \le \left\lceil \frac{n-1}{\lceil \frac{p}{2} \rceil} \right\rceil.$$

Let $P = (p_{ij}) \in B_n$ be the permutation matrix with $p_{i,i+1} = p_{n,1} = 1$ for $i = 1, \cdots, n-1$. A primitive circulant matrix $C = C\langle a_1, a_2, \cdots, a_p; n\rangle$ is a matrix of the form $C = P^{a_1} + \cdots + P^{a_p}$, where $0 \le a_1 < \cdots < a_p < n$ and $p \ge 2$. It is known that $D(C)$ is Cay $(Z_n, \{a_1, a_2, \cdots, a_p\})$ (see [29]). Let CP_n denote the set of all primitive circulant digraphs on n vertices. It is easy to see that $L_{n,n} \in CP_n$ and $k_m(L_{n,n}) = n - \lceil \frac{n-m+1}{2} \rceil = \lfloor \frac{m+n-1}{2} \rfloor$ by Theorem 2.2. Then by Corollary 2.5 and the above arhuments, we have

Corollary 2.7. *Let $D \in CP_n$ and $1 \le m \le n$. Then $k_m(D) \le \lfloor \frac{m+n-1}{2} \rfloor$, and equality holds when $D \cong L_{n,n}$.*

Corollary 2.8[22]**.** *Let $D \in CP_n$. Then $\exp(CP_n) = k_n(CP_n) \le n - 1$.*

3 Appendix: Primitive Digraphs with Girth s

In [12], Kim studied the generalized competition index of the primitive digraphs with girth s and showed the following Theorem 2.6. Now we will obtain a weaker result by a different method.

Theorem 2.6. *Let $D \in P_n$ with girth s and $1 \le m \le n$. Then*

$$k_m(D) \le \begin{cases} n + \left(\dfrac{m+n-4}{2}\right)s, & \text{if } n+m \text{ is even;} \\ n - 1 + \left(\dfrac{m+n-3}{2}\right)s, & \text{if } n+m \text{ is odd.} \end{cases}$$

Theorem 2.7. *Let $D \in P_n$ with girth s and $1 \le m \le n$. Then $k_m(D) \le n - s + s\lfloor \frac{m+n-1}{2} \rfloor$.*

Proof. Let C_s be a directed cycle of length s in D and x, y be any two different vertices in $V(D)$. Then there exist vertices u_1 and u_2 of C_s such that $x \stackrel{n-s}{\longrightarrow} u_1$ and $y \stackrel{n-s}{\longrightarrow} u_2$. Since D is primitive, the digraph D^s is primitive as well. Also u_1 and u_2 are loop vertices in D^s. By Lemma 2.1,

$$\left|R^{D^s}_{\lfloor \frac{m+n-1}{2} \rfloor}(\{u_i\})\right| \ge \min\left\{\left\lfloor \frac{m+n-1}{2} \right\rfloor + 1, n\right\} = \left\lfloor \frac{m+n-1}{2} \right\rfloor + 1$$

for $i = 1, 2$. Thus

$$\left|\bigcap_{i=1}^{2} R^{D^s}_{\lfloor \frac{m+n-1}{2} \rfloor}(\{u_i\})\right| \ge 2\Big(\left\lfloor \frac{m+n-1}{2} \right\rfloor + 1\Big) - n \ge m.$$

This implies that there exist m different vertices $v_1, v_2, \cdots, v_m \in V(D^s)$ such that

$$u_i \stackrel{\lfloor \frac{m+n-1}{2} \rfloor}{\longrightarrow} v_1, v_2, \cdots, v_m \quad \text{in} \quad D^s.$$

Hence, $x, y \stackrel{n-s}{\longrightarrow} u_i \stackrel{s\lfloor \frac{m+n-1}{2} \rfloor}{\longrightarrow} v_1, v_2, \cdots, v_m$ in D and thus $k_m(D) \le n - s + s\lfloor \frac{m+n-1}{2} \rfloor$. □

Remark 2.1. Comparing Kim's result and Theorem 2.7, we can see the two results are the same when $n + m$ is even, and Kim's result is better than Theorem 2.7 when $n + m$ is odd by the fact $n - s + s\lfloor \frac{m+n-1}{2} \rfloor - [n - 1 + (\frac{m+n-3}{2})s] = 1$. But the two proofs of the above two results are unique and very concise.

By Theorem 2.7 and the facts $\exp(D) = k_n(D)$ and the girth $s \le n - 1$ since $D \in P_n$, we have the following corollary immediately.

Corollary 2.9[4]**.** *Let $D \in P_n$, we have*

(1) *Let s be the girth of D. Then* $\exp(D) = k_n(D) \le n + (n-2)s$.

(2) $\exp(D) \le n^2 - 2n + 2$.

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