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Existence of Positive Solutions of Second-order Periodic Boundary Value Problems with Sign-Changing Green's Function

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Abstract In this paper, we consider the existence of positive solutions of second-order periodic boundary value problem

$$u'' + \Big(\frac{1}{2}+\varepsilon\Big)^2 u = \lambda g(t) f(u), \qquad t\in[0,2\pi],\quad u(0)=u(2\pi),\quad u'(0)=u'(2\pi),$$

where $0<\varepsilon<\frac{1}{2}$, $g:[0,2\pi]\to\mathbb{R}$ is continuous, $f:[0,\infty)\to\mathbb{R}$ is continuous and $\lambda>0$ is a parameter.

Keywords periodic boundary value problem; existence; positive solutions; sign-changing Green's functions
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1 Introduction

In recent years, nonlinear periodic boundary value problems have been widely studied by many authors, see, for instance, [1,3–8] and the references therein. However, in most of these papers, the existence of positive solution was mainly dependent on the positivity of Green's function (see, for instance [3–8]). In particular, by using Krasnosel'skii's fixed point theorem, Jiang[4], Zhang and Wang[8] discussed the existence and multiplicity of positive solutions to the periodic boundary value problem

$$\begin{cases} u''+\rho^2 u = f(t,u), \qquad 0<t<2\pi, \qquad 0<\rho<\frac{1}{2},\\ u(0)=u(2\pi), \qquad u'(0)=u'(2\pi). \end{cases}$$

One can see $0<\rho<\frac{1}{2}$ is the optimal condition to guarantee the Green's function $K(t,s)>0$, since when $\rho=\frac{1}{2}$, the minimum of $K(t,s)$ is zero, where

$$K(t,s)=\begin{cases} \dfrac{\sin\rho(t-s)+\sin\rho(2\pi-t+s)}{2\rho(1-\cos 2\rho\pi)}, & 0\le s\le t\le 2\pi,\\ \dfrac{\sin\rho(s-t)+\sin\rho(2\pi-s+t)}{2\rho(1-\cos 2\rho\pi)}, & 0\le t\le s\le 2\pi, \end{cases} \tag{1.1}$$

for $\rho\ne 1,2,\cdots$. Thus, under the condition $0<\rho<\frac{1}{2}$, Krasnosel'skii's fixed point theorem can be used to prove the existence and multiplicity of positive solutions.

In 2009, under the assumption that the Green's function has zero points, Graef, Kong and Wang[1] studied the existence of positive solutions to the periodic boundary value problems

$$\begin{cases} u''+a(t)u=\widetilde{g}(t)f(u), \qquad 0<t<2\pi,\\ u(0)=u(2\pi), \qquad u'(0)=u'(2\pi), \end{cases} \tag{1.2}$$

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where $a : [0, 2\pi] \to [0, \infty)$ is continuous, $\widetilde{g} : [0, 2\pi] \to [0, \infty)$ is continuous and $\eta = \min_{t\in[0,2\pi]} \widetilde{g}(t) > 0$.

In [2], for suitable $\lambda > 0$, D. D. Hai considered the existence of positive solutions to the following problems

$$\begin{cases} \Delta u + \lambda a(t) f(u) = 0, & t \in \Omega, \\ u = 0, & t \in \partial\Omega, \end{cases} \tag{1.3}$$

where a is continuous, changes its sign on Ω and also satisfies

$$\int_{\Omega} a^{+}(s) g(t,s) ds > k \int_{\Omega} a^{-}(s) g(t,s)\, ds$$

for some constant $k > 1$, and $g(t, s)$ is the Green's function of (1.3). It is worth noting that the Green's function $g(t, s)$ of (1.3) is positive on Ω.

Motivated by the above papers, we want to discuss the existence of positive solutions to second-order periodic boundary value problems

$$\begin{cases} u'' + \left(\frac{1}{2} + \varepsilon\right)^2 u = \lambda g(t) f(u), & 0 < t < 2\pi, \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \end{cases} \tag{1.4}$$

where $0 < \varepsilon < \frac{1}{2}$ is a constant and $\lambda > 0$ is a parameter, $g : [0, 2\pi] \to \mathbb{R}$ is continuous and changes its sign.

The Green's function of

$$\begin{cases} u'' + \left(\frac{1}{2} + \varepsilon\right)^2 u = 0, & 0 < t < 2\pi, \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi) \end{cases} \tag{1.5}$$

can be expressed by

$$G(t,s) = \begin{cases} \dfrac{\sin(\frac{1}{2}+\varepsilon)(t-s) + \sin(\frac{1}{2}+\varepsilon)(2\pi - t + s)}{2 \times (\frac{1}{2}+\varepsilon)(1 - \cos 2(\frac{1}{2}+\varepsilon)\pi)}, & 0 \le s \le t \le 2\pi, \\ \dfrac{\sin(\frac{1}{2}+\varepsilon)(s-t) + \sin(\frac{1}{2}+\varepsilon)(2\pi - s + t)}{2 \times (\frac{1}{2}+\varepsilon)(1 - \cos 2(\frac{1}{2}+\varepsilon)\pi)}, & 0 \le t \le s \le 2\pi, \end{cases} \tag{1.6}$$

see [1]. By direct computing, one can see the Green's function $G(t, s)$ defined in (1.6) changes its sign on $[0, 2\pi] \times [0, 2\pi]$, and this will be done in Section 2.

To get the existence of positive solutions of (1.4), we make the following assumptions:

(H1) $f : [0, \infty) \to \mathbb{R}$ is continuous and $f(0) > 0$.

(H2) $g : [0, 2\pi] \to \mathbb{R}$ is continuous, $g \not\equiv 0$, and there exists a number $k > 1$ such that

$$\int_0^{2\pi} (G(t,s)g(s))^{+} ds \ge k \int_0^{2\pi} (G(t,s)g(s))^{-} ds, \qquad t \in [0, 2\pi],$$

where

$$(G(t,s)g(s))^{+} = \begin{cases} G(t,s)g(s), & G(t,s)g(s) \ge 0, \\ 0, & G(t,s)g(s) < 0, \end{cases} \quad \text{for } t \in [0, 2\pi],$$

$$(G(t,s)g(s))^{-} = \begin{cases} -G(t,s)g(s), & G(t,s)g(s) \le 0, \\ 0, & G(t,s)g(s) > 0, \end{cases} \quad \text{for } t \in [0, 2\pi].$$

The main result of this paper is:

Theorem 1.1. *Suppose that (H1), (H2) hold. Then there exists a positive number λ^* such that* (1.4) *has a positive solution for $\lambda < \lambda^*$.*

The rest of this paper is organized as follows. In Section 2, we will discuss the properties of $G(t,s)$ in (1.6) and give some notations and preliminary results. Finally, in Section 3, we will prove Theorem 1.1.

2 Preliminaries

Let $C[0,2\pi] := \{u|u$ is continuous on $[0,2\pi]\}$, and the norm of $C[0,2\pi]$ is the maximum norm $\|u\|_0 = \max\limits_{t\in[0,2\pi]} |u(t)|$. Throughout the paper, we assume that $f(u) = f(0)$ for $u \le 0$.

Lemma 2.1. *The Green's function $G(t,s)$ defined in (1.6) satisfied the following properties:*

(i) $G(t,s)$ changes its sign on $[0,2\pi]\times[0,2\pi]$, specifically, it can be expressed by the following graph.

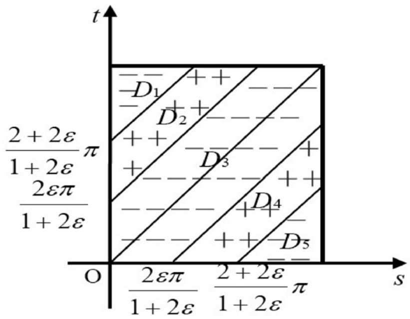

Fig.1.

In the above graph, '+' denotes $G(t,s) > 0$ on this interior, '−' denotes $G(t,s) < 0$ on this interior.

(ii) $\int_0^{2\pi} G(t,s)\,ds > 0$.

Proof. For convenience, let $\Delta = 2\times\left(\frac{1}{2}+\varepsilon\right)\left(1-\cos 2\left(\frac{1}{2}+\varepsilon\right)\pi\right)$. Then $\Delta > 0$, since $0 < \varepsilon < \frac{1}{2}$.

Now, this proof will be divided into two cases.

Case 1. $s \le t$, then

$$-\Big(\frac{1}{2}+\varepsilon\Big)\pi \le \Big(\frac{1}{2}+\varepsilon\Big)(\pi - t + s) \le \Big(\frac{1}{2}+\varepsilon\Big)\pi$$

and

$$\begin{aligned} G(t,s) =& \frac{1}{\Delta}\Big(\sin\Big(\frac{1}{2}+\varepsilon\Big)(t-s) + \sin\Big(\frac{1}{2}+\varepsilon\Big)(2\pi - t + s)\Big) \\ =& \frac{2}{\Delta}\sin\Big(\frac{1}{2}+\varepsilon\Big)\pi \times \cos\Big(\Big(\frac{1}{2}+\varepsilon\Big)(\pi - t + s)\Big). \end{aligned}$$

It is not difficult to see that $\sin\left(\frac{1}{2}+\varepsilon\right)\pi > 0$ for $0 < \varepsilon < 1/2$. Now, the proof can be completed in the following three cases.

Case 1.1. $G(t,s) = 0$ if and only if $\cos\left((\frac{1}{2}+\varepsilon)(\pi - t + s)\right) = 0$, i.e., t, s satisfy

$$\Big(\frac{1}{2}+\varepsilon\Big)(\pi - t + s) = -\frac{\pi}{2}, \qquad \text{or} \qquad \Big(\frac{1}{2}+\varepsilon\Big)(\pi - t + s) = \frac{\pi}{2},$$

which implies t, s satisfy

$$t-s=\frac{(2+2\varepsilon)\pi}{1+2\varepsilon}, \qquad \text{or} \qquad t-s=\frac{2\varepsilon\pi}{1+2\varepsilon}.$$

Case 1.2. $G(t,s)<0$ if and only if $\cos\left(\left(\frac{1}{2}+\varepsilon\right)(\pi-t+s)\right)<0$. Furthermore,

$$-\left(\frac{1}{2}+\varepsilon\right)\pi<\left(\frac{1}{2}+\varepsilon\right)(\pi-t+s)<-\frac{\pi}{2}, \qquad \text{or} \qquad \frac{\pi}{2}<\left(\frac{1}{2}+\varepsilon\right)(\pi-t+s)<\left(\frac{1}{2}+\varepsilon\right)\pi.$$

Thus, $G(t,s)<0$ if and only if

$$\frac{2+2\varepsilon}{1+2\varepsilon}\pi<t-s<2\pi, \qquad \text{or} \qquad 0<t-s<\frac{2\varepsilon\pi}{1+2\varepsilon},$$

which implies $(t,s)\in D_1$ or $(t,s)\in D_3$.

Case 1.3. $G(t,s)>0$ if and only if $\cos\left(\left(\frac{1}{2}+\varepsilon\right)(\pi-t+s)\right)>0$. Furthermore,

$$-\frac{\pi}{2}<\left(\frac{1}{2}+\varepsilon\right)(\pi-t+s)<\frac{\pi}{2}.$$

Thus, $G(t,s)>0$ if and only if

$$\frac{2\varepsilon\pi}{1+2\varepsilon}<t-s<\frac{2+2\varepsilon}{1+2\varepsilon}\pi,$$

which implies $(t,s)\in D_2$.

Case 2. $t\le s$.

By using the similar methods, we can get that

Case 2.1. If t, s satisfy $s-t=\frac{(2+2\varepsilon)\pi}{1+2\varepsilon}$ or $s-t=\frac{2\varepsilon\pi}{1+2\varepsilon}$, then $G(t,s)=0$.

Case 2.2. If $(t,s)\in D_3$ or $(t,s)\in D_5$, then $G(t,s)<0$;

Case 2.3. If $(t,s)\in D_4$, then $G(t,s)>0$.

(ii)

$$\begin{aligned}\int_0^{2\pi}G(t,s)\,ds=&\frac{1}{\Delta}\int_0^t\sin\left(\frac{1}{2}+\varepsilon\right)(t-s)\,ds+\frac{1}{\Delta}\int_0^t\sin\left(\frac{1}{2}+\varepsilon\right)(2\pi-t+s)\,ds\\&+\frac{1}{\Delta}\int_t^{2\pi}\sin\left(\frac{1}{2}+\varepsilon\right)(s-t)\,ds+\frac{1}{\Delta}\int_t^{2\pi}\sin\left(\frac{1}{2}+\varepsilon\right)(2\pi-s+t)\,ds\\=&\frac{8}{(1+2\varepsilon)\Delta}\sin^2\left(\frac{1}{2}+\varepsilon\right)\pi>0.\end{aligned}$$

□

For $u\in C[0,2\pi]$, define the operator T by $Tu(t)=\lambda\int_0^{2\pi}(G(t,s)g(s))^+f(u(s))\,ds$. It's not difficult to see that $T:C[0,2\pi]\to C[0,2\pi]$ is completely continuous.

Lemma 2.2. *Let $0<\delta<1$. Then there exists a positive number $\overline{\lambda}$ such that, for $0<\lambda<\overline{\lambda}$, the equation $u(t)=Tu(t)$ has a positive solution $\widetilde{u}_\lambda$ with $\|\widetilde{u}_\lambda\|_0\to 0$ as $\lambda\to 0$, and $\widetilde{u}_\lambda(t)\ge\lambda\delta f(0)p(t)$, $t\in[0,2\pi]$, where $p(t)=\int_0^{2\pi}(G(t,s)g(s))^+ds$.*

Proof. We shall apply the Leray-Schauder fixed point theorem to prove that T has a fixed point for λ small. Let $\eta>0$ be such that $f(u)\ge\delta f(0)$, for $0\le u\le\eta$. Suppose that $\lambda<\eta/2\|p\|_0\widetilde{f}(\eta)$, where $\widetilde{f}(t)=\max_{0\le s\le t}f(s)$. Then there exists $A_\lambda\in(0,\eta)$ such that

$$\frac{\widetilde{f}(A_\lambda)}{A_\lambda}=\frac{1}{2\lambda\|p\|_0}.$$

Let $u \in C[0, 2\pi]$ and $\theta \in (0, 1)$ such that $u = \theta Tu$. Then we have

$$\|u\|_0 \le \lambda \|p\|_0 \widetilde{f}(\|u\|_0) \qquad \text{or} \qquad \frac{\widetilde{f}(\|u\|_0)}{\|u\|_0} \ge \frac{1}{\lambda \|p\|_0},$$

which implies that $\|u\|_0 \ne A_\lambda$. Note that $A_\lambda \to 0$ as $\lambda \to 0$. By the Leray-Schauder fixed point theorem, T has a fixed point $\widetilde{u}_\lambda$ with $\|\widetilde{u}_\lambda\|_0 \le A_\lambda < \eta$. Consequently, $\widetilde{u}_\lambda(t) \ge \lambda \delta f(0) p(t)$, $t \in [0, 2\pi]$, and the proof is completed. □

3 Proof of the Main Result

Proof of Theorem 1.1. Let $q(t) = \int_0^{2\pi} (G(t,s)g(s))^- ds$. By (H2), there exist two positive numbers $\alpha, \gamma \in (0, 1)$ such that

$$q(t)|f(s)| \le \gamma p(t) f(0) \tag{3.1}$$

for $s \in [0, \alpha]$, $t \in [0, 2\pi]$. Fix $\delta \in (\gamma, 1)$ and let $\lambda^* > 0$ such that

$$\|\widetilde{u}_\lambda\|_0 + \lambda \delta f(0) \|p\|_0 \le \alpha \tag{3.2}$$

for $\lambda < \lambda^*$, where $\widetilde{u}_\lambda$ is given by Lemma 2.2, and

$$|f(x) - f(y)| \le f(0) \Big(\frac{\delta - \gamma}{2} \Big) \tag{3.3}$$

for $x, y \in [-\alpha, \alpha]$ with $|x - y| \le \lambda^* \delta f(0) \|p\|_0$.

Let $\lambda < \lambda^*$. We look for a solution u_λ of (1.4) of the form $\widetilde{u}_\lambda + v_\lambda$. Thus, v_λ satisfies

$$v_\lambda(t) = \lambda \int_0^{2\pi} G(t,s) g(s) f(\widetilde{u}_\lambda + v_\lambda) - \lambda \int_0^{2\pi} (G(t,s)g(s))^+ f(\widetilde{u}_\lambda)\, ds, \qquad t \in [0, 2\pi].$$

For each $w \in C[0, 2\pi]$, let $v = Hw$ be the solution of

$$v_\lambda = \lambda \int_0^{2\pi} G(t,s) g(s) f(\widetilde{u}_\lambda + w)\, ds - \lambda \int_0^{2\pi} (G(t,s)g(s))^+ f(\widetilde{u}_\lambda)\, ds.$$

Then $H : C[0, 2\pi] \to C[0, 2\pi]$ is completely continuous. Let $v \in C[0, 2\pi]$ and $\theta \in (0, 1)$ such that $v = \theta Hv$. Then we have

$$v = \theta\lambda \int_0^{2\pi} G(t,s) g(s) f(\widetilde{u}_\lambda + v)\, ds - \theta\lambda \int_0^{2\pi} (G(t,s)g(s))^+ f(\widetilde{u}_\lambda)\, ds.$$

We claim that $\|v\|_0 \ne \lambda \delta f(0) \|p\|_0$. Suppose on the contrary that $\|v\|_0 = \lambda \delta f(0) \|p\|_0$. Then, by (3.2) and (3.3), we obtain $\|\widetilde{u}_\lambda + v\|_0 \le \|\widetilde{u}_\lambda\|_0 + \|v\|_0 \le \alpha$ and $|f(\widetilde{u}_\lambda + v) - f(\widetilde{u}_\lambda)| \le f(0) \frac{\delta - \gamma}{2}$, which together with (3.1) implies that

$$|v(t)| \le \lambda \frac{\delta - \gamma}{2} f(0) p(t) + \lambda \gamma f(0) p(t) = \lambda \frac{\delta + \gamma}{2} f(0) p(t), \qquad t \in [0, 2\pi]. \tag{3.4}$$

In particular $\|v\|_0 \le \lambda \frac{\delta+\gamma}{2} f(0) \|p\|_0 < \lambda \delta f(0) \|p\|_0$, a contradiction, and the claim is proved. By the Leray-Schauder fixed point theorem, H has a fixed point v_λ with $\|v_\lambda\|_0 \le \lambda \delta f(0) \|p\|_0$. Hence v_λ satisfies (3.4) and, using Lemma 2.2, we obtain $u_\lambda(t) \ge \widetilde{u}_\lambda(t) - |v_\lambda(t)| \ge \lambda \delta f(0) p(t) - \lambda \frac{\delta+\gamma}{2} f(0) p(t) = \lambda \frac{\delta-\gamma}{2} f(0) p(t)$, i.e., u_λ is a positive solution of (1.4). This completes the proof of Theorem 1.1. □

4 Examples

Example 4.1. For application, we consider the following second-order periodic boundary value problem

$$\begin{cases} u'' + (\frac{1}{2}+\varepsilon)^2 u = \lambda(u+1), & t \in [0, 2\pi], \\ u(0) = u(2\pi), & u'(0) = u'(2\pi). \end{cases} \tag{4.1}$$

It is not difficult to see that f satisfies (H1). On the other hand, by Lemma 2.1, we get $\int_0^{2\pi} G(t,s)ds > 0$. So, by theorem 1.1, there exists a positive number λ^* such that (4.1) has a positive solution for $\lambda < \lambda^*$.

It is worth to note that we can choose $\lambda^* = (1/2+\varepsilon)^2$ in this case. In fact, by using the method of the variation of constant, we get

(i) if $\lambda = (1/2+\varepsilon)^2$, then (4.1) doesn't have a periodic solution.

(ii) if $\lambda \in (0, (1/2+\varepsilon)^2)$, then (4.1) has only one positive solution $u = \lambda/((1/2+\varepsilon)^2 - \lambda)$ on $[0, 2\pi]$.

(iii) if $\lambda > (1/2+\varepsilon)^2$, then (4.1) has only one negative solution $u = \lambda/((1/2+\varepsilon)^2 - \lambda)$ on $[0, 2\pi]$.

Example 4.2. Consider the following second order periodic boundary value problem

$$\begin{cases} u'' + \left(\frac{1}{2}+\varepsilon\right)^2 u = \lambda(u^2+1), & t \in [0, 2\pi], \\ u(0) = u(2\pi), & u'(0) = u'(2\pi). \end{cases} \tag{4.2}$$

It is not difficult to see that f satisfies (H1). On the other hand, by Lemma 2.1, we get $\int_0^{2\pi} G(t,s)\,ds > 0$. So, by Theorem 1.1, there exists a positive number λ^* such that (4.2) has a positive solution for $\lambda < \lambda^*$.

Under this case, we can take $\lambda^* = 1/2 \times \left(\frac{1}{2}+\varepsilon\right)^2$. This can be obtained by the Proposition 3.1, Lemma 3.1, Lemma 3.2 of [3].

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