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A Short Note on the Derivation of the Elastic Von Kármán Shell Theory

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Abstract We derive the Γ-limit of scaled elastic energies $h^{-4}E^h(u^h)$ associated with deformations u^h of a family of thin shells $S^h = \{z = x + t\vec{n}(x);\ x \in S,\ -g_1^h(x) < t < g_2^h(x)\}$. The obtained von Kármán theory is valid for a general sequence of boundaries g_1^h, g_2^h converging to 0 in an appropriate manner as h vanishes. Our analysis relies on the techniques and extends the results in [10] and [11].

Keywords nonlinear elasticity; Γ convergence; calculus of variations
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1 Introduction

In the context of *Mathematical Theory of Elasticity*, the derivation of thin shell models is a fundamental question with a long history[1,13]. Despite a large body of engineering literature, relatively little is known about the mathematical rigorous justification of various plate and shell theories. Of even more concern is that some of the existing theories seem to be incompatible with each other. Recently, substantial analytical progress has been made possible due to the seminal work of Frieseke, James and Müller[5,6], followed by other observations and results. The novel approach is based on refined methods in Calculus of Variations (notably the so-called Γ-convergence).

Given a 2-dimensional surface S in $\mathbb{R}^3$, define a family of thin shells

$$S^h = \big\{z = x + t\vec{n}(x);\ x \in S,\ -g_1^h(x) < t < g_2^h(x)\big\},$$

where $\vec{n}(x)$ is the unit normal to S at the point x, and g_1^h, g_2^h are two sequences of positive functions representing the shell's boundary. The total energy of a deformation $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ is given by

$$J^h(u^h) = \frac{1}{h}\int_{S^h} W(\nabla u^h) - \frac{1}{h}\int_{S^h} f^h \cdot u^h, \tag{1.1}$$

where $W : \mathbb{R}^{3\times 3} \to \mathbb{R}_+$ is the elastic energy density function, satisfying physically relevant conditions, and $f^h \in L^2(S^h, \mathbb{R}^3)$ represents an external force acting on the shell.

Given (1.1), now one wants to study the asymptotic behavior of J^h as $h \to 0$. Since the condition for W generally imply that the first term in $J^h(u^h)$ is non-convex in its argument u^h, while the second term in $J^h(u^h)$ is linear, the main variational analysis concerns the elastic energy

$$E^h(u^h) = \frac{1}{h}\int_{S^h} W(\nabla u^h).$$

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and it attempts computing the Γ-limit I_β of the sequence of scaled energies $h^{-\beta}E^h$, where the exponent β is determined by the scaling α of the external force f^h. Namely, one can prove that if $f^h \sim h^\alpha$, the elastic energy $E^h(u^h)$ at minimizers of J^h scales like h^β, where $beta = \alpha$ if $0 \leq \alpha \leq 2$, and $\beta = 2\alpha - 2$ if $\alpha \geq 2$. The limiting energy I_β to be determined plays hence the role of the 2d counterpart of the 3d energy functionals E^h, which is ensured by the fundamental property of Γ-convergence.

Recall[3,4], that a sequence of functionals $F_n : X \to [-\infty, +\infty]$ defined on a metric space X Γ-converges to the limit functional $F : X \to [-\infty, +\infty]$ whenever

(i) (Lower bound) For any converging sequence $x_n \to x$ in X, one has:

$$F(x) \leq \liminf_{n\to\infty} F_n(x_n).$$

(ii) (Recovery sequence) For any $x \in X$, there exists a sequence x_n converging to x, such that $\lim_{n\to\infty F_n(x_n)=F(x)}$.

The fundamental property of Γ-convergence is the following. If x_n is a sequence of approximate minimizers of F_n in X

$$\lim_{n\to\infty} \left\{ F_n(x_n) - \inf_X F_n \right\} = 0, \tag{1.2}$$

and if $x_n \to x$, then x is a minimizer of F. In turn, any recovery sequence associated to a minimizer of F is an approximate minimizing sequence for F_n. In the context of shells (plates), the sequence of minimizing deformations for J^h can be hence recovered from minimizers of the Γ-limit of $h^{-\beta}J^h$ whose crucial term is provided by I_β.

When S^h is a plate with uniform thickness ($S \subset \mathbb{R}^2$ and $g_1^h = g_2^h = h/2$), such Γ-convergence was first established by LeDret and Raoult[8] for $\beta = 0$, then by Friesecke, James and Müller[5,6] for all $\beta \geq 2$. In the case of $0 < \beta < 5/3$, the related result was obtained by Conti and Maggi[2]. The regime $5/3 \leq \beta < 2$ remains open and is proposed to be relevant to crumpling of elastic sheets.

If S^h is a shell with uniform thickness (S is an arbitrary surface and $g_1^h = g_2^h = h/2$), the Γ-convergence was first obtained in [9] for $\beta = 0$. The model is that of a *membrane shell* and the limit I_0 depends only on the stretching and shearing produced by the deformation on the surface S. Another study is due to Friesecke, James, Mora and Müller in [7], who analyzed the case $\beta = 2$. This scaling corresponds to a *flexural shell model*, where the only admissible deformations are those preserving the metric on S. The energy I_2 depends then on the change of curvature produced by the deformation. Further, Lewicka, Mora and Pakzad studied the situation $\beta \geq 4$ in [10]. For $\beta = 4$, the Γ-limit obtained therein is a generalization of the *von Kármán theory* for plates, which for $\beta > 4$ reduces to the linearized flexural shell model.

Either for plates or shells with more general thickness given by g_1^h, g_2^h, all appropriate limiting theories still need to be derived. In [11], Lewicka, Mora and Pakzad studied varying thickness shells with

$$g_1^h = hg_1, \qquad g_2^h = hg_2,$$

where g_1 and g_2 are two positive Lipschitz functions. The contribution of this paper is a study of the case $\beta = 4$ with given sequences g_1^h, g_2^h of positive $\mathcal{C}^\infty$ functions satisfying properties (2.2). The method utilized also provides a way to study other plate and shell theories in the varying thickness setting.

The main result of this paper is the Γ-convergence of the scaled energies $h^{-4}E^h(u^h)$ to the *generalized von Kármán functional* introduced in [11], which will be presented in section 2. In section 3, we prove compactness of u^h and the lower bound of $h^{-4}E^h(u^h)$, where a crucial geometric rigidity estimate is established. The upper bound of the scaled energy and the construction of the recovery sequence is discussed in section 4. The novel character of the present proofs is that we rescale the deformations to a common domain of constant thickness, which also yields simpler proofs in the setting of [11].

2 Overview of the Main Results

Consider a 2-dimensional surface S embedded in $\mathbb{R}^3$, which is compact, connected, oriented, and of class $\mathcal{C}^{1,1}$, and whose boundary ∂S is the union of finitely many (possibly none) Lipschitz continuous curves. For any $x \in S$, let $\vec{n}$ be the unit normal vector at x, T_xS the tangent space of S at x, and $\Pi(x) = \nabla\vec{n}(x)$ the shape operator on S.

Let S^h be a family of thin shells around S, having the following form

$$S^h = \{z = x + t\vec{n}(x);\ x \in S, -g_1^h(x) < t < g_2^h(x)\}, \tag{2.1}$$

where $g_1^h, g_2^h : S \to \mathbb{R}_+$ are two sequences of positive $\mathcal{C}^1$ functions for whom there exist positive functions $g_1, g_2 : S \longrightarrow \mathbb{R}_+$ such that

$$\lim_{h\to 0} g_1^h/h = g_1 \qquad \text{and} \qquad \lim_{h\to 0} g_2^h/h = g_2 \quad \text{in } \ \mathcal{C}^1(S). \tag{2.2}$$

In particular, we see that for some constants $C_1, C_2 > 0$ independent of h, there hold

$$C_1 h < g_1^h(x) < C_2 h, \qquad C_1 h < g_2^h(x) < C_2 h, \qquad \forall\, x \in S. \tag{2.3}$$

It is also convenient to define the universal domain

$$S^* = \{z = x + t\vec{n}(x);\ x \in S,\ -h_0/2 < t < h_0/2\}, \tag{2.4}$$

where h_0 is a positive number such that

$$1/2 < \det(\mathrm{Id} + t\Pi(x)) < 3/2, \quad 1/2 < |\mathrm{Id} + t\Pi(x)| < 3/2 \qquad \forall\, x \in S,\ \ \forall\, t \in (-h_0/2, h_0/2). \tag{2.5}$$

Such defined S^* plays the role of the common domain of the rescaled deformations. We also define the projection $\pi : S^* \to S$ along $\vec{n}(x)$ as $\pi(z) = x$ for $z = x + t\vec{n}(x) \in S^*$.

Consider a deformation of a thin shell $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$. The elastic energy (scaled per unit thickness) of u^h is given by the nonlinear functional

$$E^h(u^h) = \frac{1}{h}\int_{S^h} W(\nabla u^h), \tag{2.6}$$

where the stored-energy density $W : \mathbb{R}^{3\times 3} \to [0, \infty]$ is $\mathcal{C}^2$ in a neighborhood of the special orthogonal group $SO(3)$, and satisfies below frame invariance, normalization and growth conditions

$$\begin{aligned} &[(\mathrm{i})]\ W(RF) = W(F), \qquad \forall\, F \in \mathbb{R}^{3\times 3},\ \ \forall\, R \in SO(3),\\ &[(\mathrm{ii})]\ W(\mathrm{Id}) = 0,\\ &[(\mathrm{iii})]\ W(F) \geq c\ \mathrm{dist}^2(F, SO(3)), \quad \text{with some constant} \quad c > 0 \quad \text{independent of } \ F. \end{aligned} \tag{2.7}$$

This paper studies the asymptotic behavior of the sequence $h^{-4}E^h(u^h)$ as $h \to 0$. To this end, corresponding to each $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$, we shall introduce the rescaled deformations

$$y^h(x + t\vec{n}(x)) = u^h\big(x + s^h(t, x)\vec{n}(x)\big), \qquad \forall\, x \in S,\ \ \forall\, t \in (-h_0/2, h_0/2), \tag{2.8}$$

where $\int s^h(t, x) = t/h_0(g_1^h(x) + g_2^h(x)) + 1/2(g_2^h(x) - g_1^h(x))$ maps the interval $(-g_1^h, g_2^h)$ homeomorphically onto the interval $(-h_0/2, h_0/2)$, for each $x \in S$. Direct calculation shows that $y^h \in W^{1,2}(S^*, \mathbb{R}^3)$. For each y^h, define the scaled average displacement

$$V^h[y^h](x) = \frac{1}{h}\int_{-h_0/2}^{h_0/2} \big(y^h(x + t\vec{n}(x)) - \big(x + s^h(t, x)\vec{n}(x)\big)\big)\, dt, \qquad \forall\, x \in S. \tag{2.9}$$

To introduce the main theorems, especially the *generalized von Kármán functional*, we will use the following notation: Given a matrix field $M \in L^2(S, \mathbb{R}^{3\times 3})$, by sym$M$, we mean a bilinear form on T_xS given by $((\mathrm{sym}M(x)\tau)\eta) = 1/2[(M(x)\tau)\eta + (M(x)\eta)\tau]$, for all $\tau, \eta \in T_xS$. By $M_{\tan}(x)$, we denote the tangential minor of M at $x \in S$, that is $[(M(x)\tau)\eta]$ for all $\tau, \eta \in T_xS$. Then, the *generalized von Kármán functional* on S (see [11]) takes the form

$$\begin{aligned}\mathcal{I}(V, B_{\tan}) =& \frac{1}{2}\int_S (g_1+g_2)\mathcal{Q}_2\Big(x, B_{\tan} - \frac{1}{2}(A^2)_{\tan} - \frac{1}{2}\mathrm{sym}\ \big(A\nabla((g_2-g_1)\vec{n})\big)\Big)\\ &+ \frac{1}{24}\int_S (g_1+g_2)^3\ \mathcal{Q}_2\big(x, (\nabla(A\vec{n}) - A\Pi)_{\tan}\big), \end{aligned} \tag{2.10}$$

for $V \in \mathcal{V}$ and $B_{\tan} \in \mathcal{B}$. The space $\mathcal{V}$ is the space of $W^{2,2}$ first order infinitesimal isometries. A $W^{2,2}$ vector field $V \in \mathcal{V}$, provided that there exists a matrix field $A \in W^{1,2}(S, \mathbb{R}^{3\times 3})$ such that

$$\partial_\tau V(x) = A(x)\tau, \quad A(x)^T = -A(x), \qquad \forall\, \text{a.e. } x \in S, \ \ \forall\, \tau \in T_xS.$$

The finite strain space $\mathcal{B}$ (see [10]), consists of the following symmetric matrix fields

$$\mathcal{B} = \Big\{L^2 - \lim_{h\to 0} \mathrm{sym}\nabla w^h;\ w^h \in W^{1,2}(S, \mathbb{R}^3)\Big\}.$$

Notice that, $\mathcal{B}$ contains all weak L^2 limits of symmetric gradients of $W^{1,2}$ vector fields on S. In the expression of $I(V, B_{\tan})$, the quadratic forms $\mathcal{Q}_2(x,\cdot)$ are defined as follows

$$\mathcal{Q}_2(x, F_{\tan}) = \min\{\mathcal{Q}_3(\widetilde{F}); (\widetilde{F} - F)_{\tan} = 0\}, \qquad \mathcal{Q}_3(F) = D^2W(\mathrm{Id})(F, F).$$

Both $\mathcal{Q}_2$ and $\mathcal{Q}_3$ are quadratic forms; $\mathcal{Q}_3$ defined for all $F \in \mathbb{R}^{3\times 3}$, while for a given $x \in S$, $\mathcal{Q}_2(x,\cdot)$ is defined on tangential minors $F_{\tan}$ of such matrices. Actually, $\mathcal{Q}_\ni$ depends only on the symmetric part of its argument, and hence $\mathcal{Q}_\in$ depends on its first argument and the symmetric part of its second argument. Moreover, both forms are positive definite on the space of symmetric matrices.

We now state the first theorem, which shows compactness of the deformations and the lower bound of the scaled energies.

Theorem 2.1. *For any sequence of deformations $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ satisfying*

$$E^h(u^h) \le Ch^4, \tag{2.11}$$

there exist sequences $Q^h \in SO(3)$ and $c^h \in \mathbb{R}^3$ such that the normalized scaled deformations $\widetilde{y}^h(x + t\vec{n}(x)) = (Q^h)^T y^h(x + t\vec{n}(x)) - c^h$ defined on S^* satisfy

(i) $\widetilde{y}^h$ converges in $W^{1,2}(S^*)$ to the projection π.

(ii) The related scaled average displacements $V^h[\widetilde{y}^h]$ converge (up to a subsequence) in $W^{1,2}(S)$ to some $V \in \mathcal{V}$.

(iii) $h^{-1}sym\nabla V^h[\widetilde{y}^h]$ converges (up to a subsequence) weakly in $L^2(S)$ to some $B_{\tan} \in \mathcal{B}$.

(iv) $\liminf_{h\to 0} h^{-4}E^h(u^h) \ge \mathcal{I}(V, B_{\tan})$.

We further prove that this lower bound in (iv) above is optimal.

Theorem 2.2. *For every $V \in \mathcal{V}$ and every $B_{\tan} \in \mathcal{B}$, there exists a sequence of deformations $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ such that after the rescaling (2.8), we will have*

(i) y^h converges in $W^{1,2}(S^*)$ to the projection π.

(ii) The related scaled average displacement $V^h[y^h]$ converges in $W^{1,2}(S)$ to V.

(iii) $h^{-1}sym\nabla V^h[y^h]$ converges in $L^2(S)$ to $B_{\tan}$.

(iv) $\lim\limits_{h\to 0} h^{-4}E^h(u^h) = \mathcal{I}(V, B_{\tan})$.

Recall the definition of Γ-convergence[3]:

Definition 2.1. *Let $\mathcal{F}^h : X \to \bar{\mathbb{R}}$ be a sequence of functionals defined on a metric space X. We say that $\mathcal{F}^h$ Γ-converge, as $h \to 0$, to some $\mathcal{F} : X \to \overline{\mathbb{R}}$ provided that*

(i) For any converging sequence $\{x^h\}$ in X one has:

$$\mathcal{F}\Big(\lim_{h\to 0} x^h\Big) \le \liminf_{h\to 0} \mathcal{F}^h(x^h).$$

(ii) For every $x \in X$, there exists a sequence $\{x^h\}$ converging to x, such that:

$$\mathcal{F}(x) = \lim_{h\to 0} \mathcal{F}^h(x^h).$$

With this, we restate the main results in the following Corollary.

Corollary 2.3. *Define a sequence of functionals*

$$\mathcal{F}^h : W^{1,2}(S^*, \mathbb{R}^3) \times W^{1,2}(S, \mathbb{R}^3) \times L^2(S, \mathbb{R}^{2\times 2}) \longrightarrow \overline{\mathbb{R}},$$

$$\mathcal{F}^h(y^h, V^h, B^h_{\tan}) = \begin{cases} \dfrac{1}{h^4} E^h(y^h), & \text{if } \ V^h = V^h[y^h] \quad \text{and} \quad B^h_{\tan} = \dfrac{1}{h} sym \nabla V^h \\ +\infty, & \text{otherwise} . \end{cases}$$

Then $\mathcal{F}^h$ Γ-converge, as $h \to 0$, to

$$\mathcal{F}(y, V, B_{\tan}) = \begin{cases} \mathcal{I}(V, B_{\tan}), & \text{if } y = \pi, \ V \in \mathcal{V} \quad \text{and} \quad B_{\tan} \in \mathcal{B} \\ +\infty, & \text{otherwise.} \end{cases}$$

For the detailed proof of Corollary 2.3, please refer to Appendix B in [10].

In all the estimates in the proof, by C we cumulatively denote a positive constant which does not depend on h.

3 Compactness and the Lower Bound of the Energy - A Proof of Theorem 2.1

To prove Theorem 2.1, we shall introduce an important rigidity estimate based on the following one obtained by Friesecke, James and Müller[5]:

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain with Lipschitz boundary. Then, for every $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ one has*

$$\min_{R\in SO(n)} \|\nabla u(x) - R\|_{L^2(\Omega)} \le C \|\mathrm{dist}(\nabla u, SO(3))\|_{L^2(\Omega)},$$

where the constant C depends only on Ω. In particular, C is invariant under dilations of Ω and it is uniform for the uniform bilipschitz images of a unit ball in $\mathbb{R}^n$.

The rigidity estimate we are applying says:

Lemma 3.2. *Let $u \in W^{1,2}(S^h, \mathbb{R}^3)$ and assume that $h^{-3}E(u, S^h)$ is sufficiently small. Then there exist a matrix field $R \in W^{1,2}(S, \mathbb{R}^{3\times 3})$ with $R(x) \in SO(3)$ for each $x \in S$ and a matrix $Q \in SO(3)$ satisfying the following properties*

(i) $\|\nabla u - R\pi\|_{L^2(S^h)} \le C \|dist(\nabla u, SO(3))\|_{L^2(S^h)}$,

(ii) $\|\nabla R\|_{L^2(S)} \le C h^{-3/2} \|dist(\nabla u, SO(3))\|_{L^2(S^h)}$,

(iii) $\|Q^T R - Id\|_{L^p(S)} \le C h^{-3/2} \|dist(\nabla u, SO(3))\|_{L^2(S^h)}$, *for all $p \in [1, +\infty)$,*

where C is independent of u and h (but may depend on p).

Proof. For each $x \in S$, define 'balls' in S and the corresponding 'cylinders' in S^h

$$D_{x,h} = B(x,h) \cap S, \qquad B_{x,h} = \pi^{-1}(D_{x,h}) \cap S^h.$$

Then, Theorem 3.1 is applied on each set $B_{x,h}$, giving constant matrices $R_{x,h} \in SO(3)$ with the property

$$\|\nabla u(z) - R_{x,h}\|_{L^2(B_{x,h})} \leq C\|\mathrm{dist}(\nabla u, SO(3))\|_{L^2(B_{x,h})}, \tag{3.1}$$

where the constant C depends only on the region $B_{x,h}$.

The crucial observation is that C is actually a universal constant. This follows from g_i^h being Lipschitz bounded above by $C_2 h$ below by $C_1 h$ and S being of class $\mathcal{C}^{1,1}$ (with boundary ∂S consisting of finitely many Lipschitz curves). Indeed, under these assumptions, $B_{x,h}$ is (for every $x \in S$) bilipschitz equivalent to $B(0,h)$ with controlled Lipschitz constants. Besides, the constant C is invariant under dilations, thus independent of h. Hence, C in (3.1) is uniform with respect to x and h.

Finally, we may finish the rest of the proof following exactly that of Lemma 8.1 in [10]. □

With this lemma, the proof of Theorem 2.1 proceeds through several steps as:

1. Thanks to (2.7), (2.11) and Lemma 3.2, there exists a matrix field $R^h \in W^{1,2}(S, \mathbb{R}^{3\times 3})$ such that $R^h(x) \in SO(3)$, for each $x \in S$ and a matrix $Q^h \in SO(3)$ satisfying

$$\|\nabla u^h - R^h\|_{L^2(S^h)} \leq Ch^{5/2}, \quad \|\nabla R^h\|_{L^2(S)} \leq Ch, \quad \|(Q^h)^T R^h - \mathrm{Id}\|_{L^2(S)} \leq Ch. \tag{3.2}$$

We thus have

$$\lim_{h\to 0} (Q^h)^T R^h = \mathrm{Id}, \qquad \text{in } \ W^{1,2}(S). \tag{3.3}$$

Moreover, as shown in Lemma 3.2 (see [10]), there exists a $W^{1,2}$ skew-symmetric matrix field A on S such that up to a subsequence

$$\lim_{h\to 0} \frac{1}{h}\big((Q^h)^T R^h - \mathrm{Id}\big) = A \quad \text{weakly in} \quad W^{1,2}(S) \qquad \text{and} \qquad \text{strongly in } L^2(S); \tag{3.4}$$

$$\lim_{h\to 0} \frac{1}{h^2}\mathrm{sym}\,\big((Q^h)^T R^h - \mathrm{Id}\big) = \frac{1}{2}A^2 \quad \text{in} \ \ L^2(S). \tag{3.5}$$

Recall the rescaling (2.8), we further denote

$$\nabla_h y^h(x + t\vec{n}) = \nabla u^h\big(x + s^h(t,x)\vec{n}(x)\big). \tag{3.6}$$

With these notations, by direct calculation, for each $x \in S$, $t \in (-h_0/2, h_0/2)$, we obtain

$$\begin{aligned} \partial_\tau y^h(x + t\vec{n}) &= \nabla_h y^h(x + t\vec{n})\big(\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x))\big)(\mathrm{Id} + t\Pi)^{-1}\tau, \\ \partial_{\vec{n}} y^h(x + t\vec{n}) &= \frac{1}{h_0}(g_1^h + g_2^h)\,\nabla_h y^h(x + t\vec{n})\vec{n}(x) \end{aligned} \tag{3.7}$$

and

$$E^h(u^h) = \int_S \frac{1}{h}(g_1^h + g_2^h)\int_{-h_0/2}^{h_0/2} W\big(\nabla_h y^h(x + t\vec{n})\big)\det\big(\mathrm{Id} + s^h(t,x)\Pi\big)\,dtdx. \tag{3.8}$$

Thus, in view of (3.2), (3.4), (2.5) and (3.7), there follow

$$\|\nabla_h y^h - R^h\pi\|_{L^2(S^*)} \leq Ch^2, \tag{3.9}$$

$$\lim_{h\to 0} \frac{1}{h}((Q^h)^T\nabla_h y^h - \mathrm{Id}) = A\pi \quad \text{in} \quad L^2(S^*) \quad \text{up to subseqence.} \tag{3.10}$$

2. Define

$$\widetilde{y}^h = (Q^h)^T y^h - c^h, \tag{3.11}$$

where c^h is chosen, such that $\fint_S V^h[\widetilde{y}^h]\,dx = 0$.

3. Through direct calculation, the definitions of $V^h[y^h]$ in (2.9) and $\widetilde{y}^h$ in (3.11) imply

$$\begin{aligned}\nabla V^h[\widetilde{y}^h](x) =& \Big(\frac{1}{h}\int_{-h_0/2}^{h_0/2}(Q^h)^T\big(\nabla_h y^h(x+t\vec{n}) - R^h\big)\big(\mathrm{Id} + \nabla_x\big(s^h(t,x)\vec{n}(x)\big)\big)dt\Big)_{\tan}\\ &+\Big(\frac{1}{h}\big((Q^h)^T R^h - \mathrm{Id}\big)\int_{-h_0/2}^{h_0/2}\big(\mathrm{Id} + \nabla_x\big(s^h(t,x)\vec{n}(x)\big)\big)dt\Big)_{\tan}. \end{aligned}\tag{3.12}$$

Based on (2.2) and (3.9), we have

$$\Big\|\frac{1}{h}\int_{-h_0/2}^{h_0/2}(Q^h)^T(\nabla_h y^h(x+t\vec{n}) - R^h)\big(\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x))\big)dt\Big\|_{L^2(S)} \le Ch \to 0. \tag{3.13}$$

Concerning the second term of (3.12), (2.2) and (3.4) indicate that up to a subsequence

$$\Big\|\frac{1}{h}\big((Q^h)^T R^h(x) - \mathrm{Id}\big)\int_{-h_0/2}^{h_0/2}\big(\mathrm{Id} + \nabla_x\big(s^h(t,x)\vec{n}(x)\big)\big)\,dt - A\Big\|_{L^2(S)} \to 0, \qquad \text{as } h \to 0. \tag{3.14}$$

Thus, $\nabla V^h[\widetilde{y}^h] \to A_{\tan}$ in $L^2(S)$. Since $\fint_S V^h[\widetilde{y}^h]\,dx = 0$, we apply Poincaré inequality on S to deduce $V^h[\widetilde{y}^h]$ is bounded, furthermore $V^h[\widetilde{y}^h] \to V$ in $W^{1,2}(S)$ up to a subsequence, that is (ii).

4. It follows from (3.7) that

$$\begin{aligned}&\big\|\partial_{\vec{n}}\widetilde{y}^h(x+t\vec{n})\big\|_{L^2(S^*)} \le C\big(\|g_1^h\|_{L^\infty(S)} + \|g_2^h\|_{L^\infty(S)}\big),\\ &\big\|\partial_\tau\widetilde{y}^h - (Q^h)^T R^h\big(\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x))\big)(\mathrm{Id}+t\Pi)^{-1}\tau\big\|_{L^2(S^*)} \le Ch^2.\end{aligned}$$

Thus by (2.2) and (3.3), $\lim\limits_{h\to 0}\partial_\tau\widetilde{y}^h = (\mathrm{Id}+t\Pi)^{-1}\tau = \partial_\tau\pi$, $\lim\limits_{h\to 0}\partial_{\vec{n}}\widetilde{y}^h = 0$ in $L^2(S^*)$. Hence $\nabla\widetilde{y}^h \to \nabla\pi$ in $L^2(S^*)$. Applying the weighted Poincaré inequality, as in Lemma 3.5 (see [10], implies (i)).

5. For each y^h, let finite strain

$$G^h = \frac{1}{h^2}\big((R^h)^T\nabla_h y^h - \mathrm{Id}\big). \tag{3.15}$$

According to (3.9), G^h is bounded in $L^2(S^*)$, thus has a subsequence, converging weakly in $L^2(S^*)$ to a matrix field G. Furthermore, the tangential minor of G is affine in the $\vec{n}$ direction. More precisely

$$\forall\,\tau \in T_xS, \qquad G(x+t\vec{n})\tau = G_0(x)\tau + \frac{t}{h_0}(g_1+g_2)(\nabla(A\vec{n}) - A\Pi)\tau, \tag{3.16}$$

where $G_0(x) = \fint_{-h_0/2}^{h_0/2} G(x+t\vec{n})dt$.

Similar to the proof of Lemma 3.6 (see [10]), the above statement can be proven through finding the $W^{1,2}$ limit of the auxiliary sequence of vector fields

$$f^{s,h}(x+t\vec{n}) = \frac{h_0}{sh^2}\big[\widetilde{y}^h\big(x+(t+s)\vec{n}\big) - \widetilde{y}^h\big(x+t\vec{n}\big)\big] - \frac{1}{h^2}\big(g_1^h + g_2^h\big)\vec{n},$$

in view of the identity that $\nabla\big((g_1+g_2)A\vec{n}\big) - A\nabla\big((g_1+g_2)\vec{n}\big) = (g_1+g_2)\big(\nabla(A\vec{n}) - A\Pi\big)$.

6. By the frame invariance and $\mathcal{C}^2$ regularity of W, there follows a formal Taylor expansion in $\Omega_h = \{x \in S^*;\ h|G^h(x)| \leq 1\}$,

$$\begin{aligned}\frac{1}{h^4}W(\nabla_h y^h) =& \frac{1}{2}D^2W(\mathrm{Id})(G^h, G^h)\\ &+ \int_0^1 (1-s)\big[D^2W(\mathrm{Id} + sh^2G^h) - D^2W(\mathrm{Id})\big]\, ds(G^h, G^h)\\ =& \frac{1}{2}\mathcal{Q}_3(G^h) + o(1)|G^h|^2. \end{aligned} \tag{3.17}$$

where, $o(1)$ is the Landau symbol denoting any quantity uniformly converging to 0, as $h \to 0$. We thus obtain

$$\begin{aligned}\liminf_{h\to 0} \frac{1}{h^4}E^h(y^h) \geq& \frac{1}{2}\int_S (g_1+g_2)\mathcal{Q}_2\big(x, (\mathrm{sym}G_0)_{\tan}\big)\mathrm{d}x\\ &+ \frac{1}{24}\int_S (g_1+g_2)^3\mathcal{Q}_2\Big(x, \big(\nabla(A\vec{n}) - A\Pi\big)_{\tan}\Big)dx.\end{aligned}$$

7. Based on (3.7) and (3.15), there follows

$$\begin{aligned}\nabla V^h[\tilde{y}^h] =& h\Big[\fint_{-h_0/2}^{h_0/2} ((Q^h)^T R^h)G^h(x+t\vec{n})\, dt\\ &+ \fint_{-h_0/2}^{h_0/2} ((Q^h)^T R^h)G^h(x+t\vec{n})\nabla\big((g_1^h+g_2^h)t + \frac{1}{2}(g_2^h - g_1^h)\vec{n}\big)dt\\ &+ \frac{1}{h^2}((Q^h)^T R^h - \mathrm{Id}) + \frac{1}{2h}((Q^h)^T R^h - \mathrm{Id})\Big(\nabla\Big(\frac{g_2^h - g_1^h}{h}\vec{n}\Big)\Big)\Big]_{\tan}.\end{aligned}$$

Meanwhile, (3.3) and (3.16) imply

$$\Big(\fint_{-h_0/2}^{h_0/2} ((Q^h)^T R^h)G^h(x+t\vec{n})dt)_{\tan} \rightharpoonup G_0(x)_{\tan} \qquad \text{in} \quad L^2(S).$$

By (2.2) and (3.4),

$$\fint_{-h_0/2}^{h_0/2} (Q^h)^T R^h G^h(x+t\vec{n})\,\nabla\big((g_1^h+g_2^h)t + \frac{1}{2}(g_2^h - g_1^h)\vec{n}\big)dt \to 0 \qquad \text{in} \quad L^2(S)$$

and

$$\frac{1}{2h}\big((Q^h)^T R^h - \mathrm{Id}\big)\big(\nabla((h^{-1}g_2^h - h^{-1}g_1^h)\vec{n})\big) \to \frac{1}{2}A\big(\nabla(g_2 - g_1)\vec{n}\big), \qquad \text{in} \quad L^2(S). \tag{3.18}$$

Recalling (3.5), we get $\lim\limits_{h\to 0} \frac{1}{h}\mathrm{sym}\nabla V^h[\tilde{y}^h] = \big(\mathrm{sym}G_0 + \frac{1}{2}A^2 + \frac{1}{2}\mathrm{sym}(A\nabla((g_2-g_1)\vec{n}))\big)_{\tan}$ weakly in $L^2(S)$. This finishes the proof of the theorem. □

4 The Recovery Sequence—A Proof of Theorem 2.2

1. As in Section 6 (see [10]), we define the linear map c and approximate $B_{\tan}$, V. With an abuse of notation, one can write $Q_2(x, F_{\tan}) = \min\{\mathcal{Q}_3(F_{\tan} + c\otimes\vec{n}(x) + \vec{n}(x)\otimes c);\ c \in \mathbb{R}^3\}$. In view of properties of $\mathcal{Q}_3$, it has a unique minimizer $c(x, F_{\tan})$, which realizes $\mathcal{Q}_\in(\S, \mathcal{F}_{\tan})$. The uniqueness implies that c is linear in its second argument.

For any $B_{\tan} \in \mathcal{B}$, there is a sequence of vector fields $w^h \in W^{1,2}(S, \mathbb{R}^3)$ enjoying the property that $\mathrm{sym}\nabla w^h$ converge in $L^2(S)$ to $B_{\tan}$. Without loss of generality, we assume that w^h are smooth, and $\lim_{h\to 0} \sqrt{h}\|w^h\|_{W^{2,\infty}(S)} = 0$. For any $V \in \mathcal{V}$, let $v^h \in W^{2,\infty}(S, \mathbb{R}^3)$ satisfy $\lim_{h\to 0} \|v^h - V\|_{W^{2,2}(S)} = 0$, $h\|v^h\|_{W^{2,\infty}} \le \varepsilon_0$, $\lim_{h\to 0} \frac{1}{h^2}\mu\{x \in S;\ v^h(x) \neq V(x)\} = 0$. To justify the existence of such v^h, we may use a partition of unity and a truncation argument as a special case of Lusin-type result for Sobolev functions in [12].

Define the sequence of rescaled deformations

$$\begin{aligned} y^h(x + t\vec{n}(x)) =& x + \frac{1}{2}(g_2^h - g_1^h)\vec{n}(x) + hv^h(x) + h^2 w^h(x) + \frac{t}{h_0}(g_1^h + g_2^h)\vec{n}(x) \\ &+ \frac{t}{h_0} h(g_1^h + g_2^h)\big(\Pi v^h_{\tan} - \nabla(v^h\vec{n})\big)(x) - \frac{t}{h_0} h^2(g_1^h + g_2^h)(\nabla w^h(x))^T\vec{n}(x) \\ &+ \frac{t}{h_0} h^2(g_1^h + g_2^h)d^{0,h}(x) + \frac{1}{2}\frac{t^2}{{h_0}^2} h(g_1^h + g_2^h)^2 d^{1,h}(x), \end{aligned}$$

where the vector fields $d^{0,h}, d^{1,h} \in W^{1,\infty}(S, \mathbb{R}^3)$ are such that

$$\lim_{h\to 0} \sqrt{h}\big(\|d^{0,h}\|_{W^{1,\infty}(S)} + \|d^{1,h}\|_{W^{1,\infty}(S)}\big) = 0 \tag{4.1}$$

and

$$\begin{aligned} \lim_{h\to 0} d^{0,h}(x) =& 2c\Big(x, B_{\tan} - \frac{1}{2}(A^2)_{\tan} - \frac{1}{2}\mathrm{sym}\big(A\nabla\big((g_2 - g_1)\vec{n}\big)\big)\Big) \\ &+ A^2\vec{n} - \frac{1}{2}(\vec{n}^T A\vec{n}) + \frac{1}{2}\big(A\nabla\big((g_2 - g_1)\vec{n}\big)\big)^T\vec{n}, \qquad \text{in } L^2(S), \\ \lim_{h\to 0} d^{1,h}(x) =& 2c\left(x, \mathrm{sym}\big(\nabla(A\vec{n}) - A\Pi\big)_{\tan}\right) + (A\Pi - \nabla(A\vec{n}))^T\vec{n}, \qquad \text{in } L^2(S). \end{aligned} \tag{4.2}$$

2. By (2.2), for such defined y^h, (i) holds. Also, we have

$$V^h[y^h](x) = v^h(x) + hw^h(x) + \frac{1}{24}\big(g_1^h(x) + g_2^h(x)\big)^2 d^{1,h}(x),$$

which implies (ii). Furthermore,

$$\frac{1}{h}\mathrm{sym}(\nabla V^h[y^h]) = \frac{1}{h}\mathrm{sym}(\nabla v^h) + \mathrm{sym}\,(\nabla w^h) + \frac{1}{24h}(g_1^h + g_2^h)^2\mathrm{sym}\,(\nabla d^{1,h}).$$

Exactly as in the proof of Lemma 6.1 in [10], we can show $\|h^{-1}\mathrm{sym}(\nabla v^h)\|_{L^2(S)} \to 0$, as $h \to 0$. Thus (iii) follows.

3. As in section 6 of [10], we will prove that

$$\limsup_{h\to 0} \frac{1}{h^4} E^h(y^h) \le I(V, B_{\tan}) + \eta, \tag{4.3}$$

where η denotes an error quantity, with the property

$$\eta \to 0, \qquad \text{as } \varepsilon_0 \to 0. \tag{4.4}$$

This will give (iv) for the recovery sequence obtained through a diagonal argument, when $\varepsilon_0 \to 0$.

The exposition of proof is divided into several sub-steps. First is to study $\nabla_h y^h$, then the formal Taylor expansion is introduced, and next the limit of scaled strain $\frac{1}{2h^2}K^h$ in $L^2(S^*)$ is established. Finally, we estimate the upper bound for $h^{-4}E^h(u^h)$ to get (4.3). While calculating the limit of $\frac{1}{2h^2}K^h$, we shall bear more cautiousness, for provided $\Phi_t(x) = x + s^h(t,x)\vec{n}(x)$, $T_{\Phi_t(x)}\Phi_t(S) \neq T_xS$, whereas K^h is defined through $\nabla_h y^h$, for a fixed t, the natural basis is $\{T_{\Phi_t(x)}\Phi_t(S), \vec{n}(x)\}$, instead of the preferred $\{T_xS, \vec{n}(x)\}$.

3.1. By (3.7), there follows

$$\begin{aligned}&\nabla_h y^h(x+t\vec{n})\vec{n} = h_0(g_1^h+g_2^h)^{-1}\partial_{\vec{n}} y^h(x+t\vec{n})\\ =&\vec{n}(x) + h\big(\Pi v^h_{\tan} - \nabla(v^h\vec{n})\big) - h^2(\nabla w^h)^T\vec{n} + h^2 d^0(x) + \frac{t}{h_0}h(g_1^h+g_2^h)d^1(x).\end{aligned} \tag{4.5}$$

For each $\tau \in T_xS$,

$$\begin{aligned}&\nabla_h y^h(x+t\vec{n})\big(\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x))\big)\tau = \nabla y^h(x+t\vec{n})(\mathrm{Id}+t\Pi)\tau\\ =&\Big(\mathrm{Id} + h\nabla v^h + h^2\nabla w^h(x) + \nabla_x\big(s^h(t,x)\vec{n}(x)\big)\\ &+\frac{t}{h_0}h\nabla\big((g_1^h+g_2^h)\big(\Pi v^h_{\tan} - \nabla\big(v^h\vec{n}\big)\big)\big) - \frac{t}{h_0}h^2\nabla\big((g_1^h+g_2^h)(\nabla w^h)^T\vec{n}\big)\\ &+\frac{t}{h_0}h^2\nabla\big((g_1^h+g_2^h)d^{0,h}(x)\big) + \frac{1}{2}\frac{t^2}{h_0^2}h\nabla\big((g_1^h+g_2^h)^2 d^{1,h}(x)\big)\Big)\tau.\end{aligned}$$

3.2. By assumptions on w^h, v^h, and assumptions given in (4.1), (4.2) on $d^{0,h}, d^{1,h}$, one has the bound $\|\nabla_h y^h - \mathrm{Id}\|_{L^\infty(S^*)} \leq C\epsilon_0$. Then, when ϵ_0 is sufficiently small, the polar decomposition theorem implies that $\nabla_h y^h$ is a product of a proper rotation and the well defined square root of $(\nabla_h y^h)^T\nabla_h y^h$. The frame invariance property of W and Taylor expansion tell us

$$W(\nabla_h y^h) = W\big(\sqrt{(\nabla_h y^h)^T\nabla_h y^h}\big) = W\big(\mathrm{Id} + \frac{1}{2}K^h + \mathcal{O}\big(|K^h|^2\big)\big), \tag{4.6}$$

where $K^h = (\nabla_h y^h)^T\nabla_h y^h - \mathrm{Id}$. Clearly,

$$\|K^h\|_{L^\infty(S^*)} \leq C\varepsilon_0. \tag{4.7}$$

Similar to (3.17), (4.6) yields

$$\frac{1}{h^4}W(\nabla_h y^h) = \frac{1}{2}\mathcal{Q}_3\Big(\frac{1}{2h^2}K^h + \frac{1}{h^2}\mathcal{O}(|K^h|^2)\Big) + \frac{1}{h^4}\eta\mathcal{O}(|K^h|^2), \tag{4.8}$$

where η depends only on ε_0 and satisfies (4.4).

By Error and error we will cumulatively denote terms with properties

$$\lim_{h\to 0}\frac{1}{h^2}\|\mathrm{error}\|_{L^2(S^*)} = 0 \tag{4.9}$$

and

$$\lim_{h\to 0}\|\mathrm{error}\|_{L^\infty(S^*)} = 0 \tag{4.10}$$

respectively.

3.3. Recalling the expansion (4.8), in this part we calculate the limit of $\frac{1}{2h^2}K^h$. For the sake of clarity, we again divide the exposition into several sub-steps.

3.3.1. Decomposition of K^h with respect to changing basis: $\{T_{\Phi_t(x)}\Phi_t(S), \vec{n}(x)\}$.

3.3.1.1. The tangential minor of K^h. For any $\tau_1, \tau_2 \in T_xS$,

$$(\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x)))\tau_1, (\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x)))\tau_2 \in T_{\Phi_t(x)}\Phi_t(S).$$

Based on Lemma 3.7:

$$\begin{aligned}&\big(\big(\mathrm{Id}+\nabla_x(s^h(t,x)\vec{n}(x))\big)\tau_1\big)^T K^h\big((\mathrm{Id}+\nabla_x(s^h(t,x)\vec{n}(x)))\tau_2\big)\\=&2h^2\tau_1^T\Big(\mathrm{sym}(\nabla w^h)+\frac{t}{h_0}\mathrm{sym}\big(\nabla((h^{-1}g_1^h+h^{-1}g_2^h)(\Pi v_{\tan}^h-\nabla(v^h\vec{n})))\big)\\&+\frac{1}{2}h(\nabla v^h)^T(\nabla v^h)+\frac{1}{h}\mathrm{sym}\big((\nabla_x(s^h(t,x)\vec{n}(x)))^T\nabla v^h\big)\Big)\tau_2+\mathrm{Error}.\end{aligned}$$

Thus, (2.2) and the assumptions on w^h and v^h imply

$$\begin{aligned}&\lim_{h\to 0}\frac{1}{2h^2}\big((\mathrm{Id}+\nabla_x(s^h(t,x)\vec{n}(x)))\tau_1\big)^T K^h\big((\mathrm{Id}+\nabla_x(s^h(t,x)\vec{n}(x)))\tau_2\big)\\=&\tau_1^T\Big(B_{\tan}-\frac{1}{2}A^2-\frac{1}{2}\mathrm{sym}(A\nabla((g_2-g_1)\vec{n}))\\&+\frac{t}{h_0}(g_1+g_2)\mathrm{sym}(\nabla(A\vec{n})-A\Pi)\Big)\tau_2,\qquad \text{in}\quad L^2(S^*).\end{aligned}\tag{4.11}$$

3.3.1.2. The normal minor of K^h.

$$\begin{aligned}\vec{n}^TK^h\vec{n}=&h^2\big(\Pi v_{\tan}^h-\nabla(v^h\vec{n})\big)^T\big(\Pi v_{\tan}^h-\nabla(v^h\vec{n})\big)-2h^2\vec{n}^T\mathrm{sym}\,(\nabla w^h)\vec{n}\\&+2h^2\vec{n}^Td^{0,h}(x)+2h^2\frac{t}{h_0}(h^{-1}g_1^h+h^{-1}g_2^h)\vec{n}^Td^{1,h}(x)+\mathrm{Error}.\end{aligned}$$

By (2.2), the assumptions on w^h, v^h and the assumption (4.1), (4.2) on $d^{0,h}, d^{1,h}$, there follows

$$\begin{aligned}\lim_{h\to 0}\frac{1}{2h^2}\vec{n}^TK^h\vec{n}=&2\vec{n}^T\Big(c\Big(x,B_{\tan}-\frac{1}{2}\big(A^2\big)_{\tan}-\frac{1}{2}\mathrm{sym}\big(A\nabla\big((g_2-g_1)\vec{n}\big)\big)\big)\\&+\frac{t}{h_0}c\big(x,(g_1+g_2)\mathrm{sym}\big(\nabla(A\vec{n})-A\Pi\big)_{\tan}\big)\Big),\qquad\text{in}\quad L^2(S^*).\end{aligned}\tag{4.12}$$

3.3.1.3. The mixed terms of K^h. For each $\tau\in T_xS$,

$$\begin{aligned}&\vec{n}^TK^h\big(\mathrm{Id}+\nabla_x\big(s^h(t,x)\vec{n}(x)\big)\big)\tau\\=&\Big(\frac{t}{h_0}h^2\vec{n}\nabla\big((h^{-1}g_1^h+h^{-1}g_2^h)\big(\Pi v_{\tan}^h-\nabla(v^h\vec{n})\big)\big)\\&+h^2\big(\Pi v_{\tan}^h-\nabla(v^h\vec{n})\big)^T\nabla v^h+h^2\big(\Pi v_{\tan}^h-\nabla(v^h\vec{n})\big)^T\nabla_x\Big(\frac{1}{h}s^h(t,x)\vec{n}(x)\Big)\\&-h^2(d^{0,h})^T+\frac{t}{h_0}h^2(h^{-1}g_1^h+h^{-1}g_2^h)(d^{1,h}(x))^T\Big)\tau+\mathrm{Error}.\end{aligned}\tag{4.13}$$

Again, (2.2), the assumptions on w^h, v^h, $d^{0,h}, d^{1,h}$ give us

$$\begin{aligned}&\lim_{h\to 0}\frac{1}{2h^2}\vec{n}^TK^h\big(\mathrm{Id}+\nabla_x\big(s^h(t,x)\vec{n}(x)\big)\big)\tau\\=&\Big(c\Big(x,B_{\tan}-\frac{1}{2}(A^2)_{\tan}-\frac{1}{2}\mathrm{sym}\big(A\nabla\big((g_2-g_1)\vec{n}\big)\big)\Big)\\&+\frac{t}{h_0}c\big(x,(g_1+g_2)\mathrm{sym}(\nabla(A\vec{n})-A\Pi)_{\tan}\big)\Big)^T\tau,\qquad\text{in}\quad L^2(S^*).\end{aligned}\tag{4.14}$$

Most estimates in *Error* are straightforward and you may refer to [10] for some difficult terms.

3.3.2. Decomposition of K^h with respect to fixed basis $\{T_xS,\vec{n}(x)\}$.

Let $\tau_1, \tau_2 \in T_xS$ be such that $\{\tau_1, \tau_2, \vec{n}\}$ forms an orthonormal basis of $\mathbb{R}^3$. Consequently, from the convergence Assumption (2.2) of g_1^h, g_2^h, vectors in

$$\big\{(\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x)))\tau_1, (\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x)))\tau_2, \vec{n}\big\} \tag{4.15}$$

form a basis of $\mathbb{R}^3$ as well (when h is sufficiently small). Though the basis (4.15) may not be orthonormal, the transition matrix B^h belongs to a small neighborhood of $SO(3)$, and thus is bounded in $L^\infty(S^*)$. Note that the matrix field K^h, with respect to the basis (4.15) is a symmetric matrix $M^h = (m_{ij}^h)_{3\times 3}$, where

$$\begin{aligned}
&m_{ij}^h = m_{ji}^h = \big((\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x)))\tau_i\big)^T K^h \big((\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x)))\tau_j\big), \qquad \text{for} \quad i,j = 1,2;\\
&m_{i3}^h = m_{3i}^h = \big((\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x))\tau_i\big)^T K^h \vec{n}, \qquad \text{for} \quad i = 1,2;\\
&m_{33}^h = \vec{n}^T K^h \vec{n}.
\end{aligned}$$

Hence, according to (4.11), (4.12) and (4.14), $\frac{1}{2h^2}M^h$ converges in $L^2(S^*)$. Meanwhile, the transition matrix B^h is bounded in $L^\infty(S^*)$, thus the matrix of K^h with respect to the basis $\{\tau_1, \tau_2, \vec{n}\}$, denoted as $\widetilde{M}^h$, has the property that $\frac{1}{2h^2}\widetilde{M}^h$ is bounded in $L^2(S^*)$. Therefore $\frac{1}{2h^2}K^h$ is bounded in $L^2(S^*)$.

Let us calculate K^h with respect to the normal vector $\vec{n}$ and the tangent vectors of the surface S. Assume $\tau_1, \tau_2 \in T_xS$ be any tangent vectors of S at the point x, then we have

$$\begin{aligned}
\tau_1^T K^h \tau_2 =& \big((\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x)))\tau_1\big)^T \big((\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x)))^{-1}\big)^T K^h\\
&\cdot \big(\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x))\big)^{-1} \big(\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x))\big)\tau_2\\
=& \big((\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x)))\tau_1\big)^T (\mathrm{Id} + \mathrm{error}) K^h (\mathrm{Id} + \mathrm{error}) \big(\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x))\big)\tau_2\\
=& \big((\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x)))\tau_1\big)^T K^h \big(\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x))\big)\tau_2 + \mathrm{Error},
\end{aligned}$$

where we used the fact that $\frac{1}{2h^2}K^h$ is bounded in $L^2(S^*)$, Taylor expansion and (2.2). Moreover, we have the following convergence

$$\begin{aligned}
\lim_{h\to 0} \frac{1}{2h^2}\tau_1^T K^h \tau_2 = \tau_1^T \Big(& B_{\tan} - \frac{1}{2}A^2 - \frac{1}{2}\mathrm{sym}\big(A\nabla((g_2 - g_1)\vec{n})\big)\\
&+ \frac{t}{h_0}(g_1 + g_2)\mathrm{sym}\big(\nabla(A\vec{n}) - A\Pi\big)\Big)\tau_2, \qquad \text{in} \quad L^2(S^*).
\end{aligned} \tag{4.16}$$

For the mixed terms

$$\begin{aligned}
\vec{n}^T K^h \tau =& \vec{n}^T K^h \big(\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x))\big)^{-1} \big(\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x))\big)\tau\\
=& \vec{n}^T K^h (\mathrm{Id} + \mathrm{error}) \big(\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x))\big)\tau = \vec{n}^T K^h \big(\mathrm{Id} + \nabla_x(s^h(t,x)\vec{n}(x))\big)\tau + \mathrm{Error}.
\end{aligned}$$

Hence

$$\begin{aligned}
\lim_{h\to 0} \frac{1}{2h^2}\vec{n}^T K^h \tau = \Big(& c\Big(x, B_{\tan} - \frac{1}{2}(A^2)_{\tan} - \frac{1}{2}\mathrm{sym}\big(A\nabla((g_2 - g_1)\vec{n})\big)\Big)\\
&+ \frac{t}{h_0}c\big(x, (g_1 + g_2)\mathrm{sym}(\nabla(A\vec{n}) - A\Pi)_{\tan}\big)\Big)^T, \tau \qquad \text{in} \quad L^2(S^*).
\end{aligned} \tag{4.17}$$

3.3.3. Limit of $\frac{1}{2h^2}K^h$ in $L^2(S^*)$.

By the above calculation and (4.12), we obtain (with a slight abuse of notation)

$$\lim_{h\to 0} \frac{1}{2h^2}K^h = K_1(x)_{\tan} + \frac{t}{h_0}K_2(x)_{\tan} + (\zeta \otimes \vec{n} + \vec{n} \otimes \zeta), \qquad \text{in} \quad L^2(S^*), \tag{4.18}$$

where the symmetric matrix fields $K_1(x)_{\tan}, K_2(x)_{\tan} \in L^2(S, \mathbb{R}^{2\times 2})$ and the vector field $\zeta \in L^2(S^*, \mathbb{R}^3)$ are given by:

$$\begin{aligned}
K_1(x)_{\tan} =& B_{\tan} - \frac{1}{2}(A^2)_{\tan} - \frac{1}{2}\mathrm{sym}\big(A\nabla((g_2 - g_1)\vec{n})\big)\\
K_2(x)_{\tan} =& (g_1 + g_2)\mathrm{sym}\big(\nabla(A\vec{n}) - A\Pi\big)_{\tan}\\
\zeta(x + t\vec{n}) =& c\Big(x, B_{\tan} - \frac{1}{2}(A^2)_{\tan} - \frac{1}{2}\mathrm{sym}\big(A\nabla((g_2 - g_1)\vec{n})\big)\Big)\\
&+ \frac{t}{h_0}c\big(x, (g_1 + g_2)\mathrm{sym}(\nabla(A\vec{n}) - A\Pi)_{\tan}\big).
\end{aligned} \tag{4.19}$$

3.4. Bearing the convergence (2.2) in mind, with similar calculation as in Section 6 (see [10]), we finally obtain (4.3), and finish the proof of (iv), thus the proof of Theorem 2.2. □

5 Appendix: Remarks about the required convergence of g_1^h and g_2^h

In this section, we propose a weaker assumption for g_1^h and g_2^h. Theorem 2.1 holds under this assumption. Hopefully, it is also enough for Theorem 2.2. In remark 5.1 the weaker assumption will be stated and a brief analysis will be given. In remark 5.2, an example will be presented to show that the proposed assumption is indeed weaker than the original one.

Remark 5.1. Instead of g_1^h, g_2^h satisfying (2.2), in order to have Theorem 2.1 hold, we only need the following:

There exists $g_1, g_2 \in \mathcal{C}^1(S)$ bounded, such that

$$\begin{aligned}
&\lim_{h\to 0} h^{-1}g_i^h = g_i, \qquad \text{in } \ L^\infty, \ \text{ for } \ i = 1, 2,\\
&\lim_{h\to 0} h^{-1}\nabla g_i^h = \nabla g_i, \qquad \text{weakly in } \ L^2(S), \qquad \text{for } \ i = 1, 2,
\end{aligned}$$

and there exists a uniform constant C, such that

$$\|h^{-1}\nabla g_i^h\|_{L^\infty(S)} \leq C, \qquad \text{for } \ i = 1, 2.$$

This change mainly influences the convergence (3.18), however it is enough for it to be weak. Indeed, under the changed assumption it is weak in $L^2(S)$.

Remark 5.2. One example showing the assumption in Remark 5.1 is indeed weaker than (2.2).

Let $S = (0, 2\pi) \times (0, 2\pi)$, $h_n = 1/n$, define $g_1^{h_n}(x, y) = g_2^{h_n}(x, y) = -\cos nx/n^2 + 1/n$. Note that

$$\lim_{h_n\to 0} g_i^{h_n} = \lim_{n\to+\infty} \frac{-1/n^2 \cos nx + 1/n}{1/n} = \lim_{n\to+\infty} -\frac{1}{n}\cos nx + 1 = 1, \qquad \text{in } \ L^\infty(S) \quad \text{for } i = 1, 2,$$

and $\nabla g_i^{h_n} = \big[1/n \sin nx, 0\big]^T$. Thus, $h_n^{-1}\nabla g_i^{h_n} = [\sin nx, \ 0]^T \rightharpoonup [0, \ 0]^T$ in $L^2(S)$ as $h_n \to 0$. Furthermore $\|{h_n}^{-1}\nabla g_i^h\|_{L^\infty(S)} \leq 1$. Therefore let $g_1(x, y) = g_2(x, y) = 1$ for every $(x, y) \in S$. Then $g_1^{h_n}$ and $g_2^{h_n}$ satisfy the assumption in Remark 5.1. However, it is obvious that the sequence $h_n^{-1}g_i^{h_n}$ does not converge to the function g_i in $\mathcal{C}^1(S)$ (for $i = 1, 2$).

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