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On the Metric Dimension of Barycentric Subdivision of Cayley Graphs

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Abstract In a connected graph G, the distance $d(u,v)$ denotes the distance between two vertices u and v of G. Let $W=\{w_1,w_2,\cdots,w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The *representation* $r(v|W)$ of v with respect to W is the k-tuple $(d(v,w_1),d(v,w_2),\cdots,d(v,w_k))$. The set W is called a *resolving set* or a *locating set* if every vertex of G is uniquely identified by its distances from the vertices of W, or equivalently, if distinct vertices of G have distinct representations with respect to W. A resolving set of minimum cardinality is called a *metric basis* for G and this cardinality is the *metric dimension* of G, denoted by $\beta(G)$. Metric dimension is a generalization of affine dimension to arbitrary metric spaces (provided a resolving set exists). In this paper, we study the metric dimension of barycentric subdivision of Cayley graphs Cay $(Z_n\oplus Z_2)$. We prove that these subdivisions of Cayley graphs have constant metric dimension and only three vertices chosen appropriately suffice to resolve all the vertices of barycentric subdivision of Cayley graphs Cay $(Z_n\oplus Z_2)$.

Keywords metric dimension; basis; resolving set; barycentric subdivision; Cayley graph
2000 MR Subject Classification 05C12

1 Introduction and Preliminary Results

Metric dimension is a parameter that has appeared in various applications of graph theory, as diverse as, pharmaceutical chemistry[4], robot navigation[16], combinatorial optimization[18] and sonar and coast guard Loran[19], to name a few. Metric dimension is a generalization of affine dimension to arbitrary metric spaces (provided a resolving set exists).

In a connected graph G, the *distance* $d(u,v)$ between two vertices $u,v\in V(G)$ is the size of a shortest path between them. Let $W=\{w_1,w_2,\cdots,\ w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The *representation* $r(v|W)$ of v with respect to W is the k-tuple $(d(v,w_1),d(v,w_2),\ d(v,w_3),\cdots,d(v,w_k))$. The set W is called a *resolving set*[4] or *locating set*[19] if every vertex of G is uniquely identified by its distances from the vertices of W, or equivalently, if distinct vertices of G have distinct representations with respect to W. A resolving set of minimum cardinality is called a *basis* for G and this cardinality is the *metric dimension* or *location number* of G, denoted by $\beta(G)$ (see [2]).

For a given ordered set of vertices $W=\{w_1,w_2,\cdots,w_k\}$ of a graph G, the i^{th} component of $r(v|W)$ is 0 if and only if $v=w_i$. Thus, to show that W is a resolving set it suffices to verify that $r(x|W)\neq r(y|W)$ for each pair of distinct vertices $x,y\in V(G)\backslash W$.

A useful property in finding $\beta(G)$ is the following lemma:

Lemma 1[20]. *Let W be a resolving set for a connected graph G and $u,v\in V(G)$. If $d(u,w)=d(v,w)$ for all vertices $w\in V(G)\setminus\{u,v\}$, then $\{u,v\}\cap W\neq\emptyset$.*

Let $\mathcal{F}$ be a family of connected graphs $G_n:\mathcal{F}=(G_n)_{n\geq 1}$ depending on n as follows: the order $|V(G)|=\varphi(n)$ and $\lim\limits_{n\to\infty}\varphi(n)=\infty$. If there exists a constant $C>0$ such that

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$\beta(G_n) \leq C$ for every $n \geq 1$, then we shall say that $\mathcal{F}$ has bounded metric dimension; otherwise $\mathcal{F}$ has unbounded metric dimension.

If all graphs in $\mathcal{F}$ have the same metric dimension (which does not depend on n), $\mathcal{F}$ is called a family with constant metric dimension[13]. Some classes of *regular graphs* with constant metric dimension have been studied in [1,8,13] recently while metric dimension of some classes of *convex polytopes* has been determined in [9] and [11].

Other families of graphs have unbounded metric dimension: if W_n denotes a *wheel* with n spokes and J_{2n} the graph deduced from the wheel W_{2n} by alternately deleting n spokes, then $\beta(W_n) = \lfloor \frac{2n+2}{5} \rfloor$ for every $n \geq 7$ (see [2] and $\beta(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$ (see [21]) for every $n \geq 4$. The generalized Petersen graphs $P(n, 3)$ have bounded metric dimension[10].

The graphs having metric dimension 1 are characterized in the following theorem.

Theorem 1[4]**.** *The metric dimension of a graph G is 1 if and only if $G \cong P_n$, where P_n denotes a path of length $n-1$ or G is one-way infinite path.*

The next theorem gives a property of the graphs with metric dimension 2.

Theorem 2[15]**.** *Let G be a graph with metric dimension 2 and let $\{v_1, v_2\} \subseteq V(G)$ be a metric basis in G, then the degree of both v_1 and v_2 is at most 3.*

Geometrically, subdividing an edge is an operation that inserts a new vertex into the edge that results in splitting that edge into two edges. *Subdividing a graph G* means performing a sequence of edge-subdivision operations. The resulting graph is called a *subdivision of the graph G*. The operation of subdivision can be used to convert a general graph into a simple graph. The *barycentric subdivision* of a graph G is the subdivision in which one new vertex is inserted in the interior of each edge.

The following propositions give some results related to barycentric subdivision of a graph[6].

- The barycentric subdivision of any graph is a bipartite graph.
- The barycentric subdivision of any graph yields a loopless graph.
- The barycentric subdivision of any loopless graph yields a simple graph.

A graph G is *planar* if it can be drawn in the plane without edge crossings. Subdivision of graphs play a very important role in characterization of planar graphs. A graph G is planar if and only if every subdivision of G is planar. Two graphs are said to be homeomorphic if they are subdivisions of same graph G. The next theorem, known as Kuratoski's theorem, gives a characterization of planar graphs.

Theorem 3 ([Kuratowski's Theorem [6]). *A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.*

Note that the problem of determining whether $\beta(G) < k$ is an NP-complete problem[5].

In this paper, we study the metric dimension of barycentric subdivision of Cayley graphs $\mathrm{Cay}\,(Z_n \oplus Z_2)$. We prove that these subdivisions of Cayley graphs have constant metric dimension and only three vertices chosen appropriately suffice to resolve all the vertices of these subdivision of Cayley graphs $\mathrm{Cay}\,(Z_n \oplus Z_2)$.

2 The Metric Dimension of Barycentric Subdivision of Cayley Graphs Cay $(Z_n \oplus Z_2)$

Let G be a semigroup, and let S be a nonempty subset of G. The Cayley graph $\mathrm{Cay}\,(G, S)$ of G relative to S is defined as the graph with vertex set G and edge set $E(S)$ consisting of those ordered pairs (x, y) such that $sx = y$ for some $s \in S$. The Cayley graphs of groups are significant both in group theory and in constructions of graphs with interesting properties. The Cayley graph $\mathrm{Cay}\,(G, S)$ of a group G is symmetric or undirected if and only if $S = S^{-1}$.

The Cayley graph $\mathrm{Cay}\,(Z_n \oplus Z_2)$, $n \geq 3$ is a cubic graph which can be obtained as the cartesian product $P_2 \square C_n$ of a path on two vertices with a cycle on n vertices. The Cayley graph $\mathrm{Cay}\,(Z_n \oplus Z_2)$, $n \geq 3$ consists of an outer n-cycle $y_1 y_2 \cdots y_n$, an inner n-cycle $x_1 x_2 \cdots x_n$, and a set of n spokes $x_i y_i$, $i = 1, 2, \cdots, n$. We have $|V(\mathrm{Cay}\,(Z_n \oplus Z_2)| = 2n$, $|E(\mathrm{Cay}\,(Z_n \oplus Z_2)| = 3n$ and $|F(\mathrm{Cay}\,(Z_n \oplus Z_2)| = n+2$, where $|V(\mathrm{Cay}(Z_n \oplus Z_2)|$, $|E(\mathrm{Cay}(Z_n \oplus Z_2)|$ and $|F(\mathrm{Cay}(Z_n \oplus Z_2)|$ denote the number of vertices, edges and faces of the Cayley graph $\mathrm{Cay}\,(Z_n \oplus Z_2)$, respectively. The metric dimension of Cayley graph $\mathrm{Cay}\,(Z_n \oplus Z_2)$ has been determined in [3] while the metric dimension of Cayley graphs Cay $(\mathbb{Z}_n : S)$ for all $n \geq 7$ and $S = \{\pm 1, \pm 3\}$ has been determined in [14].

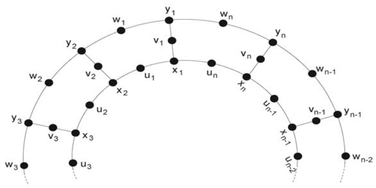

Fig. 1 The Barycentric Subdivision of Cayley Graph $Cay(Z_n \oplus Z_2)$

The barycentric subdivision graph $S(\text{Cay}\,(Z_n \oplus Z_2))$ can be obtained by adding a new vertex u_i between x_i and x_{i+1}, adding a new vertex v_i between x_i and y_i and adding a new vertex w_i between y_i and y_{i+1}, modulo n. Clearly, $S(\text{Cay}\,(Z_n \oplus Z_2))$ has $5n$ vertices and $6n$ edges.

The metric dimension of $P_m \square C_n$ has been determined in [3]. In the next theorem, we prove that the metric dimension of the barycentric subdivision $S(\text{Cay}\,(Z_n \oplus Z_2))$ is constant and only three vertices appropriately chosen suffice to resolve all the vertices of the $S(\text{Cay}\,(Z_n \oplus Z_2))$.

For our purpose, we call the cycle induced by $\{x_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$, the inner cycle, the cycle induced by $\{y_i : 1 \leq i \leq n\} \cup \{w_i : 1 \leq i \leq n\}$, the outer cycle and set of vertices $\{v_i : 1 \leq i \leq n\}$, the set of interior vertices. Note that the choice of appropriate basis vertices (also refereed to as landmarks in [15] is the core of the problem).

Theorem 4. *Let $S(Cay\,(Z_n \oplus Z_2))$ be the barycentric subdivision of Cayley graphs $Cay\,(Z_n \oplus Z_2)$; then $\beta(S(Cay\,(Z_n \oplus Z_2))) = 3$ for every $n \geq 4$.*

Proof. We will prove the above equality by double inequalities.

Case 1. When n is even.

We can write $n = 2k,\ k \geq 2,\ k \in \mathbf{Z}^+$. Let $W = \{x_1, x_2, x_{k+1}\} \subset V(S(\text{Cay}\,(Z_n \oplus Z_2)))$, we show that W is a resolving set for $S(\text{Cay}\,(Z_n \oplus Z_2))$ in this case. For this we give representations of any vertex of $V(S(\text{Cay}\,(Z_n \oplus Z_2))) \setminus W$ with respect to W.

Representations for the vertices of the inner cycle of $S(\text{Cay}\,(Z_n \oplus Z_2))$ are

$$r(x_i|W) = \begin{cases} (2i-2, 2i-4, 2k-2i+2), & 3 \leq i \leq k; \\ (4k-2i+2, 4k-2i+4, 2i-2k-2), & k+2 \leq i \leq 2k \end{cases}$$

$$r(u_i|W) = \begin{cases} (1, 1, 2k-1), & i = 1; \\ (2i-1, 2i-3, 2k-2i+1), & 2 \leq i \leq k; \\ (2k-1, 2k-1, 1), & i = k+1; \\ (4k-2i+1, 4k-2i+3, 2i-2k-1), & k+2 \leq i \leq 2k. \end{cases}$$

Representations for the set of interior vertices of $S(\text{Cay}\,(Z_n \oplus Z_2))$ are

$$r(v_i|W) = \begin{cases} (1, 3, 2k+1), & i = 1; \\ (2i-1, 2i-3, 2k-2i+3), & 2 \leq i \leq k+1; \\ (4k-2i+3, 4k-2i+5, 2i-2k-1), & k+2 \leq i \leq 2k. \end{cases}$$

Representations for the vertices on the outer cycle of $S(\mathrm{Cay}\,(Z_n \oplus Z_2))$ are

$$r(y_i|W) = \begin{cases} (2,4,2k+2), & i=1; \\ (2i,2i-2,2k-2i+4), & 2 \le i \le k+1; \\ (4k-2i+4,4k-2i+6,2i-2k), & k+2 \le i \le 2k \end{cases}$$

$$r(w_i|W) = \begin{cases} (3,3,2k+1), & i=1; \\ (2i+1,2i-1,2k-2i+3), & 2 \le i \le k; \\ (2k+1,2k+1,3), & i=k+1; \\ (4k-2i+3,4k-2i+5,2i-2k+1), & k+2 \le i \le 2k. \end{cases}$$

We note that there are no two vertices having the same representations implying that $\beta(S(\mathrm{Cay}\,(Z_n \oplus Z_2))) \le 3$.

On the other hand, we show that $\beta(S(\mathrm{Cay}\,(Z_n \oplus Z_2))) \ge 3$. Suppose on contrary that $\beta(S(\mathrm{Cay}\,(Z_n \oplus Z_2))) = 2$, then there are the following possibilities to be discussed.

(1) Both vertices are in the inner cycle. Here are the following subcases.

• Both vertices belong to the set $\{x_i : 1 \le i \le n\}$. Without loss of generality, we can suppose that one resolving vertex is x_1. Suppose that the second resolving vertex is x_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(u_n|x_1,x_t) = r(v_1|x_1,x_t) = (1,2t-1)$, and for $t=k+1$, we have $r(u_1|x_1,x_{k+1}) = r(u_n|x_1,x_{k+1}) = (1,2k-1)$, a contradiction.

• Both vertices belong to the set $\{u_i : 1 \le i \le n\}$. Without loss of generality, we can suppose that one resolving vertex is u_1. Suppose that the second resolving vertex is u_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(u_n|u_1,u_t) = r(v_1|u_1,u_t) = (2,2t)$, and for $t=k+1$, we have $r(x_1|u_1,u_{k+1}) = r(x_2|u_1,u_{k+1}) = (1,2k-1)$, a contradiction.

• One vertex is in the set $\{x_i : 1 \le i \le n\}$ and the other one is in the set $\{u_i : 1 \le i \le n\}$. Without loss of generality, we can suppose that one resolving vertex is x_1. Suppose that the second resolving vertex is u_t $(1 \le t \le k+1)$. Then for $1 \le t \le k$, we have $r(u_n|x_1,u_t) = r(v_1|x_1,u_t) = (1,2t)$, and for $t=k+1$, we have $r(u_1|x_1,u_{k+1}) = r(v_1|x_1,u_{k+1}) = (1,2k)$, a contradiction.

(2) Both vertices are the interior vertices. Without loss of generality, we can suppose that one resolving vertex is v_1. Suppose that the second resolving vertex is v_t $(2 \le t \le k+1)$. Then for $2 \le t \le k+1$, we have $r(x_1|v_1,v_t) = r(y_1|v_1,v_t) = (1,2t-1)$, a contradiction.

(3) Both vertices are in the outer cycle. Due to the symmetry of the graph, this case is analogous to case (1).

(4) One vertex is in the inner cycle and the other one is in the set of interior vertices. Here are the two subcases.

• One vertex is in the set $\{x_i : 1 \le i \le n\}$ and the other one is in the set of interior vertices. Without loss of generality, we can suppose that one resolving vertex is x_1. Suppose that the second resolving vertex is v_t $(1 \le t \le k+1)$. Then for $t=1$, we have $r(u_1|x_1,v_1) = r(u_n|x_1,v_1) = (1,2)$. For $2 \le t \le k$, $r(u_n|x_1,v_t) = r(v_1|x_1,v_t) = (1,2t)$ and for $t=k+1$, we have $r(u_1|x_1,v_{k+1}) = r(u_n|x_1,v_{k+1}) = (1,2k)$, a contradiction.

• One vertex is in the set $\{u_i : 1 \le i \le n\}$ and the other one is in the set of interior vertices. Without loss of generality, we can suppose that one resolving vertex is u_1. Suppose that the second resolving vertex is v_t $(1 \le t \le k+1)$. Then for $t=1$, we have $r(w_1|u_1,v_1) = r(w_n|u_1,v_1) = (2,4)$. For $2 \le t \le k$, $r(u_n|u_1,v_t) = r(v_1|u_1,v_t) = (2,2t)$ and for $t=k+1$, we have $r(u_n|u_1,v_{k+1}) = r(v_2|u_1,v_{k+1}) = (2,2k)$, a contradiction.

(5) One vertex is in the outer cycle and the other one is in the set of interior vertices. Due to the symmetry of the graph, this case is analogous to case (4).

(6) One vertex is in the inner cycle and the other one is in the outer cycle. We have the following subcases.

• One vertex is in the set $\{x_i : 1 \le i \le n\}$ and the other one is in the set $\{y_i : 1 \le i \le n\}$. Without loss of generality, we can suppose that one resolving vertex is x_1. Suppose that the second resolving vertex is y_t $(1 \le t \le k+1)$. Then for $t=1$, we have $r(u_1|x_1,y_1) = r(u_n|x_1,y_1) = (1,3)$. For $2 \le t \le k+1$, we have $r(u_1|x_1,y_t) = r(v_1|x_1,y_t) = (1,2t-1)$, a contradiction.

• One vertex is in the set $\{x_i : 1 \le i \le n\}$ and the other one is in the set $\{w_i : 1 \le i \le n\}$. Without loss of generality, we can suppose that one resolving vertex is x_1. Suppose that the second resolving vertex is w_t $(1 \le t \le k+1)$. Then for $t = 1$, we have $r(u_1|x_1, w_1) = r(u_n|x_1, w_1) = (1, 4)$. For $2 \le t \le k+1$, we have $r(u_1|x_1, w_t) = r(v_1|x_1, w_t) = (1, 2t)$, a contradiction.

• One vertex is in the set $\{u_i : 1 \le i \le n\}$ and the other is in the set $\{y_i : 1 \le i \le n\}$. Due to the symmetry of the graph, this subcase is analogous to above subcase.

• One vertex is in the set $\{u_i : 1 \le i \le n\}$ and the other one is in the set $\{w_i : 1 \le i \le n\}$. Without loss of generality, we can suppose that one resolving vertex is u_1. Suppose that the second resolving vertex is w_t $(1 \le t \le k+1)$. Then for $t = 1$, we have $r(x_1|u_1, w_1) = r(x_2|u_1, w_1) = (1, 3)$. For $t = 2$, $r(v_3|u_1, w_2) = r(w_1|u_1, w_2) = (4, 2)$ and when $3 \le t \le k+1$, we have $r(v_3|u_1, w_t) = r(w_2|u_1, w_t) = (4, 2t-4)$, a contradiction.

Hence from above it follows that there is no resolving set with two vertices for $V(S(\text{Cay}\,(Z_n \oplus Z_2)))$ implying that $\beta(S(\text{Cay}\,(Z_n \oplus Z_2))) = 3$.

Case 2. When n is odd.

We can write $n = 2k+1$, $k \ge 2$, $k \in \mathbf{Z}^+$. Let $W = \{x_1, x_2, u_{k+1}\} \subset V(S(\text{Cay}\,(Z_n \oplus Z_2))) \setminus W$, we show that W is a resolving set for $S(\text{Cay}\,(Z_n \oplus Z_2))$. For this we give representations of any vertex of $V(S(\text{Cay}\,(Z_n \oplus Z_2))) \setminus W$ with respect to W.

Representations for the vertices on the inner cycle of $S(\text{Cay}\,(Z_n \oplus Z_2))$ are

$$r(x_i|W) = \begin{cases} (2i-2, 2i-4, 2k-2i+3), & 3 \le i \le k+1; \\ (2k, 2k, 1), & i = k+2; \\ (4k-2i+4, 4k-2i+6, 2i-2k-3), & k+3 \le i \le 2k+1 \end{cases}$$

$$r(u_i|W) = \begin{cases} (1, 1, 2k), & i = 1; \\ (2i-1, 2i-3, 2k-2i+2), & 2 \le i \le k; \\ (4k-2i+3, 4k-2i+5, 2i-2k-2), & k+2 \le i \le 2k+1. \end{cases}$$

Representations for the set of interior vertices of $S(\text{Cay}\,(Z_n \oplus Z_2))$ are

$$r(v_i|W) = \begin{cases} (1, 3, 2k-2), & i = 1; \\ (2i-1, 2i-3, 2k-2i+4), & 2 \le i \le k+1; \\ (2k-3, 2k-1, 2), & i = k+2; \\ (4k-2i+1, 4k-2i+3, 2i-2k-2), & k+3 \le i \le 2k+1. \end{cases}$$

Representations for the vertices on the outer cycle of $S(\text{Cay}\,(Z_n \oplus Z_2))$ are

$$r(y_i|W) = \begin{cases} (2, 4, 2k-1), & i = 1; \\ (2i, 2i-2, 2k-2i+5), & 2 \le i \le k+1; \\ (2k-2, 2k, 3), & i = k+2; \\ (4k-2i+2, 4k-2i+4, 2i-2k-1), & k+3 \le i \le 2k+1 \end{cases}$$

$$r(w_i|W) = \begin{cases} (3, 3, 2k), & i = 1; \\ (2i+1, 2i-1, 2k-2i+4), & 2 \le i \le k; \\ (2k-1, 2k+1, 3), & i = k+1; \\ (4k-2i+1, 4k-2i+3, 2i-2k), & k+2 \le i \le 2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representations which implies that $\beta(S(\text{Cay}\,(Z_n \oplus Z_2))) \le 3$. On the other hand, suppose that $\beta(S(\text{Cay}\,(Z_n \oplus Z_2))) = 2$, then there are the same possibilities as in Case (1) and contradictions can be deduced analogously. This implies that $\beta(S(\text{Cay}\,(Z_n \oplus Z_2))) = 3$, which completes the proof. □

3 Conclusion

The problem of determining whether $\beta(G) < k$ is an NP-complete problem. In this paper, we have studied the metric dimension of barycentric subdivision of Cayley graphs Cay $(Z_n \oplus Z_2)$. We proved that these subdivisions of Cayley graphs have constant metric dimension and only three vertices chosen appropriately suffice to resolve all the vertices of subdivisions of Cayley graphs Cay $(Z_n \oplus Z_2)$. It is natural to ask for characterization of graph classes with constant metric dimension. We close this section by raising a question that naturally arises from the text.

Open Problem. Let G be a non trivial connected graph and $S(G)$ denotes its barycentric subdivision. Characterize all those graphs G for which $\beta(G) = \beta(S(G))$.

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