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A Fixed Point Method for the Linear Complementarity Problem Arising from American Option Pricing

Xian-Jun SHI, Lei YANG, Zheng-Hai HUANG†

Department of Mathematics, School of Science, Tianjin University, Tianjin 300072, China
(†E-mail: huangzhenghai@tju.edu.cn)

Abstract For American option pricing, the Black-Scholes-Merton model can be discretized as a linear complementarity problem (LCP) by using some finite difference schemes. It is well known that the Projected Successive Over Relaxation (PSOR) has been widely applied to solve the resulted LCP. In this paper, we propose a fixed point iterative method to solve this type of LCPs, where the splitting technique of the matrix is used. We show that the proposed method is globally convergent under mild assumptions. The preliminary numerical results are reported, which demonstrate that the proposed method is more accurate than the PSOR for the problems we tested.

Keywords American option pricing; finite difference method; fixed point method; linear complementarity problem

2000 MR Subject Classification 90C33; 91G20

1 Introduction

Option is one of the most important financial tools in finance market today. It gives the holder of the option the right to do something in the future, but the holder does not have to exercise this right. For European options, the analytic solution is relatively easier to be obtained by Black-Scholes formula. But, there is no exact analytical solution for American option, because the American option may be early exercised before the expiry date. Thus, the numerical method for pricing American options has important practical significance and is currently regarded as one of the important topics in Finance. The classical model for option pricing is the Black-Scholes model proposed in 1973 by Black and Scholes who assumed that the price of the underlying asset following a geometric Brownian motion with constant interest rate and volatility[2].

One of the main and simple numerical methods for American option pricing is the binomial tree method introduced by Cox, Ross, and Rubinstein[7]. On the other hand, Brennan and Schwartz[3,4] and Schwartz[17] applied finite difference methods in solving American options. Up to now, there are various kinds of schemes extended from the original finite difference scheme such as Crank-Nicolson method, upwind finite difference method[9], compact finite difference[24] method, and so on. After the discretion of the model by finite difference or finite element methods, American option pricing can be solved as a linear complementarity problem (LCP)[9] or a variational inequality[14].

In order to solve the discretized LCP arising in American option pricing, many methods have been proposed in the literature. One of the popular and effective methods is the projected successive over relaxation method (PSOR)[18]. Other methods include the improved PSOR (IPSOR)[15], penalty methods[16,23,25], operator splitting methods[11], componentwise splitting

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†Corresponding author.

methods[12], Newton method[5], the artificial boundary method[8,10,20] and several new active-set methods[1,19].

In this paper, we propose a new method based on fixed point (FP) iteration for the resulted LCP. To this end, the Black-Scholes partial differential equation (PDE) is first transformed to a standard heat equation. As shown by Tavella and Randall[21], this transformation leads to the flattening of the eigenvalue distribution of the discrete PDE operator. This means that any numerical scheme applied to the heat equation has better stability range than the case when it is used to discretize the Black-Scholes equation. We use the Crank-Nicolson scheme to discretize the standard heat equation; and obtain a symmetric tridiagonal P-matrix with equal subdiagonal elements and equal diagonal elements. These characteristics are significant for the convergence of the FP method to solve the LCP. We can accelerate the algorithm by taking advantage of the sparsity of the matrix and the equality of elements. Furthermore, the proposed FP method is proved to be globally convergent under mild assumptions.

The numerical results given in this paper show that the FP method for solving the LCP arising in American option pricing is more effective than PSOR with nearly equal time, higher accuracy and less iterations. Besides, with the increasing of the length of interval and the number of grid points, the FP method presents higher accuracy compared with PSOR. To compare the efficiency of these methods, we take the results of binomial tree with 5,000 steps as the true values.

The rest of this paper is organized as follows. In Section 2, the Black-Scholes-Merton model and the process of transforming to a standard heat equation are presented and a finite difference scheme for the space discretization and the time discretization scheme are described. In Section 3, the specific fixed point method for the LCP and its proof of convergence are given. In the last section, we present the results of some numerical tests and comparisons between different methods.

Now, we briefly introduce some essential notations used in this paper. For a vector $x \in \Re^n$ or a matrix $A \in \Re^{n\times n}$, $|x|$ and $|A|$ represent the corresponding vector and matrix whose components are the absolute value of the corresponding components of x and A. The identity matrix is denoted by I. For $i = 1, 2, \dots, n$, the symbol x_+ stands for the vector with elements $(x_+)_i = \max\{0, x_i\}$ and x_- stands for the vector with elements $(x_-)_i = \max\{0, -x_i\}$.

2 Transformation and Discretization

2.1 Transformation of Black-Scholes-Merton Model

In this paper, we focus on the one factor option model. Suppose that the financial market consists of a risky asset with fluctuant price (S_t) and fixed volatility $(\sigma > 0)$ in a risk neutral economy with constant interest rate $(r > 0)$. Besides, K denotes the strike price, T denotes the maturity time and μ stands for the expected return on underlying asset per year. Assuming that the asset pays out the dividend $\delta S_t dt$ during a time interval dt, the price of underlying asset S_t satisfies the stochastic differential equation: $\frac{dS_t}{S_t} = (\mu - \delta)dt + \sigma dW$, where dW is a standard Wiener process with mean zero and variance dt. Let $G(S,t)$ and $V(S,t)$ denote the given payoff function of an American option and the value of this option correspondingly. By a standard no-arbitrage argument, $V(S,t)$ must satisfy the following complementarity conditions:

$$
\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV \le 0,\\
&V(S,t) \ge G(S,t),\\
&\Big(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta)\, S \frac{\partial V}{\partial S} - rV\Big)(V(S,t) - G(S,t)) = 0,
\end{aligned}
$$

where

$$S > 0, \qquad 0 \le t \le T, \tag{2.1}$$

$$V(S,T) = \max\{S_T - K, 0\}, \tag{2.2}$$

$$G(S,t) = \begin{cases} \max\{S_t - K, 0\}, & \text{for call}, \\ \max\{K - S_t, 0\}, & \text{for put}. \end{cases}$$

For better stability range as shown in [21], we let

$$\begin{cases} S = Ke^x, \quad t = T - \dfrac{2\tau}{\sigma^2}, \quad h := \dfrac{2r}{\sigma^2}, \quad h_\delta = \dfrac{2(r-\delta)}{\sigma^2}, \\ V(S,t) = V\Big(Ke^x, T - \dfrac{2\tau}{\sigma^2}\Big) =: v(x,\tau), \\ G(S,t) = G\Big(Ke^x, T - \dfrac{2\tau}{\sigma^2}\Big) =: g(x,\tau), \\ v(x,\tau) =: K\exp\Big\{ -\dfrac{1}{2}(h_\delta - 1)x - \Big(\dfrac{1}{4}(h_\delta-1)^2 + h\Big)\tau\Big\} y(x,\tau). \end{cases} \tag{2.3}$$

A property of the transformation (2.3) is the parameters r, σ and δ must be constants. From the transformation of the expiration time $t = T$ is determined in the new time by $\tau = 0$, and $t = 0$ is transformed to $\tau = \frac{1}{2}\sigma^2 T$. Up to the scaling by $\frac{1}{2}\sigma^2$, the time τ represents the remaining life time of the American option. And the original domain of the half strip (2.1) becomes the strip: $-\infty < x < \infty,\ 0 \le \tau \le \frac{1}{2}\sigma^2 T$, on which we will derive a solution $y(x,\tau)$ and then get the value function $V(S,t)$. Under the transformation, the terminal condition (2.2) becomes an initial condition for $y(x,0)$. From the transformation we find $V(S,T) = K\exp\big\{-\frac{x}{2}(h_\delta - 1)\big\}\,y(x,0)$, and then $y(x,0) = \max\big\{e^{\frac{x}{2}(h_\delta+1)} - e^{\frac{x}{2}(h_\delta-1)}, 0\big\}$. For the put option, we derive that

$$y(x,0) = \max\{e^{\frac{x}{2}(h_\delta-1)} - e^{\frac{x}{2}(h_\delta+1)}, 0\}. \tag{2.4}$$

Then, combining (2.3) with (2.4) and (2.5), the original Black-Scholes-Merton model changes to the following linear complementarity problem:

$$\Big(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\Big)(y - g) = 0, \qquad \frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} \ge 0, \qquad y - g \ge 0. \tag{2.5}$$

Here the transformed payoff function $g(x,\tau)$ is given by

- for put:

$$g(x,\tau) = \exp\Big\{\frac{\tau}{4}((h_\delta-1)^2 + 4h)\Big\}\max\{e^{\frac{x}{2}(h_\delta-1)} - e^{\frac{x}{2}(h_\delta+1)}, 0\};$$

- for call:

$$g(x,\tau) = \exp\big\{\frac{\tau}{4}((h_\delta-1)^2 + 4h)\big\}\max\{e^{\frac{x}{2}(h_\delta+1)} - e^{\frac{x}{2}(h_\delta-1)}, 0\}.$$

The initial and boundary conditions become:

$$y(x,0) = g(x,0), \qquad \lim_{x\to\pm\infty} y(x,\tau) = \lim_{x\to\pm\infty} g(x,\tau).$$

As outlined in (2.5), the American option pricing can be regarded as an order complementarity problem (OCP) and the next step is to discretize that OCP.

2.2 Discretization with Finite Difference Scheme

We apply a finite difference approximation scheme to the partial differential complementarity system (2.5). We perform a full discretization for both x-axis and τ-axis leading to a two dimensional grid. Let $\Delta\tau$ and Δx be the mesh sizes of the discretizations of τ and x. The step in τ is $\Delta\tau := \tau_{\max}/m$ for $\tau_{max} := \frac{1}{2}\sigma^2 T$ and a suitable integer m. We use a finite interval $a \le x \le b$ to replace the infinite interval $-\infty < x < \infty$. Here the end values a and b should be chosen corresponding $S_{\min} = Ke^a$ and $S_{\max} = Ke^b$ and the interval $S_{min} \le S \le S_{\max}$ is the main factor that affects the sufficiency of the approximation. We can discretize the interval of x into n equal parts with step length $\Delta x := (b-a)/n$. The additional notations for the grid are given as following: $\tau_j := j \cdot \Delta\tau$ for $j = 0, 1, \cdots, m$; $x_i := a + i\Delta x$ for $i = 0, 1, \ldots, n$; $y_{ij} := y(x_i, \tau_j)$ and $g_{ij} := g(x_i, \tau_j)$; we employ $u_{ij} := u(x_i, \tau_j)$ to approximate y_{ij}.

We use a parameter θ to combine the backward difference, the explicit and the Crank-Nicolson method into one formula:

$$\frac{u_{i,j+1} - u_{ij}}{\Delta j} = \theta \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{\Delta x^2} + (1-\theta)\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2},$$

with the choice $\theta = 0$ (explicit), $\theta = \frac{1}{2}$ (Crank-Nicolson) and $\theta = 1$ (backward difference method). Then, the differential inequality $\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} \ge 0$ is boiled down to the discrete formulation:

$$\begin{aligned} &u_{i,j+1} - \lambda\theta(u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}) \\ &- u_{i,j} - \lambda(1-\theta)(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) \ge 0, \end{aligned} \tag{2.6}$$

where we use the abbreviation $\lambda := \frac{\Delta\tau}{\Delta x^2}$. Now, we introduce the following notations:

$$\begin{cases} b^{(j)} := (b_{1,j}, \ldots, b_{n-1,j})^T, \\ u^{(j)} := (u_{1,j}, \ldots, u_{n-1,j})^T, \\ g^{(j)} := (g_{1,j}, \ldots, g_{n-1,j})^T, \\ b_{i,j} := u_{i,j} + \lambda(1-\theta)(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}), i = 2, \ldots, n-2, \\ b_{1,j} = u_{1,j} + \lambda(1-\theta)(u_{2,j} - 2u_{1,j} + g_{0,j}) + \lambda\theta g_{0,j+1}, \\ b_{n-1,j} = u_{n-1,j} + \lambda(1-\theta)(g_{n,j} - 2u_{n-1,j} + u_{n-2,j}) + \lambda\theta g_{n,j+1}; \end{cases} \tag{2.7}$$

$$A := \begin{pmatrix} 1+2\lambda\theta & -\lambda\theta & & 0 \\ -\lambda\theta & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix} \in \Re^{(n-1)\times(n-1)}. \tag{2.8}$$

Therefore, with the notations in (2.7) and (2.8), (2.6) and the inequality $y - g \ge 0$ can be written in vector form as

$$Au^{(j+1)} \ge b^{(j)}, \qquad u^{(j)} \ge g^{(j)}. \tag{2.9}$$

Meanwhile, the equation $(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2})(y - g) = 0$ becomes

$$(Au^{(j+1)} - b^{(j)})^T(u^{(j+1)} - g^{(j+1)}) = 0 \tag{2.10}$$

with the initial and boundary conditions:

$$\begin{aligned} &u_{i,0} = g_{i,0}, \qquad i = 1, \cdots, n-1, \qquad u^{(0)} = g(0); \\ &u_{0,j} = g_{0,j}, \qquad u_{n,j} = g_{n,j}, \qquad j \ge 1. \end{aligned} \tag{2.11}$$

At last, we attain the LCP in the form as showing (2.7)–(2.11) together. We conclude the formulation of the problem as:

$$(Au - b)(u - g) = 0, \qquad Au - b \geq 0, \qquad u - g \geq 0. \tag{2.12}$$

3 Algorithm and Convergence

In this section, we will give the equivalence between original problems and fixed point problems, then prove that the algorithm converges to the unique solution of the original LCP under a mild condition. We use the transformation $x = u - g$ and $q = b - Ag$, then Problem (2.12) becomes:

$$(Ax - q)x = 0, \qquad Ax - q \geq 0, \qquad x \geq 0. \tag{3.1}$$

Then we will propose a fixed point algorithm for Problem (3.1) and show the convergence of the algorithm.

Lemma 3.1. *Problem (3.1) has a unique solution. Let $\alpha > 0$ be given. If x^* is a solution of*

$$x = x_+ - \alpha(Ax_+ - q), \tag{3.2}$$

then x_+^* is a solution of (3.1), and hence, $x_+^* + g$ is a solution of Problem (2.12).

Proof. Noticing that the matrix A is symmetric, strictly diagonally dominant and has positive diagonal entries, we can get A is positive definite. Since A is symmetric and positive definite, it is well known that A is a P-matrix. Thus, the LCP (3.1) has a unique solution[6].

Since x^* is a solution of (3.2), then

$$x_-^* = x_+^* - x^* = \alpha(Ax_+^* - q).$$

where $\alpha > 0$. Noticing that

$$(x_-^*)^T x_+^* = 0, \qquad x_+^* \geq 0, \qquad x_-^* \geq 0,$$

we obtain that x_+^* is a solution of Problem (3.1), and hence, $x_+^* + g$ is a solution of (2.12). □

In the following, k denotes the number of iterations for every time step, n and m stand for the numbers of the grid points of space and time correspondingly. For the matrix A, we use the decomposition $A = D - L - U$, where D, L and U are diagonal, strictly lower and upper triangular matrices obtained from A, respectively.

Algorithm 3.1 (Fixed Point Algorithm for Problem (3.2)).

Initialization *Given $x^{(0)} \in \Re^{n-1}$, error $\varepsilon > 0$, $\alpha > 0$, and $A = D - L - U$. Set $k = 0$.*

repeat *until* $\|x^{(k+1)} - x^{(k)}\| < \varepsilon$

$x_+^{(k)} = \max\{x^{(k)}, 0\}$

for $i = 1, \cdots, n-1$

$$x_i^{(k+1)} = (x_+^{(k)})_i - \alpha[(D - U)x_+^{(k)} - Lx_+^{(k+1)} - q]_i \tag{3.3}$$

end

$k = k + 1$

end repeat

In Algorithm 3.1, α is the relaxation parameter as ω in PSOR.

The following lemma can be found in [22].

Lemma 3.2. *Suppose that $Z = (z_{i,j}) \in R^{n\times n}$ is an arbitrary matrix. Then*

$$\rho(Z) \le \max_{1\le i\le n} \sum_{j=1}^{n} |z_{i,j}|,$$

where $\rho(Z)$ denotes the spectral radius of Z.

Now, we show the global convergence of Algorithm 3.1.

Theorem 3.1. *Let $A \in \Re^{(n-1)\times(n-1)}$ be defined as (2.8) and $q \in \Re^{n-1}$. If $0 < \alpha < \frac{2}{1+4\lambda\theta}$ holds, then the sequence $\{x^{(k)}\}$ generated by Algorithm 3.1 converges to the unique solution x^* for Problem (3.2) (and hence, x_+^* is the unique solution for Problem (3.1)).*

Proof. By iterative formula (3.3), we have

$$x^{k+1} = x_+^k - \alpha[(D-U)x_+^k - Lx_+^{k+1} - q],$$

and hence,

$$\begin{aligned} x^{k+1} - x^k =&(x_+^k - x_+^{k-1}) - \alpha[(D-U)(x_+^k - x_+^{k-1}) - L(x_+^{k+1} - x_+^k)] \\ =&[I - \alpha(D-U)](x_+^k - x_+^{k-1}) + \alpha L(x_+^{k+1} - x_+^k). \end{aligned}$$

Note that

$$(x+y)_+ \le x_+ + y_+, \qquad |x| = x_+ + x_- = x_+ + (-x)_+,$$

we further obtain that

$$\begin{aligned} |x^{k+1} - x^k| =&(x^{k+1} - x^k)_+ + (x^k - x^{k+1})_+ \\ =&\{[I-\alpha(D-U)](x_+^k - x_+^{k-1}) + \alpha L(x_+^{k+1} - x_+^k)\}_+ \\ &+ \{[I-\alpha(D-U)](x_+^{k-1} - x_+^k) + \alpha L(x_+^k - x_+^{k+1})\}_+ \\ \le&\{[I-\alpha(D-U)](x_+^k - x_+^{k-1})\}_+ + \{\alpha L(x_+^{k+1} - x_+^k)\}_+ \\ &+ \{[I-\alpha(D-U)](x_+^{k-1} - x_+^k)\}_+ + \{\alpha L(x_+^k - x_+^{k+1})\}_+ \\ =&|[I-\alpha(D-U)](x_+^k - x_+^{k-1})| + |\alpha L(x_+^{k+1} - x_+^k)| \\ \le&|I-\alpha(D-U)||x_+^k - x_+^{k-1}| + \alpha|L||x_+^{k+1} - x_+^k| \\ \le&|I-\alpha(D-U)||x^k - x^{k-1}| + \alpha|L||x^{k+1} - x^k|. \end{aligned}$$

Thus,

$$|x^{k+1} - x^k| \le P^{-1}Q|x^k - x^{k-1}|$$

and

$$\|x^{k+1} - x^k\|_2 \le \|P^{-1}Q|x^k - x^{k-1}|\|_2 \le \|P^{-1}Q\|_2\|x^k - x^{k-1}\|_2$$

hold, where $P = I - \alpha|L|$ and $Q = |I - \alpha(D-U)|$.

If the spectral radius satisfies the condition: $\rho(P^{-1}Q) = \|P^{-1}Q\|_2 < 1$, then the sequence $\{x^{(k)}\}$ converges.

Since

$$P^{-1} = (I - \alpha|L|)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \alpha\lambda\theta & 1 & 0 & 0 & 0 & \cdots \\ (\alpha\lambda\theta)^2 & \alpha\lambda\theta & 1 & 0 & 0 & \cdots \\ (\alpha\lambda\theta)^3 & (\alpha\lambda\theta)^2 & \alpha\lambda\theta & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$Q = |I - \alpha(D-U)| = \begin{pmatrix} |1-\alpha(1+2\lambda\theta)| & \alpha\lambda\theta & 0 & \cdots \\ 0 & |1-\alpha(1+2\lambda\theta)| & \alpha\lambda\theta & \cdots \\ 0 & 0 & |1-\alpha(1+2\lambda\theta)| & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

we have

$$P^{-1}Q = (I - \alpha|L|)^{-1}|I-\alpha(D-U)| = \begin{pmatrix} a_1 & b_1 & 0 & 0 & \cdots \\ a_2 & b_2 & b_1 & 0 & \cdots \\ a_3 & b_3 & b_2 & b_1 & \cdots \\ a_4 & b_4 & b_3 & b_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

whose entries are given by

$$\begin{aligned} &a_1 = |1-\alpha(1+2\lambda\theta)|; \\ &a_i = (\alpha\lambda\theta)^{i-1}|1-\alpha(1+2\lambda\theta)|, \qquad i = 2,3,\cdots,n-1; \\ &b_1 = \alpha\lambda\theta; \\ &b_j = (\alpha\lambda\theta)^j + (\alpha\lambda\theta)^{j-2}|1-\alpha(1+2\lambda\theta)|, \qquad j = 2,3,\cdots,n-1. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{i=1}^{n-1}|a_i| =& |1-\alpha(1+2\lambda\theta)|(1+(\alpha\lambda\theta)+(\alpha\lambda\theta)^2+\cdots+(\alpha\lambda\theta)^{n-2}) \\ =& |1-\alpha(1+2\lambda\theta)|\frac{1-(\alpha\lambda\theta)^{n-1}}{1-\alpha\lambda\theta}, \\ \sum_{j=1}^{n-1}|b_j| =& \alpha\lambda\theta + [(\alpha\lambda\theta)^2 + |1-\alpha(1+2\lambda\theta)|] + \cdots \\ &+ [(\alpha\lambda\theta)^{n-1} + (\alpha\lambda\theta)^{n-3}|1-\alpha(1+2\lambda\theta)|] \\ =& [\alpha\lambda\theta + (\alpha\lambda\theta)^2 + \cdots + (\alpha\lambda\theta)^{n-1}] \\ &+ |1-\alpha(1+2\lambda\theta)|[1+\alpha\lambda\theta+\cdots+(\alpha\lambda\theta)^{n-3}] \\ =& \frac{\alpha\lambda\theta-(\alpha\lambda\theta)^{n-1}}{1-\alpha\lambda\theta} + |1-\alpha(1+2\lambda\theta)|\,\frac{1-(\alpha\lambda\theta)^{n-2}}{1-\alpha\lambda\theta}. \end{aligned}$$

From the condition $0 < \alpha < \frac{2}{1+4\lambda\theta}$, we have $\alpha\lambda\theta < \frac{2\lambda\theta}{1+4\lambda\theta} < 1$. This implies $\sum_{i=1}^{n-1}|a_i| \le \sum_{i=1}^{n-1}|b_i|$ and hence the second column is the maximum of the column sums of the moduli of the entries of the matrix $P^{-1}Q$.

When $0 < \alpha \le \frac{1}{1+2\lambda\theta}$, we have

$$\begin{aligned} &\frac{\alpha\lambda\theta}{1-\alpha\lambda\theta} + |1-\alpha(1+2\lambda\theta)|\frac{1}{1-\alpha\lambda\theta} \\ =& \frac{\alpha\lambda\theta}{1-\alpha\lambda\theta} + \frac{1-\alpha(1+2\lambda\theta)}{1-\alpha\lambda\theta} = 1 - \frac{\alpha}{1-\alpha\lambda\theta} < 1; \end{aligned}$$

and when

$$\frac{1}{1+2\lambda\theta} < \alpha < \frac{2}{1+4\lambda\theta},$$

we have

$$\begin{aligned} &\frac{\alpha\lambda\theta}{1-\alpha\lambda\theta}+|1-\alpha(1+2\lambda\theta)|\,\frac{1}{1-\alpha\lambda\theta}\\ =&\frac{\alpha\lambda\theta}{1-\alpha\lambda\theta}-\frac{1-\alpha(1+2\lambda\theta)}{1-\alpha\lambda\theta}\\ =&1-\frac{2-\alpha(1+4\lambda\theta)}{1-\alpha\lambda\theta}<1. \end{aligned}$$

Hence, when $0<\alpha<\frac{2}{1+4\lambda\theta}$, inequality

$$\frac{\alpha\lambda\theta}{1-\alpha\lambda\theta}+|1-\alpha(1+2\lambda\theta)|\frac{1}{1-\alpha\lambda\theta}<1$$

holds.

Furthermore, by applying Lemma 3.2, we have

$$\rho(P^{-1}Q)\le\sum_{j=1}^{n-1}|b_j|<\frac{\alpha\lambda\theta}{1-\alpha\lambda\theta}+|1-\alpha(1+2\lambda\theta)|\frac{1}{1-\alpha\lambda\theta}<1.$$

Therefore, the sequence $\{x^k\}$ is convergent. This, together with (3.3) and Lemma 3.1, implies that the desired result holds. □

4 Numerical Experiments

In this section, we report the results of numerical experiments comparing the PSOR and the algorithm proposed in this paper (Algorithm 3.1) on the standard Black-Scholes-Merton model. The comparisons are based on the accuracy of approximating option values and the total computation cost (the CPU time). Since the analytical solution of option values are unknown, we use the binomial method with large steps (5000) as the exact option values for each instance, of which the results are considered very accurate with large steps. All experiments are done at a PC with CPU of 2.80GHz and RAM of 4.0GB, and all codes are executed in MATLAB 7.8.0.

Throughout our numerical experiments, we value (find the fair value) an American put option based on a single underlying asset S, with exercise price K, interest rate r, volatility σ, dividend yield δ, expiry time T and the interval $[a,b]$ of x. A variety of grids, consisting of a number of x-nodes (n) combined with a number of time steps (m), have been applied in the discretization process. All algorithms are terminated when two consecutive iterations satisfy $\|x^{(k+1)}-x^{(k)}\|<10^{-6}$. In Algorithm 3.1, we set the vector $x^{(0)}=\mathbf{0}$ as the initial point at the each time step in LCP. For the PSOR method, the result in [13] shows that the relaxation parameter

$$\omega=\frac{2}{1+\sqrt{1-\rho^2}}$$

here $\rho\approx\max\frac{1}{A_{i,i}}\sum_{j\neq i}|A_{i,j}|$, is the most effective one. However, for FP method, we choose the parameter α as

$$\alpha=\frac{1.2+0.1\lambda\theta}{1+2\lambda\theta},$$

where $\lambda=\frac{\Delta\tau}{\Delta x^2}$ with $\Delta\tau=\frac{\frac{1}{2}\sigma^2T}{m}$ and $\Delta x=\frac{b-a}{n}$ respectively being the mesh sizes of the discretizations of τ and x, and θ is the parameter of difference scheme with the choice $\theta=0$ (explicit), $\theta=\frac{1}{2}$ (Crank-Nicolson) and $\theta=1$ (backward difference method).

Tables 1–6 present the results for the three methods under the CN ($\theta = 0.5$) scheme with different intervals of x, different values of T and different volatilities σ. The specific problem settings are given in each table, respectively. For each method, we report the grid steps of both x-direction and τ-direction (Grid (n, m)), the fair value of American option (V), the errors between the true values and the values returned by FP or PSOR (Error), the total number of iterations (Iter) and the total computing time (CPU). In each table, binomial denotes the exact value computed by binomial method with 5000 iterations.

According to the results in Tables 1–6, we can draw some desirable conclusions. First, we can see the FP method is more accurate than the PSOR method with nearly same time costs and iterations. Second, while the computing time and the number of iterations of the new method increasing considerably for increasing grid points, the accuracy of FP is increasing, however, the accuracy of PSOR decreases. Thirdly, by comparing Table 1 and Table 2, we aware that with longer expiry time T, the FP method have higher accuracy with less iterations. At last, comparison between Table 4 and Table 5 shows that the volatility σ has the same effect on accuracy of FP as T.

Table 1. $T = 0.5,\ S_0 = K = 100,\ \sigma = 0.2,\ \delta = 0,\ r = 0.05,\ a = -1.5,\ b = 1.5$

Method	Grid (n, m)	V	Error	Iter	CPU (s)
Binomial		4.655561	0		
FP	(400,200)	4.653438	2.12e-3	1795	0.61
	(600,400)	4.654667	8.93e-4	3597	1.30
	(800,600)	4.655110	4.50e-4	5397	2.48
	(800,800)	4.655116	4.44e-4	6399	3.16
	(1000,1000)	4.655318	2.42e-4	8996	5.18
PSOR	(400,200)	4.653160	2.40e-3	1698	0.59
	(600,400)	4.653713	1.84e-3	3539	1.40
	(800,600)	4.653845	1.71e-3	5827	2.54
	(800,800)	4.653955	1.60e-3	6970	3.21
	(1000,1000)	4.653714	1.84e-3	9626	5.31

Table 2. $T = 0.25,\ S_0 = K = 100,\ \sigma = 0.2,\ \delta = 0,\ r = 0.05,\ a = -1.5,\ b = 1.5$

Method	Grid (n, m)	V	Error	Iter	CPU (s)
Binomial		3.479755	0		
FP	(400,200)	3.476662	3.09e-3	1399	0.38
	(600,400)	3.478429	1.32e-3	2799	1.11
	(800,600)	3.479050	7.04e-4	4199	2.21
	(800,800)	3.479053	7.01e-4	5599	3.02
	(1000,1000)	3.479340	4.14e-4	6999	4.68
PSOR	(400,200)	3.476466	3.28e-3	1251	0.38
	(600,400)	3.478090	1.66e-3	2689	1.12
	(800,600)	3.478226	1.52e-3	4253	2.27
	(800,800)	3.478521	1.23e-3	5244	2.99
	(1000,1000)	3.477894	1.86e-3	6893	4.71

Table 3. $T = 0.5,\ S_0 = K = 100,\ \sigma = 0.2,\ \delta = 0,\ r = 0.05,\ a = -2,\ b = 2$

Method	Grid (n, m)	V	Error	Iter	CPU (s)
Binomial		4.655561	0		
FP	(600,300)	4.653916	1.64e-3	2398	0.90
	(800,400)	4.654667	8.93e-4	3597	1.71
	(1000,500)	4.655030	5.30e-4	4497	2.58
	(1500,1500)	4.655397	1.63e-4	11999	11.27
	(2000,2000)	4.655483	7.77e-5	16000	20.01
PSOR	(600,300)	4.653369	2.19e-3	2348	1.02
	(800,400)	4.653713	1.84e-3	3539	1.73
	(1000,500)	4.653671	1.88e-3	4893	2.62
	(1500,1500)	4.652573	2.98e-3	13267	11.31
	(2000,2000)	4.652570	2.99e-3	20556	21.75

Table 4. $T = 0.25,\ S_0 = K = 100,\ \sigma = 0.3,\ \delta = 0,\ r = 0.05,\ a = -3,\ b = 3$

Method	Grid (n, m)	V	Error	Iter	CPU (s)
Binomial		5.442150	0		
FP	(800,400)	5.440199	1.95e-3	3198	1.54
	(1600,800)	5.441780	3.69e-4	7197	6.16
	(2000,1000)	5.441972	1.77e-4	8998	9.60
	(2400,1500)	5.442081	6.81e-5	13498	17.29
	(3000,2500)	5.442161	1.13e-5	20000	35.60
PSOR	(800,400)	5.439731	2.41e-3	2738	1.51
	(1600,800)	5.440519	1.63e-3	7433	6.19
	(2000,1000)	5.440161	1.98e-3	10249	9.81
	(2400,1500)	5.439489	2.66e-3	15177	17.69
	(3000,2500)	5.437334	4.81e-3	24480	36.58

Table 5. $T = 0.25,\ S_0 = K = 100,\ \sigma = 0.2,\ \delta = 0,\ r = 0.05,\ a = -3,\ b = 3$

Method	Grid (n, m)	V	Error	Iter	CPU (s)
Binomial		3.479755	0		
FP	(800,400)	3.476744	3.01e-3	2400	1.49
	(1600,800)	3.479053	7.01e-4	5599	5.97
	(2000,1000)	3.479340	4.14e-4	6999	9.31
	(2400,1500)	3.479498	2.56e-4	10499	17.11
	(3000,2500)	3.479629	1.25e-5	15000	34.93
PSOR	(800,400)	3.476485	3.26e-3	1968	1.50
	(1600,800)	3.478521	1.23e-3	5344	6.17
	(2000,1000)	3.477894	1.86e-3	6893	9.52
	(2400,1500)	3.477718	2.03e-3	10324	17.14
	(3000,2500)	3.477480	2.27e-3	17142	35.41

Table 6. $T = 0.5,\ S_0 = K = 100,\ \sigma = 0.2,\ \delta = 0,\ r = 0.05,\ a = -4,\ b = 4$

Method	Grid (n, m)	V	Error	Iter	CPU (s)
Binomial		4.655561	0		
FP	(800,400)	4.651795	3.76e-3	2799	1.64
	(1600,800)	4.654695	8.65e-4	5599	6.48
	(3000,2000)	4.655399	1.61e-4	14000	29.55
	(4000,2000)	4.655483	7.77e-5	16000	41.34
	(4000,3000)	4.655532	3.77e-5	21001	59.84
PSOR	(800,400)	4.651576	3.98e-3	2118	1.64
	(1600,800)	4.653859	1.70e-3	5489	6.33
	(3000,2000)	4.652425	3.13e-3	15758	30.10
	(4000,2000)	4.652570	2.99e-3	20556	43.61
	(4000,3000)	4.652389	3.17e-3	26166	62.81

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