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# Observability Inequality for the Petrovsky Equation

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**Abstract** The aim of this paper is to study the exact controllability of the Petrovsky equation. Under some checkable geometric assumptions, we establish the observability inequality via the multiplier method for the Dirichlet control problem.

**Keywords** exact controllability; compact uniqueness; multiplier method
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## 1 Introduction

Consider the initial/boundary value problem:

$$\begin{cases} u'' + \Delta^2 u + u = \chi_G(x)h(x,t), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = \triangle u = 0, & (x,t) \in \Gamma \times (0,T), \\ u(0) = u_0, u'(0) = u_1, & x \in \Omega, \end{cases} \tag{1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with the boundary $\Gamma$, $T$ is a positive constant, $G$ is a sub domain, $\chi_G(x)$ is the eigenfunction of the set $G, \chi_G(x)h(x,t)$ is the locally distributed control.

In this paper, we use the geometric assumptions and the piecewise multiplier method to establish inequality for the Petrovsky equation. For the conservative systems, How to choose the control subregion so that the system has exact controllability, this problem has been studied extensively[3,6]. The locally distributed control for the Petrovsky equation was studied by Liu[5]. Very recently, a novel control approach is utilized to solve the control problem[4,7].

We study the Petrovsky equation, and the geometric conditions on $\Omega$ and $G$ were introduced by Liu[5]. In the system (1), due to the increase of the general term $u$ and the initial value condition, this makes its analysis much more complicated than that of the Petrovsky system with the simply supported boundary condition[5]. So we are faced with new difficulties: to set up appropriate geometric conditions for the observability estimates and to derive the control inequality for the solution $u$ of (1).

Set

$$V = H_0^2(\Omega), \qquad \|v\|_V = \Big(\int_\Omega |\triangle v|^2 dx\Big)^{\frac{1}{2}}, \qquad H = L^2(\Omega), \qquad \|v\|_H = \Big(\int_\Omega |v|^2 dx\Big)^{\frac{1}{2}},$$

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where

$$H_0^2(\Omega) = \{u \in H^2(\Omega),\ u = \partial_\nu u = 0,\ x \in \Gamma\},$$

$H^2(\Omega)$ is 2-order Sobolev space[1]. Then, $H$ and $V$ are complex *Hilbert* spaces, the corresponding derived inner product denoted as $\lessdot, \gtrdot_V$ and $\lessdot, \gtrdot_H$ . Introduce the space $U = V \times H$, with the norm $\|(u,v)\|_U = (\|u\|_V^2 + \|v\|_H^2)^{\frac{1}{2}}$. It is obvious that $U$ is also a complex Hilbert space with the inner product $\lessdot, \gtrdot_U$ .

It is clear that for each $h \in L^2(\Omega \times (0,T))$ and $(u^0, u^1) \in U$, the Problem (1) has a unique solution satisfies

$$u \in C^1([0,T]; H) \cap C([0,T]; V).$$

Define the energy of the System (1) by

$$E(t) = \frac{1}{2}\int_\Omega (|u'|^2 + |\triangle u|^2 + |u|^2)dx. \tag{2}$$

**Remark 1.1.** For $S \subset \mathbb{R}^n, \varepsilon > 0$, let

$$N_\varepsilon(S) = \bigcup_{x \in S} \{y \in R^n : |y - x| < \varepsilon\}. \tag{3}$$

**Remark 1.2.** For $x_0^j \in \mathbb{R}^n, \Omega_j \subset \Omega,\ 1 \le j \le J$, set

$$\Gamma_j = \Gamma_j(x_0^j) = \big\{x \in \partial\Omega_j : (x - x_0^j) \cdot \nu^j(x) > 0\big\}, \tag{4}$$

where $\nu^j(x)$ is the unit normal vector of $\partial\Omega_j$ at $x$, $x \cdot y$ denotes the inner product of $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. We know that when $\Omega_j$ with Lipschitz boundary, $\nu^j(x) \in L^\infty(\partial\Omega_j; \mathbb{R}^n)$[2].

**Remark 1.3.** Geometric conditions on $\Omega$ and $G$ [5]: (a) $\Omega$ is sufficiently smooth; (b) there exists $\varepsilon > 0, \Omega_j$ with *Lipschitz* boundary $\partial\Omega_j$ and point $x_0^j \in \mathbb{R}^n,\ 1 \le j \le J$, such that

$$\Omega_i \cap \Omega_j = \emptyset, \qquad 1 \le i < j \le J, \tag{5}$$

$$G \supset \Omega \cap N_\varepsilon\Big[\bigcup_{j=1}^J \Gamma_j(x_0^j) \cup (\Omega \setminus \bigcup_{j=1}^J \Omega_j)\Big]. \tag{6}$$

First, we consider dual version

$$\begin{cases} u'' + \Delta^2 u + u = 0, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = \triangle u = 0, & (x,t) \in \Gamma \times (0,T), \\ u(0) = u_0, u'(0) = u_1, & x \in \Omega. \end{cases} \tag{7}$$

Let

$$\begin{cases} D(A) = \{u \in V : \triangle^2 u \in H\}, \\ Au = \triangle^2 u, \qquad \forall\, u \in D(A). \end{cases} \tag{8}$$

And on $U$ by

$$\begin{cases} D(\Lambda) = D(A) \times V, \\ \Lambda = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}. \end{cases} \tag{9}$$

The solution of (7) is given by

$$(u(\cdot,t),u'(\cdot,t)) = e^{At}(u_0,u_1), \qquad \forall\,(u_0,u_1)\in U. \tag{10}$$

By (10), we readily have

$$u(u_0,u_1)\in C^2([0,T];H)\cap C^1([0,T];V)\cap C([0,T];D(A)), \qquad \forall\,(u_0,u_1)\in D(\Lambda). \tag{11}$$

From the regularity results on elliptic problems[2], we know that the geometric Condition (a) implies

$$D(A) = H^4(\Omega)\cap H_0^2(\Omega). \tag{12}$$

In general, if the solution of (1) and (7) are real valued, then (7) is conservative, i.e.

$$E(t) = \frac{1}{2}\int_\Omega (|u'|^2 + |\triangle u|^2 + |u|^2)dx = E(0), \qquad \forall\, t\geq 0. \tag{13}$$

## 2 The Main Results and Proofs

In this section, we give the observability inequality and its proof. First, we prove Theorem 2.1. Because of the limit of the length of the paper, some obvious derivation is omitted.

**Theorem 2.1.** *Let the function $u\in H^2_{\text{loc}}(\mathbb{R};H^4(\Omega))$ satisfy*

$$u'' + \triangle^2 u + u = 0, \qquad (x,t)\in\Omega\times(0,T), \tag{14}$$

*$q:\overline{\Omega}\longrightarrow\mathbb{R}^n$ is a vector function of calss $C^2$, then for any given $0<T<\infty$, the following identity holds true*

$$\begin{aligned}
&\int_0^T\int_\Gamma t(T-t)\big[(q\cdot\nu)u_t^2 - 2(\partial_\nu\triangle u)q\cdot\nabla u - 2(q\cdot\nu)u^2\big]\,d\Gamma dt\\
=&-2\int_0^T\int_\Omega (T-2t)u_t q\cdot\nabla u dxdt + \int_0^T\int_\Omega t(T-t)(divq)\big[u_t^2-(\triangle u)^2-2u^2\big]\,dxdt\\
&+2\int_0^T\int_\Omega t(T-t)\Big[\sum_{i=1}^n(\triangle q)(\partial_i u)(\triangle u) + 2\sum_{i,k=1}^n(\partial_i q_k)(\partial_i\partial_k u)(\triangle u)\Big]dxdt.
\end{aligned}\tag{15}$$

*Proof.* We multiply (14) by $2t(T-t)q\cdot\bigtriangledown u$ and we integrate by parts

$$0=\int_0^T\int_\Omega 2(u''+\triangle^2u+u)t(T-t)q\cdot\bigtriangledown udxdt, \tag{16}$$

$$\begin{aligned}&\int_0^T\int_\Omega 2u_{tt}t(T-t)q\cdot\bigtriangledown udxdt\\ &=-\int_0^T\int_\Omega[2(T-2t)u_tq\cdot\nabla u+2u_tt(T-t)q\cdot\bigtriangledown u_t]dxdt\\ &=-\int_0^T\int_\Omega[2(T-2t)u_tq\cdot\nabla u+t(T-t)q\cdot\bigtriangledown u_t^2]dxdt\\ &=-\int_0^T\int_\Omega 2(T-2t)u_tq\cdot\nabla udxdt-\int_0^T\int_\Gamma t(T-t)(q\cdot\nu)u_t^2d\Gamma dt\\ &\quad+\int_0^T\int_\Omega t(T-t)(\mathrm{div}q)u_t^2dxdt,\end{aligned} \tag{17}$$

$$\begin{aligned}&\int_0^T\int_\Omega 2ut(T-t)q\cdot\bigtriangledown udxdt\\ &=\int_0^T\int_\Omega 2t(T-t)q\cdot\bigtriangledown u^2dxdt\\ &=\int_0^T\int_\Gamma 2t(T-t)(q\cdot\nu)u^2d\Gamma dt-\int_0^T\int_\Omega 2t(T-t)(\mathrm{div}q)u^2dxdt.\end{aligned} \tag{18}$$

Applying

$$\triangle(q\cdot\nabla u)=\sum_{i,k=1}^n\partial_i^2(q_k\partial_ku)=2\sum_{i,k=1}^n(\partial_iq_k)(\partial_i\partial_ku)+\sum_{i=1}^n(\triangle q_k)(\partial_ku)+\sum_{i=1}^nq_k\partial_k(\triangle u),$$

We have

$$2(\triangle u)\triangle(q\cdot\nabla u)=q\cdot\nabla(\triangle u)^2+2\sum_{i=1}^n(\triangle q_i)(\partial_iu)(\triangle u)+4\sum_{i,k=1}^n(\partial_iq_k)(\partial_i\partial_ku)(\triangle u). \tag{19}$$

Use of (19) yields

$$\begin{aligned}&\int_0^T\int_\Omega 2(\triangle^2u)t(T-t)q\cdot\bigtriangledown udxdt\\ &=\int_0^T\int_\Gamma 2(\partial_\nu\triangle u)t(T-t)q\cdot\nabla ud\Gamma dt-\int_0^T\int_\Omega 2t(T-t)(\nabla\triangle u)\cdot\nabla(q\cdot\nabla u)dxdt\\ &=\int_0^T\int_\Gamma 2(\partial_\nu\triangle u)t(T-t)q\cdot\nabla ud\Gamma dt+\int_0^T\int_\Omega 2t(T-t)(\triangle u)\triangle(q\cdot\nabla u)dxdt\\ &=\int_0^T\int_\Gamma 2t(T-t)(\partial_\nu\triangle u)q\cdot\nabla ud\Gamma dt-\int_0^T\int_\Omega t(T-t)(divq)(\triangle u)^2dxdt\\ &\quad+\int_0^T\int_\Omega t(T-t)\Big[2\sum_{i=1}^n(\triangle q_i)(\partial_iu)(\triangle u)+4\sum_{i,k=1}^n(\partial_iq_k)(\partial_i\partial_ku)(\triangle u)\Big]dxdt.\end{aligned} \tag{20}$$

We deduce from (16)–(18) and (20) that (15). □

**Theorem 2.2.** *For any $T > 0$, there exists a $C = C(\Omega, T, J, x_0^j, \Omega_j, G)$, $1 \le j \le J$ such that for every solution $u$ of (7) with initial value $(u_0, u_1) \in U$ satisfying*

$$\int_0^T \int_G |u'|^2 dxdt \ge CE(0). \tag{21}$$

*Proof.* First of all, using piecewise multiplier method, will justify that for any given $T > 0$, there exists a constant $M > 0$, such that the solution $u$ of (7) satisfies

$$E(0) \le M \int_0^T \int_\Omega (u^2 + |\nabla u|^2) dxdt + M \int_0^T \int_G u_t^2 dxdt. \tag{22}$$

Putting

$$\begin{aligned} &x = (x_1, \cdots, x_n), \qquad x_0^j = (x_{01}^j, \cdots, x_{0n}^j), \\ &m^j = (m_1^j, \cdots, m_n^j), \qquad m_k^j = x_k - x_{0k}^j, \qquad 1 \le k \le n, \ \ 1 \le j \le J, \\ &\nu^j = (\nu_1^j, \cdots, \nu_n^j), \qquad 1 \le j \le J. \end{aligned}$$

We choose $\varepsilon_0, \varepsilon_1 > 0$ such that $0 < \varepsilon_1 < \varepsilon_0 < \varepsilon$, and set

$$\begin{cases} Q_0 = N_{\varepsilon_0}\Big[ \bigcup\limits_{j=1}^J \cup \Big( \Omega \setminus \bigcup\limits_{j=1}^J \Omega_j \Big) \Big], \\ Q_1 = N_{\varepsilon_1}\Big[ \bigcup\limits_{j=1}^J \cup \Big( \Omega \setminus \bigcup\limits_{j=1}^J \Omega_j \Big) \Big]. \end{cases} \tag{23}$$

Choose $\phi^j$ such that

$$\phi^j \in C_0^\infty(\mathbb{R}^n), \qquad 0 \le \phi^j \le 1, \tag{24}$$

$$\phi^j = 1, \qquad x \in \overline{\Omega_j} \setminus Q_0, \tag{25}$$

$$\beta \phi^j = 0, \qquad x \in Q_1. \tag{26}$$

For each $j$, we insert $\Omega := \Omega_j$, $h(x) = \phi^j(x) m^j(x)$ into (15), we obtain

$$\begin{aligned} &\int_0^T \int_{\partial\Omega_j} t(T-t)\big[(\phi^j m^j \cdot \nu^j) u_t^2 - 2(\phi^j m^j \cdot \nu^j) u^2\big]\, d\Gamma dt \\ &- 2 \int_0^T \int_{\partial\Omega_j} t(T-t)(\partial_{\nu^j} \triangle u)(\phi^j m^j \cdot \nabla u) d\Gamma dt \\ = &- 2 \int_0^T \int_{\Omega_j} (T-2t) u_t \phi^j m^j \cdot \nabla u dxdt \\ &+ \int_0^T \int_{\Omega_j} t(T-t)\big[\mathrm{div}(\phi^j m^j)\big]\big[u_t^2 - (\triangle u)^2 - 2u^2\big] dxdt \\ &+ 2 \int_0^T \int_{\Omega_j} t(T-t) \sum_{i=1}^n \big[\triangle(\phi^j m_i^j)\big](\partial_i u)(\triangle u) \\ &+ 4 \int_0^T \int_{\Omega_j} t(T-t) \sum_{i,k=1}^n \big[\partial_i(\phi^j m_k^j)\big](\partial_i \partial_k u)(\triangle u) dxdt. \end{aligned} \tag{27}$$

As $\Omega \setminus \big(\bigcup_{j=1}^{J} \Omega_j\big) \supset \partial\Omega_j \setminus \Gamma$, combining with (23) and (26) we find that

$$\phi^j = 0, \qquad x \in N_{\varepsilon_1}\big[(\partial\Omega_j \setminus \Gamma) \cup \Gamma_j)\big]. \tag{28}$$

We deduce from the second equation of (7) that

$$\begin{aligned}
&\int_0^T \int_{(\partial\Omega_j \setminus \Gamma_j)\cap\Gamma} t(T-t)(\partial_{\nu^j}\triangle u)(\phi^j m^j \cdot \nabla u)dxdt \\
=&\int_0^T \int_{(\partial\Omega_j \setminus \Gamma_j)\cap\Gamma} t(T-t)(\phi^j m^j \cdot \nu^j \cdot \nabla u)\triangle u dxdt = 0.
\end{aligned} \tag{29}$$

Notice

$$\partial\Omega_j = (\partial\Omega_j \setminus \Gamma) \cup (\Gamma_j \cap \Gamma) \cup ((\partial\Omega_j \setminus \Gamma_j) \cap \Gamma),$$

Utilizing the second equation of (7) and (27)–(29), we have

$$\begin{aligned}
&\int_0^T \int_{(\partial\Omega_j \setminus \Gamma_j)\cap\Gamma} t(T-t)(\phi^j m^j \cdot \nu^j)u_t^2 d\Gamma dt \\
=&-2\int_0^T \int_{\Omega_j} (T-2t)u_t \phi^j m^j \cdot \nabla u dxdt \\
&+\int_0^T \int_{\Omega_j} t(T-t)[\mathrm{div}(\phi^j m^j)][u_t^2 - (\triangle u)^2 - 2u^2]dxdt \\
&+2\int_0^T \int_{\Omega_j} t(T-t)\sum_{i=1}^{n}[\triangle(\phi^j m_i^j)](\partial_i u)(\triangle u)dxdt \\
&+4\int_0^T \int_{\Omega_j} t(T-t)\sum_{i,k=1}^{n}[\partial_i(\phi^j m_k^j)](\partial_i\partial_k u)(\triangle u)dxdt.
\end{aligned} \tag{30}$$

Using (4) and (24) we obtain that

$$\int_0^T \int_{(\partial\Omega_j \setminus \Gamma_j)\cap\Gamma} t(T-t)(\phi^j m^j \cdot \nu^j)u_t^2 dxdt \le 0.$$

Pay attention to integral on $\Omega_j \cap Q_1$ is zero, by (30) can be introduced

$$\begin{aligned}
&-2\int_0^T \int_{\Omega_j \setminus Q_1} (T-2t)u_t \phi^j m^j \cdot \nabla u dxdt \\
&+\int_0^T \int_{\Omega_j \setminus Q_1} t(T-t)[\mathrm{div}\,(\phi^j m^j)][u_t^2 - (\triangle u)^2 - 2u^2]dxdt \\
&+2\int_0^T \int_{\Omega_j \setminus Q_1} t(T-t)\sum_{i=1}^{n}[\triangle(\phi^j m_i^j)](\partial_i u)(\triangle u)dxdt \\
&+4\int_0^T \int_{\Omega_j \setminus Q_1} t(T-t)\sum_{i,k=1}^{n}[\partial_i(\phi^j m_k^j)](\partial_i\partial_k u)(\triangle u)dxdt \le 0.
\end{aligned} \tag{31}$$

Furthermore, we have

$$\sum_{j=1}^{J}\int_0^T \int_{\Omega_j \setminus Q_0} t(T-t)[nu_t^2 + (4-n)(\triangle u)^2 - 2nu^2]dxdt$$

$$
\begin{aligned}
&\leq 2\sum_{j=1}^{J}\int_0^T\int_{\Omega_j\setminus Q_1}(T-2t)u_t\phi^j m^j\cdot\nabla u dxdt\\
&\quad-\sum_{j=1}^{J}\int_0^T\int_{\Omega_j\cap Q_0\setminus Q_1}t(T-t)[\mathrm{div}\,(\phi^j m^j)][u_t^2-(\triangle u)^2-2u^2]dxdt\\
&\quad-2\sum_{j=1}^{J}\int_0^T\int_{\Omega_j\cap Q_0\setminus Q_1}t(T-t)\sum_{i=1}^{n}[\triangle(\phi^j m_i^j)](\partial_i u)(\triangle u)dxdt\\
&\quad-4\sum_{j=1}^{J}\int_0^T\int_{\Omega_j\cap Q_0\setminus Q_1}t(T-t)\sum_{i,k=1}^{n}[\partial_i(\phi^j m_k^j)](\partial_i\partial_k u)(\triangle u)dxdt\\
&\leq C_1\int_0^T\int_\Omega |u_t|\nabla u|dxdt\\
&\quad+C_1\sum_{j=1}^{J}\int_0^T\int_{\Omega_j\cap Q_0}[u_t^2+(\triangle u)^2+u^2]dxdt, \qquad (32)
\end{aligned}
$$

where $C_1=C_1(T,n,m^j,\phi^j)>0$ is a constant independent of $u$, and used here from (25)

$$
\mathrm{div}\,(\phi^j m^j)=\mathrm{div}\, m^j=n, \qquad x\in\Omega_j\setminus Q_0,
$$

$$
\frac{\partial(\phi^j m_k^j)}{\partial x_i}=\begin{cases}1, & k=i,\quad x\in\Omega_j\setminus Q_0,\\ 0, & K\neq i,\quad x\in\Omega_j\setminus Q_0.\end{cases}
$$

On the other hand, multiplying the first equation of (7) by $(n-2)t(T-t)u$, and we integrate on $\Omega\times(0,T)$, then using the second equation of (7) we obtain that

$$
-(n-2)\int_0^T\int_\Omega(T-2t)u_t u dxdt+(n-2)\int_0^T\int_\Omega t(T-t)[(\triangle u)^2-u_t^2+u^2]dxdt=0.
$$

Hence

$$
\begin{aligned}
&-(n-2)\int_0^T\int_\Gamma(T-2t)u_t u dxdt+(n-2)\int_0^T\int_{\Omega\setminus Q_0}t(T-t)[(\triangle u)^2-u_t^2+u^2]dxdt\\
&+(n-2)\int_0^T\int_{\Omega\cap Q_0}t(T-t)[(\triangle u)^2-u_t^2+u^2]dxdt=0. \qquad (33)
\end{aligned}
$$

Combing now (32), (33), and applying (5), (13) and $\bigcup_{j=1}^{J}(\Omega_j\setminus Q_0)=(\Omega\setminus Q_0)$, we discover

$$
\begin{aligned}
&2T^3E(0)-(n-2)\int_0^T\int_\Omega(T-2t)u_t u dxdt\\
&\leq\int_0^T\int_{\Omega\cap Q_0}t(T-t)\big[(4-n+C_1)(\triangle u)^2+(n+C_1)u_t^2+(4+C_1)u^2\big]dxdt\\
&\quad+C_1\int_0^T\int_\Omega|u_t|\,|\nabla u|dxdt. \qquad (34)
\end{aligned}
$$

By $G \supset \Omega \cap Q_0$, (34) implies

$$\begin{aligned} T^3E(0) \leq & M_1\Big[\int_0^T\int_\Omega(|u_tu|+|u_t||\nabla u|)dxdt+\int_0^T\int_G u_t^2dxdt \\ & +\int_0^T\int_{\Omega\cap Q_0}t(T-t)(\triangle u)^2dxdt+\int_0^T\int_{\Omega\cap Q_0}t(T-t)u^2dxdt\Big], \end{aligned} \tag{35}$$

where $M_1 > 0$ is a constant.

Let us estimate $\int_0^T\int_{\Omega\cap Q_0}t(T-t)(\triangle u)^2dxdt$. Employing (6) and (23),we have $\overline{(\mathbb{R}^n\setminus G)}\cap\overline{Q_0}=\emptyset$. So take a fixed function $\xi$ satisfies

$$\xi\in C_0^\infty(\mathbb{R}^n), \qquad 0\leq\xi\leq 1;\ \ \xi=1,\ \ x\in Q_0;\ \ \xi=0,\ \ x\in\mathbb{R}^n\setminus G.$$

We multiply (7) by $t(T-t)\xi u$ and integrate on $\Omega\times(0,T)$, we obtain

$$\int_0^T\int_\Omega(u_{tt}+\triangle^2u+u)t(T-t)\xi udxdt=0.$$

Since

$$\begin{aligned} & \int_0^T\int_\Omega u_{tt}t(T-t)\xi udxdt \\ = & -\int_0^T\int_\Omega(T-2t)\xi u_tudxdt-\int_0^T\int_\Omega t(T-t)\xi u_t^2dxdt, \\ & \int_0^T\int_\Omega(\triangle^2u)t(T-t)\xi udxdt \\ = & -\int_0^T\int_\Omega t(T-t)[\xi(\nabla\triangle u)\cdot\nabla u+u(\nabla\triangle u)\cdot\nabla\xi]dxdt \\ = & -\int_0^T\int_\Omega t(T-t)\nabla(\triangle u)(\nabla\xi\cdot u+\xi\cdot\nabla u)dxdt \\ = & \int_0^T\int_\Omega t(T-t)[2(\triangle u)(\nabla u)\cdot\nabla\xi+\xi(\triangle u)^2+u(\triangle u)\triangle\xi]dxdt, \\ & \int_0^T\int_\Omega ut(T-t)\xi udxdt=\int_0^T\int_\Omega t(T-t)\xi u^2dxdt, \end{aligned}$$

hence combing three equalities above, we deduce

$$\begin{aligned} & \int_0^T\int_\Omega t(T-t)\xi(\triangle u)^2dxdt \\ = & \int_0^T\int_\Omega(T-2t)\xi u_tudxdt \\ & +\int_0^T\int_\Omega t(T-t)\big[\xi u_t^2-2(\triangle u)(\nabla u)\cdot\nabla\xi-\xi u^2-u(\triangle u)\triangle\xi\big]dxdt. \end{aligned} \tag{36}$$

Moreover, we have

$$
\begin{aligned}
&\int_0^T \int_{\Omega\cap Q_0} t(T-t)(\triangle u)^2 dxdt \\
\le& \int_0^T \int_\Omega t(T-t)\xi(\triangle u)^2 dxdt \\
=& \int_0^T \int_G \big\{(T-2t)\xi u_t u + t(T-t)\big[\xi u_t^2 - 2(\triangle u)(\nabla u)\cdot \triangledown\xi - \xi u^2 - u(\triangle u)\triangle\xi\big]\big\} dxdt \\
\le& M_2 \int_0^T \int_\Omega |u_t u| dxdt + \int_0^T \int_G \frac{T^2}{4} u_t^2 dxdt \\
&- \int_0^T \int_G t(T-t)\big[2(\triangle u)(\nabla u)\cdot\triangledown\xi + u(\triangle u)\triangle\xi + \xi u^2\big] dxdt, \qquad (37)
\end{aligned}
$$

where $M_2 > 0$ is a positive constant. By (35) and (37), we have

$$
\begin{aligned}
T^3 E(0) \le& M_3 \int_0^T \int_\Omega (|u_t u| + |u_t||\nabla u|) dxdt + M_3 \int_0^T \int_G u_t^2 dxdt \\
&- M_1 \int_0^T \int_G t(T-t)[2(\triangle u)(\nabla u)\cdot\triangledown\xi + \xi u^2 + u(\triangle u)\triangle\xi] dxdt \\
=& M_3 \int_0^T \int_\Omega (|u_t u| + |u_t||\nabla u|) dxdt + M_3 \int_0^T \int_G u_t^2 dxdt \\
&+ M_1 \int_0^T \int_\Omega t(T-t)\xi(\triangle u)^2 dxdt \\
&- M_1 \int_0^T \int_\Omega (T-2t)\xi u_t u dxdt - M_1 \int_0^T \int_\Omega t(T-t)\xi u_t^2 dxdt \\
\le& M_3 \int_0^T \int_\Omega (|u_t u| + |u_t||\nabla u|) dxdt + M_3 \int_0^T \int_G u_t^2 dxdt \\
&+ M_1 \int_0^T \int_\Omega t(T-t)\xi(\triangle u)^2 dxdt + M_1 \int_0^T \int_\Omega |T-2t||\xi||u_t u| dxdt \\
\le& \int_0^T \int_\Omega \Big[M_3\Big(u_t^2 + \frac{u^2 + (\nabla u)^2}{2}\Big) + \frac{M_1 T^2}{4}(\triangle u)^2 + M_1 \cdot 2T \cdot \frac{u^2 + u_t^2}{2}\Big] dxdt \\
&+ M_3 \int_0^T \int_G u_t^2 dxdt \\
=& \int_0^T \int_\Omega \Big[(M_3 + M_1 T)u_t^2 + \frac{M_1 T^2}{4}(\triangle u)^2\Big] + \Big[\Big(M_1 T + \frac{M_3}{2}\Big)u^2 \\
&+ \frac{M_3}{2}(\nabla u)^2\Big] dxdt + M_3 \int_0^T \int_G u_t^2 dxdt \\
\le& 2T\delta E(0) + M_4 \int_0^T \int_\Omega (u^2 + |\nabla u|^2) dxdt + M_3 \int_0^T \int_G u_t^2 dxdt. \qquad (38)
\end{aligned}
$$

For any $\delta > 0$, where $M_3 > 0$ and $M_4 > 0$ are constants independent of $u$. By (38), we have (22).

Secondly, combing with (22) and using the compact uniqueness theorem, we shall obtain the inequality of Theorem 2.2. This completes the proof of Theorem 2.2. □

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