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Commutator Representations and Roots of Pseudo Differential Operators

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Abstract Based on the fundamental commutator representation proposed by Cao [4] we established two explicit expressions for roots of a third order differential operator. By using those expressions we succeeded in clarifying the relationship between two major approaches in theory of integrable systems: the zero curvature and the Lax representations for the KdV and the Boussinesq hierarchies. The proposed procedure could be extended to the general case of higher order of differential operators that leads to the Gel'fand-Dickey hierarchy.

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1 Introduction

Generating new integrable equations and checking if a known equation is integrable are two main tasks in the modern theory of integrable systems. There are two major approaches for establishing the integrability: the zero curvature representation and the Lax representation (see [1,13,15,21,22,25]. For example, the celebrated KdV equation

$$\frac{\partial u}{\partial t} = 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \tag{1}$$

and its hierarchy can be derived either from the zero curvature representation

$$U_{t_n} = V_x^{(n)} - [U, V^{(n)}] \tag{2}$$

or from the Lax representation

$$L_{t_n} = [P_n, L], \tag{3}$$

where U, $V^{(n)}$, L and P_n are given by

$$U = \begin{pmatrix} 0 & 1 \\ \frac{1}{4}\lambda - u & 0 \end{pmatrix}, \qquad V^{(n)} = \begin{pmatrix} a^{(n)} & b^{(n)} \\ c^{(n)} & -a^{(n)} \end{pmatrix}$$

and

$$L = \partial^2 + u, \qquad P_n = (L^{(2n+1)/2})_+, \qquad \partial = \frac{\partial}{\partial x}.$$

The symbol S_+ stands for the non-negative part of the pseudo differential operator S. A detail explanation of those notations will be given in sections below.

Questions arise naturally. What is the relationship between $V^{(n)}$ and P_n? In the zero curvature representations the matrices $V^{(n)}$ can be calculated recursively. Is there also a recursive formula among $[P_n, L]$ as seen in the Lax representations? It is known that the traces

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tr $(L^{(2n+1)/2})$ are conserved densities of the KdV hierarchy. Is there a recursive formula among those traces as we have seen in the zero curvature representations?

It is far from trivial to explore the relationship between zero curvature and Lax representations. For example, what is the zero curvature representation of the Gel'fand-Dickey (GD) hierarchy? The GD hierarchy is generated from the operator $L = \partial^m + u_1\partial^{m-2} + u_2\partial^{m-3} + \cdots + u_{m-1}$. The question become even more crucial when we study 2+1 dimensional integrable equations such as the KP hierarchy. The KP hierarchy can be reduced from the Gel'fand-Dickey hierarchy. However, to the author's knowledge the KP hierarchy does not admit a good zero curvature representation.

In the present paper we give a partial answer to the questions mentioned above. It turns out that the bi-Hamiltonian structure is a bridge between two representations. Furthermore, the fundamental commutator representation, proposed by Cao et al.[4], provides a good tool to explore bi-Hamiltonian structures. By using the commutator representation we established two explicit formulas for the roots $L^{1/3}$ and $L^{2/3}$, where $L = \partial^3 + u\partial + \partial u + v$. Based on those two formulas we derived recursive equations among $[(L^{(3m+1)/3})_+, L]$, $[(L^{(3m+2)/3})_+, L]$, tr $(L^{(3m+1)/3})$, and tr $(L^{(3m+2)/3})$.

The paper is organized as follows. The next section contains a brief introduction on pseudo differential operators, the GD hierarchy, bi-Hamiltonian structures and commutator representations. The Section 3 constitutes the main body of this paper. We solved the commutator equation for the operator $L_3 = \partial^3 + u\partial + \partial u + v$, and derived the bi-Hamiltonian structure for the Boussinesq hierarchy. Furthermore, we established formulas of roots $L_3^{1/3}$ and $L_3^{2/3}$, and generated recursive equations among $[(L_3^{(n/3)})_+, L_3]$ and $tr(L_3^{(n/3)})$. Boussinesq hierarchy is a special case of the GD hierarchy when $m = 3$. For completeness the Section 4 gives a short presentation on KdV hierarchy that is a special case of GD hierarchy when $m = 2$.

2 Preliminaries

2.1 Pseudo Differential Operators

In this section we recall some definitions in the theory of pseudo differential operators. The readers are referred to the book[7] for details. A pseudo differential operator takes the form

$$P = \sum_{i=-\infty}^{m} p_i\partial^i, \tag{4}$$

where p_i are smooth functions of $u = (u_1, \cdots, u_p)$, $u_i = u_i(x,t)$, and $\partial = \frac{\partial}{\partial x}$. The positive and the negative powers ∂^n is defined by the Leibniz rule

$$\partial^n f = \sum_{k=0}^{\infty} \binom{n}{k} (\partial^k f)\partial^{n-k},$$

where $\binom{n}{k}$ stands for the binomial coefficients

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}, \qquad n \in \mathbb{Z}.$$

For example

$$\partial^{-1} f = f\partial^{-1} - (\partial f)\partial^{-2} + (\partial^2 f)\partial^{-3} - (\partial^3 f)\partial^{-4} + \cdots$$

The m-th root $A = P^{1/m}$ of the pseudo differential operator (4) is defined by the equation $A^m = P$. It is proved that the root exists and is unique. For example let the operator $P = L_3$ where

$$L_3 = \partial^3 + 2u\partial + v + u_x. \tag{5}$$

Its roots $L_3^{1/3}$ and $L_3^{2/3}$ can be calculated by the MAPLE package PSEUDO

$$\begin{aligned} L_3^{1/3} =& \partial + \frac{2}{3}u\partial^{-1} + \Big(-\frac{1}{3}u_x + \frac{1}{3}v\Big)\partial^{-2} \\ &+ \Big(-\frac{4}{9}u^2 + \frac{1}{9}u_{xx} - \frac{1}{3}v_x\Big)\partial^{-3} + \cdots \\ L_3^{2/3} =& \partial^2 + \frac{4}{3}u + \frac{2}{3}v\partial^{-1} + \Big(-\frac{4}{9}u^2 - \frac{1}{9}u_{xx} - \frac{1}{3}v_x\Big)\partial^{-2} \\ &+ \Big(\frac{8}{9}uu_x - \frac{4}{9}uv + \frac{1}{9}v_{xx} + \frac{1}{9}u_{xxx}\Big)\partial^{-3} + \cdots. \end{aligned}$$

We define the weight of u, v and ∂ as

$$\sigma(u) = 2, \qquad \sigma(v) = 3, \qquad \sigma(\partial) = 1. \tag{6}$$

Each item in the operator L_3 is of weight 3 under the above definition. For example $\sigma(u\partial) = \sigma(u)+\sigma(\partial) = 3$. It is easily seen that each term in $L_3^{1/3}$ is of weight 1. For example, $\sigma(v_x\partial^{-3}) = \sigma(v)+\sigma(\partial)+\sigma(\partial^{-3}) = 3+1-3 = 1$. Similarly, each term in $L_3^{2/3}$ is of weight 2. Therefore we could write that

$$\sigma(L_3) = 3, \qquad \sigma(L_3^{1/3}) = 1, \qquad \sigma(L_3^{2/3}) = 2. \tag{7}$$

The weight plays a key role in some MAPLE packages for automatically generating conserved densities of an equation[24].

The positive and the negative part of (4) is defined by

$$P_+ = \sum_{i\geq 0} p_i\partial^i, \qquad P_- = \sum_{i<0} p_i\partial^i. \tag{8}$$

Note that we have $P = P_+ + P_-$.

Brunelli created a wonderful package PSEUDO[3] for operations with pseudo differential operators such as taking an n-th roots, multiplication of two operators and etc. The source code is only of 404 lines. The package has to be extended to include matrices operators.

2.2 The GD Hierarchy

Let

$$L_m = \partial^m + u_1\partial^{m-2} + u_2\partial^{m-3} + \cdots + u_{m-1}, \tag{9}$$

where u_i, $i = 1, \cdots, m-1$ are smooth functions of x and t. The Gel'fand-Dickey (GD) hierarchy[7] is given by the Lax representation

$$L_{m,t_n} = [(L_m^{(n/m)})_+, L_m], \tag{10}$$

where $L_{m,t_n} = \frac{\partial}{\partial t_n}L_m$. The special cases $m = 2, 3$ correspond to the classical KdV and Boussinesq hierarchies as we will discuss in details in the subsequent two sections.

2.3 Bi-Hamiltonian Structures

It is welll known that most 1+1 dimensional integrable hiercharchies admit bi-Hamiltonian structures in the sense that they can be written as

$$u_{t_n} = JG_{n+1} = KG_n, \tag{11}$$

where J and K are two Hamiltonian operators, and they are compatible[9,12,16]. The operator

$$R = KJ^{-1} \tag{12}$$

is frequently referred to as "recursion operator" in literature. The next two sections will show the bi-Hamitonian structures of the Boussinesq and KdV hierarchies.

To the author's knowledge all known recursive operators (12) contains an inverse differential operator ∂^{-1}. Therefore to calculate the next G_{n+1} from G_n we have to find an expression F_n such that $G_n = \partial F_n$. The operator that carries G_n to F_n is called a homotopy operator. There are rich literature on this topic[17]. The homotopy operator presented in [17] contains terms that will be canceled each other. Poole and Hereman[18] gave a simplified representation of the homotopy operator. The MAPLE procedure that is used to calculate $\partial^{-1}G$ is *Inverse Total Diff*. That procedure is contained in the package *Jet Calculus*.

2.4 Commutator Representations

Let $L(u)$ be a differential operator with coefficients depending on $u = u(x,t)$. Its derivative $(\mathcal{L}_v L)(u)$ is defined by

$$(\mathcal{L}_v L)(u) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} L(u + \varepsilon v). \tag{13}$$

The above derivative carries various names: Lie derivative, Frechet derivative, Gateaux derivative, tangent derivative etc. We will simply use "derivatives along with v", or simply "derivatives" in case no confusing arises. For many integrable hierarchies with two compatible Hamiltonian operators J and K, the following equation had been established (see [4,8,10,12,19,23]).

$$\mathcal{L}_{KG} L = [N(G), L] + (\mathcal{L}_{JG} L)L, \tag{14}$$

where N is an operator that plays a crucial role in derivation of expressions of roots. An alternative equation is given by

$$\mathcal{L}_{KG} L = [N(G), L] + L(\mathcal{L}_{JG} L). \tag{15}$$

A combination of the above two equations gives the third equation

$$\mathcal{L}_{KG} L = [N(G), L] + \frac{1}{2}\mathcal{L}_{JG}(L^2). \tag{16}$$

All the above three equations can be use to find operators K and N when the operator J is known.

3 Case $m = 3$: the Boussinesq Hierarchy

The case $m = 3$ in (9) corresponds to the Boussinesq hierarchy. For convenience we set $u_1 = 2u$ and $u_2 = v + u_x$ in (9):

$$L_3 = \partial^3 + 2u\partial + (v + u_x). \tag{17}$$

This operator can be also written as $L_3 = \partial^3 + u\partial + \partial u + v$. The corresponding hierarchies are given by

$$\begin{aligned} &L_{3,t_{3m+i}} = [P_{3m+i}, L_3], \qquad i = 1, 2, \quad m = 0, 1, 2, \cdots, \\ &P_k = (L_3^{k/3})_+. \end{aligned}$$

For late references we write down the first four sets of equations in the hierarchy.

$$\begin{aligned} u_{t_1} &= u_x, \qquad v_{t_1} = v_x, \qquad u_{t_2} = v_x, \qquad v_{t_2} = -\frac{8}{3}uu_x - \frac{1}{3}u_{xxx}, \\ u_{t_4} &= \frac{4}{3}u_x v + \frac{4}{3}uv_x + \frac{1}{3}v_{xxx}, \\ v_{t_4} &= -\frac{32}{9}u^2u_x - \frac{8}{3}u_xu_{xx} + \frac{4}{3}vv_x - \frac{4}{3}uu_{xxx} - \frac{1}{9}u_{xxxxx}, \\ u_{t_5} &= -\frac{20}{9}u^2u_x - \frac{25}{9}u_xu_{xx} + \frac{5}{3}vv_x - \frac{10}{9}uu_{xxx} - \frac{1}{9}u_{xxxxx} \\ v_{t_5} &= -\frac{1}{9}v_{xxxxx} - \frac{40}{9}vuu_x - \frac{5}{9}u_{xxx}v - \frac{10}{9}u_{xx}v_x - \frac{5}{3}u_xv_{xx} \\ &\quad - \frac{10}{9}uv_{xxx} - \frac{20}{9}v_xu^2. \end{aligned}$$

The first Hamiltonian operator J for the Bousssinesq hierarchy was found through its zero curvature representation[5,17]

$$J = \begin{pmatrix} 0, \partial \\ \partial, 0 \end{pmatrix}. \tag{18}$$

We are going to solve the commutator Equation (14) to find the second Hamiltonian operator K and the differential operator N. In this case N is of order $m - 1 = 2$. Therefore we could assume

$$N = a\partial^2 + b\partial + c, \tag{19}$$

where a, b and c are unknown functions to be determined. Moreover, G is a 2-dimensional vector $G = (r, s)^T$. We have

$$G = \begin{pmatrix} r \\ s \end{pmatrix}, \qquad JG = \begin{pmatrix} s_x \\ r_x \end{pmatrix}, \qquad \mathcal{L}_{JG}L_3 = 2s_x\partial + (r_x + s_{xx}).$$

Therefore,

$$\begin{aligned} \mathcal{L}_{KG}L_3 =& [N(G), L_3] + (\mathcal{L}_{JG}L_3)L_3 \\ =& [a\partial^2 + b\partial + c, \partial^3 + 2u\partial + v + u_x] \\ &+ (2s_x\partial + r_x + s_{xx})(\partial^3 + 2u\partial + v + u_x) \\ =& (-3a_x + 2s_x)\partial^4 + (-3b_x - 3a_{xx} + r_x + s_{xx})\partial^3 \end{aligned}$$

$$
\begin{aligned}
&+(4au_x-3c_x-2ua_x-3b_{xx}-a_{xxx}+4s_xu)\partial^2\\
&+(2bu_x+2av_x+4au_{xx}-2ub_x-3c_{xx}-b_{xxx}\\
&+2s_xv+6s_xu_x+2ur_x+2us_{xx})\partial\\
&+bv_x+bu_{xx}+av_{xx}+au_{xxx}-2uc_x-c_{xxx}+r_xv\\
&+r_xu_x+s_{xx}v+s_{xx}u_x+2s_xv_x+2s_xu_{xx}.
\end{aligned}
$$

Assume that $KG=(\widetilde{r},\widetilde{s})^T$ then $\mathcal{L}_{KG}L_3=2\,\widetilde{r}\,\partial+(\widetilde{s}+\widetilde{r}_x)$. Compare both sides of the above equations we are led to the following equations

$$
\begin{aligned}
&-3a_x+2s_x=0,\\
&-3b_x-3a_{xx}+r_x+s_{xx}=0, \qquad (20)\\
&4au_x-3c_x-2ua_x-3b_{xx}-a_{xxx}+4s_xu=0,\\
&2\,\widetilde{r}=2bu_x+2av_x+4au_{xx}-2ub_x-3c_{xx}\\
&\qquad -b_{xxx}+2s_xv+6s_xu_x+2ur_x+2us_{xx}\\
&\widetilde{s}+\widetilde{r}_x=bv_x+bu_{xx}+av_{xx}+au_{xxx}-2uc_x-c_{xxx} \qquad (21)\\
&\qquad +r_xv+r_xu_x+s_{xx}v+s_{xx}u_x+2s_xv_x+2s_xu_{xx}.
\end{aligned}
$$

We solve a, b and c from (20):

$$
a=\frac{2}{3}s, \qquad b=-\frac{1}{3}s_x+\frac{1}{3}r, \qquad c=\frac{8}{9}su+\frac{1}{9}s_{xx}-\frac{1}{3}r_x. \qquad (22)
$$

Substituting those expressions into (21) we obtain

$$
\begin{aligned}
\widetilde{r}=&\frac{1}{3}u_xr+\frac{2}{3}sv_x+\frac{2}{3}ur_x+\frac{1}{3}r_{xxx}+s_xv\\
\widetilde{s}=&\Big(-\frac{2}{9}u_{xxx}-\frac{16}{9}uu_x\Big)s+\Big(-\frac{16}{9}u^2-u_{xx}\Big)s_x\\
&+\frac{1}{3}v_xr-\frac{10}{9}us_{xxx}-\frac{5}{3}s_{xx}u_x-\frac{1}{9}s_{xxxxx}+r_xv.
\end{aligned}
$$

Therefore the operator N and the second Hamiltonian operator K are given by

$$
\begin{aligned}
N&=\frac{2}{3}s\partial^2+\Big(-\frac{1}{3}s_x+\frac{1}{3}r\Big)\partial+\Big(\frac{8}{9}su+\frac{1}{9}s_{xx}-\frac{1}{3}r_x\Big),\\
K&=\frac{1}{9}\begin{pmatrix}3u_x+6u\partial+3\partial^3 & 6v_x+9v\partial\\ 3v_x+9v\partial & K_{22}\end{pmatrix}, \qquad (23)\\
K_{22}&=(-16uu_x-2u_{xxx})-(16u^2+9u_{xx})\partial-15u_x\partial^2-10u\partial^3-\partial^5.
\end{aligned}
$$

It is easy to verify that

$$
J^*=-J, \qquad K^*=-K,
$$

where $*$ stands for the formal conjugate

$$
\Big(\sum a_i\partial^i\Big)^*=\sum(-1)^i\partial^ia_i, \qquad (M_{ij})^*=(M_{ji}^*).
$$

The recursive operator R is then given by

$$
\begin{aligned}
R&=KJ^{-1}=\frac{1}{9}\begin{pmatrix}6v_x\partial^{-1}+9v & 3u_x\partial^{-1}+6u+3\partial^2\\ R_{21} & 3v_x\partial^{-1}+9v\end{pmatrix},\\
R_{22}&=(-16uu_x-2u_{xxx})\partial^{-1}-(16u^2+9u_{xx})-15u_x\partial-10u\partial^2-\partial^4.
\end{aligned}
$$

We choose the initial vectors

$$G_{-2} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \qquad G_{-1} = \begin{pmatrix} 0 \\ \frac{3}{2}. \end{pmatrix}.$$

Applying the operator K and J we obtain

$$JG_1 = KG_{-2} = \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \qquad JG_2 = KG_{-1} = \begin{pmatrix} v_x \\ -\frac{8}{3}uu_x - \frac{1}{3}u_{xxx} \end{pmatrix}.$$

Thus

$$G_1 = \begin{pmatrix} v \\ u \end{pmatrix}, \qquad G_2 = \begin{pmatrix} -\frac{4}{3}u^2 - \frac{1}{3}u_{xx} \\ v \end{pmatrix}.$$

In general

$$JG_j = KG_{j-3}, \qquad j = 1, 2, \cdots.$$

Let

$$G_n = \begin{pmatrix} r_n \\ s_n \end{pmatrix}, \qquad n = -2, -1, 0, 1, 2, \cdots.$$

It is easy to see that

$$\sigma(r_n) = n + 2, \qquad \sigma(s_n) = n + 1, \qquad \sigma(N(G_n)) = n + 3.$$

The Boussinesq hierarchy is then given by

$$\begin{pmatrix} u_{t_n} \\ v_{t_n} \end{pmatrix} = JG_n, \qquad n = 1, 2, \cdots, \quad n \not\equiv 0 \pmod 3$$

or

$$\begin{pmatrix} u_{t_{3m+i}} \\ v_{t_{3m+i}} \end{pmatrix} = \begin{pmatrix} s_{3m+i,x} \\ r_{3m+i,x} \end{pmatrix}, \qquad i = 1, 2, \quad m = 0, 1, 2, \cdots.$$

Lemma 3.1. *The pseudo differential operator* $P_1 = \partial + \sum_{i\geq 1}^{\infty} p_i \partial^{-i}$ *is uniquely determined by the commutation condition* $[L_3, P_1] = 0$ *and* $\sigma(p_i) = i + 1$*, where the weight* σ *is defined by (6).*

Proof. It is easily verified that

$$[\partial^3, P_1] = 3p_{1x}\partial + 3p_{2x} + 3p_{1xx} + \sum_{i\geq 1}(p_{i,xxx} + 3p_{i+1,xx} + 3p_{i+2,x})\partial^{-i}$$

and

$$[2u\partial, P_1] = -2u_x\partial + \sum_{i\geq 1} F_i \partial^{-i}, \qquad [v + u_x, P_1] = -u_{xx} - v_x + \sum_{i\geq 1} H_i \partial^{-i},$$

where F_i and H_i depend on p_j, $j \leq i + 1$. Therefore the commutation condition $[L_3.P_1] = 0$ leads to the equations

$$3p_{1x} - 2u_x = 0,$$
$$3p_{2x} + 3p_{1xx} - u_{xx} - v_x = 0$$
$$3p_{i+2,x} = Q(p_{i+1}, p_i, p_{i-1}, \cdots), \qquad i = 1, 2, \cdots$$

and we obtain

$$p_1 = \frac{2}{3}u, \qquad p_2 = \frac{1}{3}(v - u_x),$$
$$p_{i+2,x} = Q(p_{i+1}, p_i, p_{i-1}, \cdots), \qquad i = 1, 2, \cdots.$$

Therefore $p_3, p_4, \cdots$ are determined by their previous terms. The requirement $\sigma(p_i) = i+1$ will force us to set the integral constant to zero when we calculate $p_{i+2} = \frac{1}{3}\partial^{-1}Q_i$. □

Similarly we can prove the following

Lemma 3.2. *The pseudo differential operator $P_2 = \partial^2 + \sum_{i\geq 0}^{\infty} p_i\partial^{-i}$ is uniquely determined by the commutation condition $[L_3, P_2] = 0$ and $\sigma(p_i) = i+2$, where the weight σ is defined by (6).*

Now we set

$$V = \sum_{j=0}^{\infty} N(G_{3j-2})L_3^{-j}, \tag{24}$$

V is commute with L_3 since we have

$$\begin{aligned}
[V, L_3] =& \sum_{j=0}^{\infty}[N(G_{3j-2}), L_3]L_3^{-j} \\
=& \sum_{j=0}^{\infty} \left(\mathcal{L}_{K(G_{3j-2})}L_3 - (\mathcal{L}_{J(G_{3j-2})}L_3)L_3\right) L_3^{-j} \\
=& \sum_{j=0}^{\infty} \left(\mathcal{L}_{J(G_{3j+1})}L_3 - (\mathcal{L}_{J(G_{3j-2})}L_3)L_3\right) L_3^{-j} \\
=& \sum_{j=0}^{\infty}(\mathcal{L}_{J(G_{3j+1})}L_3)L_3^{-j} - \sum_{j=0}^{\infty}(\mathcal{L}_{J(G_{3j-2})}L_3)L_3^{-j+1} \\
=& \sum_{j=0}^{\infty}(\mathcal{L}_{J(G_{3j+1})}L_3)L_3^{-j} - \sum_{j=-1}^{\infty}(\mathcal{L}_{J(G_{3j+1})}L_3)L_3^{-j} \\
=& -(\mathcal{L}_{J(G_{-2})}L_3)L_3.
\end{aligned}$$

V is homogeneous in weight. Each term in the summation (24) is of weight 1:

$$\sigma(V) = \sigma(N(G_{3j-2})) + \sigma(L_3^{-j}) = 3j + 1 - 3j = 1.$$

Moreover the leading term of ∂^j in V is ∂:

$$\begin{aligned}
V =& \sum_{j=0}^{\infty} N(G_{3j-2})L_3^{-j} \\
=& N(G_{-2}) + N(G_1)L_3^{-1} + \cdots \\
=& \frac{2}{3}s\partial^2 + \Big(-\frac{1}{3}s_x + \frac{1}{3}r\Big)\partial + \Big(\frac{8}{9}su + \frac{1}{9}s_{xx} - \frac{1}{3}r_x\Big)\Big|_{r=3,s=0} + o(\partial^{-1}) \\
=& \partial + \frac{2}{3}u\partial^{-1} + o(\partial^{-2}).
\end{aligned}$$

Here the symbol $o(\partial^n)$ refers to terms involving $\partial^k,\ k \leq n$. Since V and $L_3^{1/3}$ both commute with L_3 and they have the same leading terms $\partial + \frac{2}{3}u\partial^{-1} + o(\partial^{-2})$. Moreover, $\sigma(V) = \sigma(L_3^{1/3}) = 1$

we conclude by the Lemma 3.1 that

$$L_3^{1/3} = \sum_{j=0}^{\infty} N(G_{3j-2}) L_3^{-j}. \tag{25}$$

From the above formula we deduce especially

$$(L_3^{(3m+1)/3})_+ = (L_3^{1/3} L_3^m)_+ = \sum_{j=0}^{m} N(G_{3j-2}) L_3^{m-j}.$$

Therefore

$$\begin{aligned}
[(L_3^{(3m+1)/3})_+, L_3] &= \Big[\sum_{j=0}^{m} N(G_{3j-2}) L_3^{m-j}, L_3 \Big] \\
=& - \Big[\sum_{j=m+1}^{\infty} N(G_{3j-2}) L_3^{m-j}, L_3 \Big] \\
=& - \sum_{j=m+1}^{\infty} [N(G_{3j-2}), L_3] L_3^{m-j} \\
=& - \sum_{j=m+1}^{\infty} \big(\mathcal{L}_{K(G_{3j-2})} L_3 - (\mathcal{L}_{J(G_{3j-2})} L_3) L_3 \big) L_3^{m-j} \\
=& - \sum_{j=m+1}^{\infty} \big(\mathcal{L}_{J(G_{3j+1})} L_3 - (\mathcal{L}_{J(G_{3j-2})} L_3) L_3 \big) L_3^{m-j} \\
=& - \sum_{j=m+1}^{\infty} (\mathcal{L}_{J(G_{3j+1})} L_3) L_3^{m-j} + \sum_{j=m+1}^{\infty} (\mathcal{L}_{J(G_{3j-2})} L_3) L_3^{m-j+1} \\
=& - \sum_{j=m+1}^{\infty} (\mathcal{L}_{J(G_{3j+1})} L_3) L_3^{m-j} + \sum_{j=m}^{\infty} (\mathcal{L}_{J(G_{3j+1})} L_3) L_3^{m-j} \\
=& \mathcal{L}_{J(G_{3m+1})} L_3.
\end{aligned} \tag{26}$$

From the above equation we see that the Lax representation

$$L_{3,t_{3m+1}} = \big[(L_3^{(3m+1)/3})_+, L_3 \big] \tag{27}$$

is the same as $L_{3,t_{3m+1}} = \mathcal{L}_{J(G_{3m+1})} L_3$ which, in its turn, is the same as the zero curvature representation

$$\begin{pmatrix} u_{t_{3m+1}} \\ v_{t_{3m+1}} \end{pmatrix} = J(G_{3m+1}). \tag{28}$$

Thus we proved that two hierarchies (27) and (28), one comes from the Lax representation, and the other comes from the zero curvature representation, are actually the same. Moreover we have

$$\begin{aligned}
\operatorname{tr}(L_3^{(3m+1)/3}) &= \operatorname{tr}(L_3^{1/3} L^m) \\
=& \operatorname{tr}\Big(\sum_{j=0}^{\infty} N(G_{3j-2}) L_3^{m-j} \Big) \\
=& \operatorname{tr}(N(G_{3m+1}) L_3^{-1})
\end{aligned}$$

$$
\begin{aligned}
=&\operatorname{tr}\Big(\Big(\frac{2}{3}s_{3m+1}\partial^2+\cdots\Big)(\partial^{-3}+\cdots)\Big)\\
=&\operatorname{tr}\Big(\frac{2}{3}s_{3m+1}\partial^{-1}+\cdots\Big)\\
=&\frac{2}{3}s_{3m+1}.
\end{aligned}
\tag{29}
$$

They are conserved densities of the hierarchy. By a similar reasoning we have

$$
L_3^{2/3}=\sum_{j=0}^{\infty}N(G_{3j-1})L_3^{-j}. \tag{30}
$$

Note that the leading term of the right hand side is $\partial^2+\frac{4}{3}u$:

$$
\begin{aligned}
&\sum_{j=0}^{\infty}N(G_{3j-1})L_3^{-j}\\
=&N(G_{-1})+N(G_2)L_3^{-1}+\cdots\\
=&\frac{2}{3}s\partial^2+\Big(-\frac{1}{3}s_x+\frac{1}{3}r\Big)\partial+\Big(\frac{8}{9}su+\frac{1}{9}s_{xx}-\frac{1}{3}r_x\Big)\Big|_{r=0,s=3/2}+o(\partial^{-1})\\
=&\partial^2+\frac{4}{3}u+o(\partial^{-1}).
\end{aligned}
\tag{31}
$$

Similarly we have

$$
\big[(L_3^{(3m+2)/3})_+,L_3\big]=\mathcal{L}_{J(G_{3m+2})}L_3. \tag{32}
$$

It shows that the Lax representation

$$
L_{3,t_{3m+2}}=[(L_3^{(3m+2)/3})_+,L_3] \tag{33}
$$

is the same as its zero curvature representation

$$
\begin{pmatrix} u_{t_{3m+2}}\\ v_{t_{3m+2}}\end{pmatrix}=J(G_{3m+2}). \tag{34}
$$

We sum up the above results in a theorem.

Theorem 3.3. *For the third order differential operator $L_3=\partial^3+u\partial+\partial u+v$, its roots $L_3^{1/3}$ and $L_3^{2/3}$ are given by (25) and (30) where N is given by (23). The corresponding hierarchies in their zero curvature and Lax representations are respectively given by (27), (28), (33), (34). Those two representations give actually the same equations since it holds that $[(L_3^{(3m+i)/3})_+,L_3]=\mathcal{L}_{J(G_{3m+i})}L_3$. The sequence of conserved densities are given by $tr(L_3^{(3m+i)/2})=\frac{2}{3}s_{3m+i}$ where $G_n=(r_n,s_n)^T$.*

4 Case $m=2$: KdV Hierarchy

The famous KdV hierarchy corresponds to the case $m=2$. In this case the operator reduces to the classical Schrodinger operator

$$
L_2=\partial^2+u. \tag{35}
$$

The first and the second Hamiltonian operators are known to be

$$
J=\partial,\qquad K=\partial^3+4u\partial+2u_x. \tag{36}
$$

The square root of L_2 is given by [20]

$$L_2^{1/2} = \sum_{n=1}^{\infty}(2b_n\partial - b_{n,x})(4L)^{-n+1}, \tag{37}$$

where the sequence $\{b_k\}$ is defined by

$$b_1 = 1/2, \qquad Jb_{n+1} = Kb_n, \qquad n = 1, 2, \cdots. \tag{38}$$

Based on the Formula (37) we can easily derive the following equations

$$\mathrm{tr}\,(L^{(2n+1)/2}) = \frac{1}{2^{2n+1}} b_{n+2} \tag{39}$$

and

$$[L_+^{(2n+1)/2}, L] = 4^{-n} b_{n+2,x}. \tag{40}$$

Those equations show the relationship between zero curvature and Lax representations. To sum up we proved the following theorem

Theorem 4.1. *For the second order differential operator $L_2 = \partial^2 + u$ the corresponding hierarchy in its zero curvature representation is given by $u_{t_n} = Jb_{n+1}$ where $J = \partial$. It holds that $[(L_2^{(2n+1)/2})_+, L_2] = J(4^{-n}b_{n+1})$. Therefore the hierarchy in its Lax representation $L_{2,t_n} = [(L_2^{(2n+1)/2})_+, L_2]$ is the same as in its zero curvature representation. Moreover, the sequence of conserved densities are given by $tr\,(L_2^{(2n+1)/2}) = \frac{1}{2^{2n+1}}b_{n+2}$.*

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