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Further Results on Mutually Nearly Orthogonal Latin Squares

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Abstract Nearly orthogonal Latin squares are useful for conducting experiments eliminating heterogeneity in two directions and using different interventions each at each level. In this paper, some constructions of mutually nearly orthogonal Latin squares are provided. It is proved that there exist 3 MNOLS($2m$) if and only if $m \geq 3$ and there exist 4 MNOLS($2m$) if and only if $m \geq 4$ with some possible exceptions.

Keywords Latin square; orthogonal; nearly orthogonal; holey
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1 Introduction

A Latin square of order n is an $n \times n$ array in which every row and column is a permutation of a set N of n elements. We assume that $N = \{0, 1, 2, \cdots, n-1\}$. Let $L = (l_{i,j})$ and $M = (m_{i,j})$ be two Latin squares of order n, based on the set N. Define the superposition of L onto M to be the $n \times n$ array $A = (l_{i,j}, m_{i,j})$. Then L and M are said to be orthogonal if the superposition of L onto M has every ordered pair (i, j) appearing exactly once. A set of Latin squares in which each pair is orthogonal, is called a set of mutually orthogonal Latin squares. Mutually orthogonal Latin squares have been extensively studied[5].

The concept of two Latin squares being nearly orthogonal was introduced by Raghavarao et al[9]. Latin squares L and M of even order n are said to be nearly orthogonal if the superposition of L onto M has every ordered pair (i, j) appearing exactly once except for $i = j$, when the ordered pair appears 0 times and except for $i - j = n/2 \pmod{n}$, when the ordered pair appears 2 times. A set of t Latin squares of order n is called a set of mutually nearly orthogonal Latin squares, denoted by t MNOLS(n), if the t Latin squares are pairwise nearly orthogonal.

Nearly orthogonal Latin squares are useful for conducting experiments eliminating heterogeneity in two directions and using different interventions each at each level[9].

The upper bound on the number of a set of MNOLS(v) is given by Raghavarao et al.[9]. Li and van Rees[8] gave a new proof of this bound.

Lemma 1.1[8,9]. *Let $m \geq 2$ be a positive integer.*

(a) If there exists a set of t MNOLS($2m$), then $t \leq m + 1$.

(b) If m is even and there exists a set of t MNOLS($2m$), then $t < m + 1$.

For $t = 3, 4$, Li and van Rees[8] proved the following.

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Lemma 1.2[8]**.** *(i) There exist* 3 *MNOLS*$(2m)$ *for any* $3 \le m \le 10$ *and* $m \ge 179$*;*

(ii) There exist 4 *MNOLS*$(2m)$ *for* $m = 5, 6$.

In this article, we further investigate the existence of t MNOLS$(2m)$ for $t = 3, 4$ and prove that the necessary condition of the existence of 3 MNOLS$(2m)$ is also sufficient, which provides a positive answer to Conjecture 5.1 given by Li and Rees in [8]. It is also true for $t = 4$ with several undetermine cases. Specifically, we obtain the following.

Theorem 1.3. *There exist* 3 *MNOLS(*$2m$*) if and only if* $m \ge 3$.

Theorem 1.4. *There exist* 4 *MNOLS*$(2m)$ *if and only if* $m \ge 4$ *with some possible exceptions* $m \in E = \{4, 11, 13, 14, 17, 18, 19, 21, 22, 23, 25, 26, 27, 29, 31, 33, 34, 37, 38, 39, 41, 43, 44, 46\}$.

Some terminology and known results are introduced in Section 2. In Section 3, some recursive constructions of HMOLS and MNOLS are provided. The proof of our main results are given in Sections 4 and 5, respectively. Some remarks are mentioned in Section 6.

2 Preliminaries

In this section, we shall give some terminology and some known results which will be used in the proof of the main results.

A *group divisible design* (or GDD) is a triple $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ which satisfies the following properties:

1. $\mathcal{G}$ is a partition of a set $\mathcal{X}$ (of *points*) into subsets called *groups*;

2. $\mathcal{B}$ is a set of subsets of $\mathcal{X}$ (called *blocks*) such that a group and a block contain at most one common point;

3. Every pair of points from distinct groups occurs in exactly λ blocks.

The *group type* (or *type*) of GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. We shall use an "exponential" notation to describe types: so type $g_1^{u_1} \cdots g_k^{u_k}$ denotes u_i occurrences of g_i, $1 \le i \le k$, in the multiset. A GDD with block sizes from a set of positive integers K is called a (K, λ)-GDD.

A *transversal design* TD$(k, \lambda; n)$ is a (k, λ)-GDD of group type n^k. When $\lambda = 1$, we simply write TD(k, n). It is well known that a TD(k, n) is equivalent to $k - 2$ mutually orthogonal Latin squares (MOLS) of order n. The following results will be used.

Lemma 2.1[4]**.**

1. A TD$(q+1, q)$ *exists, consequently, a TD*(k, q) *exists for any positive integer* k $(k \le q+1)$*, where* q *is a prime power.*

2. A TD$(5, n)$ *exists for all* n *and* $n \notin \{2, 3, 6, 10\}$.

3. A TD$(6, n)$ *exists for all* $n \ge 5$ *and* $n \notin \{6, 10, 14, 18, 22\}$.

4. A TD$(7, n)$ *exists for all* $n \ge 7$ *and* $n \notin \{10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60\}$.

Let S be a set and $H = \{S_1, S_2, \cdots, S_n\}$ be a set of disjoint subsets of S. A incomplete Latin square having hole set H is an $|S| \times |S|$ array L, indexed by S, satisfying the following properties:

(1) Every cell of L either contains a symbol of S or is empty;

(2) Every symbol of S occurs at most once in any row or column of L;

(3) The subarrays indexed by $S_i \times S_i$ are empty for $1 \le i \le n$ (these subarrays are called holes);

(4) Symbol $s \in S$ occurs in row or column t if and only if $(s, t) \in (S \times S) \setminus \bigcup_{1 \le i \le n} (S_i \times S_i)$.

Two incomplete Latin squares L and M on symbol set S and hole set H are said to be orthogonal if their superposition yields every ordered pair in $(S \times S)\setminus \bigcup_{1\le i\le n} (S_i \times S_i)$. We shall use the notation IMOLS $(s; s_1, s_2, \cdots, s_n)$ to denote a pair of orthogonal incomplete Latin squares on symbol set S and hole set $H = \{S_1, S_2, \cdots, S_n\}$, where $s = |S|$ and $s_i = |S_i|$ for $1 \le i \le n$. If $H = \emptyset$, we obtain a pair of MOLS(s).

If $H = \{S_1, S_2, \cdots, S_n\}$ is a partition of S, then an incomplete Latin square is called a holey Latin square, denoted by HLS. The type of the HLS is defined to be the multiset $\{|S_i| : 1 \le i \le n\}$. We shall use an "exponential" notation to describe types: so type $s_1^{u_1} s_2^{u_2} \cdots s_n^{u_n}$ denotes u_i occurrences of s_i, $1 \le i \le n$, in the multiset. If any two HLS in a set of t HLS of type T are orthogonal, then we denote the set by t HMOLS (T).

Some known results on IMOLS and HMOLS are summarized in the following.

Lemma 2.2[4]. *There exist* 3 *IMOLS*$(m+u, u)$ *for* $m = 6, 8$, $u = 1, 2$ *and* 4 *IMOLS*$(12+t, t)$ *for* $t = 1, 3$.

Lemma 2.3[4,7]. *If* $h \ge 1$ *and* $n \ge 5$, *then there exist* 3 *HMOLS*(h^n), *except for* $(h, n) = (1, 6)$ *and possibly for* $(h, n) \in \{(1, 10), (3, 6), (3, 18), (3, 28), (3, 34), (6, 18)\}$.

Lemma 2.4[1,4]. *Then there exist* 4 *HMOLS*(h^n) *for* $n \ge 6$ *and* $h \in \{1, 2, 3, 10, 12, 14, 18\}$ *except possibly for the following cases:*

1. $h = 1$ *and* $n \in \{10, 14, 18, 22, 26\}$ *(except* $n = 6$*)*.
2. $h = 2$ *and* $n \in \{28, 30, 32, 33, 34, 35, 38, 39, 40\}$.
3. $h = 3$ *and* $n \in \{6, 12, 18, 24, 28, 46, 54, 62\}$.
4. $h = 10$ *and* $n \in \{32, 33, 35, 38\}$.
5. $h = 14$ *and* $n = 34$.

3 Recursive Constructions

In this section, some recursive constructions of HMOLS and MNOLS are discussed, which will be used in the proof of our main results.

Construction 3.1[1,6]. *Suppose* $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ *is a GDD and let weighting function* $w : X \to Z^+ \cup \{0\}$. *Suppose there exist* t *HMOLS of type* $\{w(x) : x \in B\}$ *for every* $B \in \mathcal{B}$. *Then there exist* t *HMOLS of type* $\{\sum_{x\in G} w(x) : G \in \mathcal{G}\}$.

Construction 3.2[1,2]. *Suppose there exist* $k+1$ *MOLS*(t), *and* k *IMOLS*$(m+u_i, u_i)$, *where* $u_i \ge 0$, $1 \le i \le t-1$. *Then there exist* k *HMOLS*$(m^t u^1)$, *where* $u = \sum_{i=1}^{t-1} u_i$.

For the construction of MNOLS, the following construction can be found in [8].

Construction 3.3[8]. *Suppose that there exist* k *MOLS*(n), k *MOLS*$(2m)$ *and* k *MNOLS*$(2m)$, *then there exist* k *MNOLS*$(2mn)$.

Construction 3.4 (Filling Holes). *Suppose there exist* k *HMOLS*$(g_1 g_2 \cdots g_t)$ *and there exist* k *MNOLS*(g_i), $1 \le i \le t$. *Then there exist* k *MNOLS*(v), *where* $v = \sum_{i=1}^{t} g_i$.

Proof. For each i, $1 \le i \le t$, by the definition of MNOLS(g_i), we know that g_i is even. Let

$$m_0 = 0, \qquad m_i = \frac{g_i}{2}, \qquad a_i = \sum_{u=0}^{i-1} m_u, \qquad 1 \le i \le t, \qquad h = \frac{v}{2}.$$

Without loss of generality, we assume that $L_1, L_2, \cdots, L_k$ are k HMOLS($g_1 g_2 \cdots g_t$) based on $S = \{0, 1, \cdots, v-1\}$, and the holes $H_1^{(i)}, H_2^{(i)}, \cdots, H_t^{(i)}$ of L_i are based on the sets $S_1, S_2, \cdots, S_t$, respectively, where

$$S_i = \bigcup_{j=0}^{m_i-1} \{a_i + j, a_i + j + h\}, \qquad i = 1, 2, \cdots, t.$$

It is readily checked that $|S_i| = g_i$, $i = 1, 2, \cdots, t$, and $S_1, S_2, \cdots, S_t$ form a partition of S.

Let $A_1^{(i)}, A_2^{(i)}, \cdots, A_k^{(i)}$ be k MNOLS(g_i) based on the set $\{0, 1, \cdots, g_i - 1\}$, $i = 1, 2, \cdots, t$. For each i, $1 \le i \le t$, let

$$\sigma_i(j) = a_i + j, \qquad \sigma_i(j + m_i) = a_i + j + h, \qquad j = 0, 1, \cdots, m_i - 1.$$

Then σ_i is a bijection from $\{0, 1, \cdots, g_i - 1\}$ to S_i. We operate σ_i on the elements of $A_s^{(i)}$ to get a Latin square $B_s^{(i)}$ based on S_i, $1 \le s \le k$.

Now for each i, $1 \le i \le k$, we fill t holes $H_1^{(i)}, H_2^{(i)}, \cdots, H_t^{(i)}$ of L_i with $B_i^{(1)}, B_i^{(2)}, \cdots, B_i^{(t)}$ to get a Latin square M_i based on S. We shall show that $M_1, M_2, \cdots, M_k$ are k MNOLS(v).

In fact, let $u, u' \in \{1, 2, \cdots, s\}$, $u \ne u'$ and $(p, q) \in S \times S$. If $(p, q) \in (S \times S) \setminus \big(\bigcup_{i=1}^{t} (S_i \times S_i)\big)$, then (p, q) appears exactly once in the superposition of M_u onto $M_{u'}$ since L_u and $L_{u'}$ are orthogonal holy Latin squares. Otherwise, there exists $i \in \{1, 2, \cdots, t\}$ such that $(p, q) \in S_i \times S_i$, it follows $(\sigma^{-1}(p), \sigma^{-1}(q)) \in S_i \times S_i$. If $p = q$, then $\sigma^{-1}(p) = \sigma^{-1}(q)$, $(\sigma^{-1}(p), \sigma^{-1}(q))$ doesn't appear in the superposition of $A_u^{(i)}$ and $A_{u'}^{(i)}$ since $A_u^{(i)}$ and $A_{u'}^{(i)}$ are nearly orthogonal, therefore (p, q) doesn't appear in the superposition of $B_u^{(i)}$ and $B_{u'}^{(i)}$. Similarly, one can show that (p, q) appears exactly twice in the superposition of $B_u^{(i)}$ and $B_{u'}^{(i)}$ if $p - q \equiv h \pmod{v}$. So $M_1, M_2, \cdots, M_s$ are k MNOLS(v). The proof is completed. □

To deal with some cases of 4 MNOLS($2m$), we need the following.

Construction 3.5. *Suppose that there exist* $(s+k+1)$ *MOLS*(n)*, s MOLS*($2m$)*, s IMOLS*($2m+h, h$)*, s MNOLS*($2m$) *and s MNOLS*($2m + kh$)*, then there exist s MNOLS*($2mn + kh$)*.*

Proof. By hypothesis, there exist $(s+k+1)$ MOLS(n), s MOLS($2m$) and s IMOLS($2m+h, h$), so there exist s HMOLS($(2m)^{n-1}(2m + kh)^1$) by Construction 5.6 in [1] (or the references therein). Since there exist s MNOLS($2m$) and s MNOLS($2m+kh$), we get s MNOLS($2mn+kh$) by Construction 3.4. □

4 Proof of Theorem 1.3

In this section, we shall give the proof of Theorem 1.3. By Lemma 1.2, we need only to prove that there exist 3 MNOLS($2m$) for all integers m such that $11 \le m \le 178$. For some small value m, we shall construct t MNOLS($2m$) directly by making use of a $(t, 2m)$-difference set.

A $(t, 2m)$-difference set is a set of $2m$ t-tuples in which the ordered differences modulo $2m$ between elements in two positions form no 0-difference, two m-differences and every other difference appears once. Raghavarao et al proved the following.

Lemma 4.1[9]**.** *If there exists a $(t, 2m)$-difference set, then there exist t MNOLS$(2m)$.*

By Lemma 4.1, to construct t MNLOS$(2m)$, it suffices to find a $(t, 2m)$-difference set.

Lemma 4.2. *There is* 3 *MNOLS*$(2m)$ *for any* $m \in M = \{11, 12, 13, 14, 17, 22\}$.

Proof. For each $m \in M$, with the aid of a computer, we find a $(3, 2m)$-difference set list below, consequently, 3 MNOLS$(2m)$ are obtained by Lemma 4.1.

$m = 11$:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 & 21 & 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 \\ 2 & 0 & 3 & 9 & 15 & 21 & 16 & 19 & 4 & 18 & 14 & 8 & 7 & 11 & 17 & 20 & 1 & 6 & 10 & 5 & 13 & 12 \end{pmatrix}.$$

$m = 12$:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 & 21 & 23 & 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 \\ 2 & 0 & 3 & 1 & 14 & 21 & 20 & 19 & 23 & 15 & 6 & 18 & 16 & 10 & 17 & 8 & 11 & 22 & 5 & 13 & 4 & 9 & 7 & 12 \end{pmatrix}.$$

$m = 13$:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 & 21 & 23 & 25 & 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 \\ 2 & 0 & 3 & 1 & 12 & 21 & 19 & 22 & 25 & 4 & 23 & 16 & 8 & 17 & 20 & 18 & 10 & 24 & 15 & 7 & 13 & 5 & 14 & 6 & 9 & 11 \end{pmatrix}.$$

$m = 14$:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 & 21 & 23 & 25 & 27 & 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 \\ 2 & 0 & 3 & 1 & 11 & 16 & 22 & 25 & 20 & 23 & 4 & 8 & 21 & 5 & 18 & 10 & 19 & 13 & 24 & 27 & 7 & 26 & 15 & 9 & 6 & 14 & 17 & 12 \end{pmatrix}.$$

$m = 17$:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 9 & 4 & 15 & 6 & 12 & 5 & 11 & 24 & 28 & 17 & 26 & 33 & 14 & 7 & 18 & 13 & 31 & 19 & 22 & 16 & 21 & 20 & 25 & 10 & 32 & 23 & 29 & 30 & 8 & 27 \\ 1 & 3 & 5 & 7 & 19 & 9 & 31 & 13 & 25 & 11 & 23 & 14 & 22 & 0 & 18 & 32 & 29 & 15 & 2 & 27 & 28 & 4 & 10 & 33 & 8 & 6 & 16 & 21 & 30 & 12 & 24 & 26 & 17 & 20 \\ 2 & 0 & 3 & 1 & 23 & 19 & 18 & 16 & 4 & 29 & 30 & 31 & 5 & 12 & 32 & 24 & 22 & 20 & 11 & 33 & 14 & 28 & 9 & 21 & 10 & 17 & 7 & 6 & 26 & 27 & 13 & 8 & 25 & 15 \end{pmatrix}.$$

$m = 22$:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 & 21 & 23 & 25 & 27 & 29 & 31 & 33 & 35 & 37 & 39 & 41 & 43 & 0 \\ 4 & 0 & 12 & 8 & 20 & 16 & 28 & 24 & 36 & 32 & 1 & 40 & 9 & 5 & 17 & 21 & 11 & 3 & 43 & 31 & 39 & 29 & 18 \end{pmatrix},$$

$$\begin{pmatrix} 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 & 41 & 42 & 43 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 30 & 32 & 34 & 36 & 38 & 40 & 42 \\ 10 & 42 & 2 & 35 & 41 & 22 & 30 & 37 & 13 & 34 & 26 & 23 & 15 & 7 & 6 & 14 & 27 & 38 & 19 & 25 & 33 \end{pmatrix}.$$

□

Lemma 4.3. *There exist* 3 *MNOLS*$(2m)$ *for all* $m \in \{15, 16, 18, 20, 21, 24–27, 30, 32, 33, 35, 36\}$.

Proof. For each $m \in \{15, 18, 21, 24, 27, 30, 33, 36\}$, we write $m = 3t$, where $5 \le t \le 12$. Since there exist 3 HMOLS(6^t) from Lemma 2.3 and 3 MNOLS(6) from Lemma 1.2, we obtain 3 MNOLS$(2m)$ by Construction 3.4.

For each $m \in \{16, 32\}$, we write $m = 4a$, $a \in \{4, 8\}$. Since there exist 3 MOLS(a), 3 MOLS(8) by Lemma 2.1 and 3 MNOLS(8) by Lemma 1.2, we get 3 MNOLS$(2m)$ by Construction 3.3.

For each $m \in \{20, 25, 35\}$, we write $m = 5t$, where $t \in \{4, 5, 7\}$. Since there exist 3 HMOLS(10^t) from Lemma 2.1 and 3 MNOLS(10) from Lemma 1.2, we obtain 3 MNOLS$(2m)$ by Construction 3.4.

For $m = 26$, applying Construction 3.2 with parameters $k = 3$, $t = 7$, $m = 6$, $u_1 = u_2 = u_3 = u_4 = 2$, $u_5 = u_6 = 1$, $u = \sum_{i=1}^{6} = 10$, we get 3 HMOLS$(6^7(10)^1)$. Here the required 4 MOLS(7) and 3 IMOLS$(6+u_i, u_i)$ comes from Lemma 2.1 and Lemma 2.2, respectively. Noting that there exist 3 MNOLS(6) and 3 MNOLS(10) by Lemma 1.2, 3 MNOLS(52) is obtained by Construction 3.4. □

Lemma 4.4. *There exist* 3 *MNOLS*$(2m)$ *for all* $m \in \{19, 23, 37\}$.

Proof. For $m = 19$, since there exist 4 MOLS(5) and 3 IMOLS$(6+2, 2)$ from Lemma 2.1 and Lemma 2.2, respectively, there exist 3 HMOLS$(6^5 8^1)$ by Construction 3.2. Applying Construction 3.4, we obtain 3 MNOLS(38), here the input 3 MNOLS(6) and 3 MNOLS(8) come from Lemma 1.2.

For $m = 23$, since there exist 4 MOLS(5), 3 MOLS(8) by Lemma 2.1, 3 IMOLS$(8+2, 2)$ by Lemma 2.2. So there exist 3 HMOLS$(8^5 6^1)$ by Construction 3.2. Applying Construction 3.4, we obtain 3 MNOLS(46).

For $m = 37$, since there exist 4 MOLS(8) and 3 IMOLS$(8+2, 2)$ from Lemma 2.1 and Lemma 2.2, respectively, there exist 3 HMOLS$(8^8 (10)^1)$ by Construction 3.2. Applying Construction 3.4, we obtain 3 MNOLS(74), here the input 3 MNOLS(8) and 3 MNOLS(10) come from Lemma 1.2. □

Lemma 4.5. *There exist* 3 *MNOLS*$(2m)$ *for all* $m \in \{28, 29, 31, 34\}$.

Proof. For $m \in \{28, 29\}$, we write $m = 25 + a$, where $a \in \{3, 4\}$. Delete $5 - a$ points from the last group of a TD(6, 5), we obtain a $\{5, 6\}$-GDD of type $5^5 a^1$. Since there exist 3 HMOLS(2^5) and 3 HMOLS(2^6) coming from Lemma 2.3, we get 3 HMOLS$((10)^5 (2a)^1)$ by Construction 3.1. By Lemma 1.2, 3 MNOLS(10) and 3 MNOLS$(2a)$. Therefore we obtain 3 MNOLS$(2m)$ by Construction 3.4.

For $m = 31$, delete 2 points from the last group of a TD(8, 8), we get a $\{7, 8\}$-GDD of type $8^7 6^1$. Applying Construction 3.1 with 3 HMOLS(1^7) and 3 HMOLS(1^8) coming from Lemma 2.3, we get 3 HMOLS$(8^7 6^1)$. Since there exist 3 MNOLS(8) and 3 MNOLS(6) coming from Lemma 1.2, we obtain 3 MNOLS(62) by Construction 3.4.

For $m = 34$, delete two points from the last two groups of a TD(9, 8), respectively, we get a $\{7, 8, 9\}$-GDD of type $8^7 6^2$. Applying Construction 3.1 with 3 HMOLS(1^t), $t \in \{7, 8, 9\}$, coming from Lemma 2.3, we obtain 3 HMOLS$(8^7 6^2)$. Since there exist 3 MNOLS(8) and 3 MNOLS(6) coming from Lemma 1.2, we get 3 MNOLS(68) by Construction 3.4. □

Lemma 4.6. *There exist* 3 *MNOLS*$(2m)$ *for all integers* $m \in [38, 178]$.

Proof. For each integer $m \in [38, 178]$, we can write $m = 5n + (x + y)$, where $n \in \{7, 9, 11, 13, 16, 17, 19, 25, 27\}$ and $x, y \in [3, n] \cup \{0\}$. The parameters are listed below.

m	n	$x+y$
$[38, 49]$	7	$[3, 14]$
$[50, 63]$	9	$[4, 18]$
$[60, 79]$	11	$[5, 22]$
$[80, 91]$	13	$[15, 26]$
$[91, 112]$	16	$[11, 32]$
$[113, 133]$	19	$[18, 38]$
$[134, 175]$	25	$[9, 50]$
$[176, 178]$	27	$[41, 43]$

By Lemma 2.1, there exists a TD$(7, n)$. Delete $n - x$ and $n - y$ points from the last two groups of the TD$(7, n)$, respectively, we get a $\{5, 6, 7\}$-GDD of type $n^5 x^1 y^1$. By Lemma 2.3, there exist 3 HMOLS(2^t) for $t \in \{5, 6, 7\}$, we get 3 HMOLS$((2n)^5 (2x)^1 (2y)^1)$ by Construction 3.1. Noting that there exist 3 MNOLS$(2n)$, 3 MNOLS$(2x)$ and 3 MNOLS$(2y)$ coming from Lemma 1.2 and Lemmas 4.2–4.5, we obtain 3 MNOLS$(2m)$ by Construction 3.4. □

Combine Lemma 1.2 and Lemmas 4.2–4.6, we get the proof of Theorem 1.3. □

5 Proof of Theorem 1.4

In this section, we shall give the proof of Theorem 1.4. For some small values m, the corresponding 4 MNOLS($2m$) are obtained by finding a $(4, 2m)$-difference set directly.

Lemma 5.1. *There exist* 4 *MNOLS*($2m$) *for all* $m \in \{7, 9, 10, 15\}$.

Proof. For each $m \in \{7, 9, 10, 15\}$, with the aid of a computer, we find a $(4, 2m)$-difference set listed below. Hence 4 MNOLS($2m$) are obtained by Lemma 4.1.

$m = 7$:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 1 & 0 & 4 & 6 & 11 & 9 & 12 & 3 & 13 & 2 & 7 & 5 & 10 & 8 \\ 5 & 2 & 11 & 0 & 7 & 12 & 13 & 9 & 6 & 1 & 4 & 10 & 8 & 3 \\ 12 & 7 & 6 & 4 & 1 & 0 & 11 & 10 & 2 & 8 & 3 & 13 & 5 & 9 \end{pmatrix}.$$

$m = 9$:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\ 4 & 3 & 11 & 16 & 2 & 15 & 14 & 10 & 5 & 8 & 17 & 12 & 0 & 6 & 1 & 9 & 7 & 13 \\ 2 & 13 & 0 & 11 & 5 & 9 & 15 & 16 & 1 & 12 & 7 & 3 & 17 & 8 & 10 & 14 & 4 & 6 \\ 11 & 2 & 5 & 17 & 13 & 7 & 1 & 12 & 14 & 3 & 8 & 0 & 16 & 10 & 4 & 6 & 15 & 9 \end{pmatrix}.$$

$m = 10$:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\ 12 & 11 & 4 & 2 & 8 & 15 & 17 & 5 & 13 & 6 & 19 & 18 & 0 & 9 & 7 & 16 & 10 & 3 & 1 & 14 \\ 16 & 5 & 12 & 4 & 9 & 2 & 0 & 14 & 3 & 19 & 18 & 13 & 11 & 6 & 17 & 1 & 8 & 15 & 7 & 10 \\ 10 & 4 & 1 & 17 & 13 & 9 & 16 & 15 & 14 & 2 & 6 & 8 & 3 & 11 & 19 & 7 & 18 & 12 & 5 & 0 \end{pmatrix}.$$

$m = 15$:

$$\begin{pmatrix} 5 & 20 & 17 & 18 & 12 & 22 & 3 & 21 & 25 & 7 & 13 & 27 & 2 & 28 & 4 & 16 & 23 & 1 & 11 & 14 & 19 & 9 & 26 & 15 & 6 & 0 & 29 & 24 & 10 & 8 \\ 20 & 17 & 18 & 12 & 5 & 3 & 21 & 25 & 7 & 22 & 27 & 2 & 28 & 4 & 13 & 23 & 1 & 11 & 14 & 16 & 9 & 26 & 15 & 6 & 19 & 29 & 24 & 10 & 8 & 0 \\ 17 & 18 & 12 & 5 & 20 & 21 & 25 & 7 & 22 & 3 & 2 & 28 & 4 & 13 & 27 & 1 & 11 & 14 & 16 & 23 & 26 & 15 & 6 & 19 & 9 & 24 & 10 & 8 & 0 & 29 \\ 18 & 12 & 5 & 20 & 17 & 25 & 7 & 22 & 3 & 21 & 28 & 4 & 13 & 27 & 2 & 11 & 14 & 16 & 23 & 1 & 15 & 6 & 19 & 9 & 26 & 10 & 8 & 0 & 29 & 24 \end{pmatrix}.$$

For $m \equiv 0 \pmod 4$, to find a $(4, 2m)$-difference set, we shall make use of the following useful construction.

Construction 5.2. *Suppose that* $m \equiv 0 \pmod 4$, *let* $C_k = (a_{4k}, a_{4k+1}, a_{4k+2}, a_{4k+3}), k = 0, 1, \cdots, \frac{m}{2} - 1$ *and* C_0, C_1, $\cdots$, $C_{\frac{m}{2}-1}$ *form a partition of* $\{0, 1, \cdots, 2m-1\}$. *Let*

$$M_1 = \bigcup_{k=1}^{\frac{m}{2}-1} \{a_{4k+1} - a_{4k}, a_{4k+2} - a_{4k+1}, a_{4k+3} - a_{4k+2}, a_{4k} - a_{4k+3}\},$$

$$M_2 = \bigcup_{k=1}^{\frac{m}{2}-1} \{a_{4k+2} - a_{4k}, a_{4k+3} - a_{4k+1}, a_{4k} - a_{4k+2}, a_{4k+1} - a_{4k+3}\},$$

here the operations are all taken modulo $2m$. *If for each* $i = 1, 2$, *each number of* $\{0, 1, \cdots, 2m-1\}$ *appears exactly once in* M_i *except for 0 and* m, *where* $0 \notin M_i$ *and* m *appears exactly twice, then there exists a* $(4, 2m)$*-difference set over* $\{0, 1, \cdots, 2m-1\}$.

Proof. For each k, $0 \le k \le \frac{m}{2} - 1$, let $\sigma = (0\ 1\ 2\ 3)$ be a component permutation of C_k. Let $A = (A_0, A_1, \cdots, A_{\frac{m}{2}-1})$, where

$$A_k = \begin{pmatrix} C_k \\ \sigma(C_k) \\ \sigma^2(C_k) \\ \sigma^3(C_k) \end{pmatrix} = \begin{pmatrix} a_{4k} & a_{4k+1} & a_{4k+2} & a_{4k+3} \\ a_{4k+1} & a_{4k+2} & a_{4k+3} & a_{4k} \\ a_{4k+2} & a_{4k+3} & a_{4k} & a_{4k+1} \\ a_{4k+3} & a_{4k} & a_{4k+1} & a_{4k+2} \end{pmatrix}, \qquad k = 0, 1, \cdots, \frac{m}{2} - 1.$$

It is easy to check that A is a $(4, 2m)$-difference set. □

Lemma 5.3. *There exist* 4 *MNOLS*($2m$) *for all* $m \in M = \{8, 12, 16, 20, 24, 28, 32\}$.

Proof. For each $m \in M$, to construct a $(4, 2m)$-difference set, by Construction 5.2, we need only to find 4-tuples $C_0, C_1, \cdots, C_{\frac{m}{2}-1}$ satisfying the conditions described in Construction 5.2. We list the 4-tuples as columns of a $4 \times \frac{m}{2}$ array below.

$m = 8$:

$$\begin{pmatrix} 0 & 2 & 4 & 6 \\ 1 & 15 & 9 & 12 \\ 3 & 10 & 5 & 11 \\ 7 & 8 & 13 & 14 \end{pmatrix}.$$

$m = 12$:

$$\begin{pmatrix} 0 & 11 & 21 & 19 & 1 & 7 \\ 12 & 3 & 9 & 18 & 14 & 8 \\ 22 & 23 & 4 & 20 & 17 & 16 \\ 2 & 6 & 15 & 5 & 10 & 13 \end{pmatrix}.$$

$m = 16$:

$$\begin{pmatrix} 11 & 30 & 14 & 23 & 15 & 12 & 22 & 2 \\ 20 & 18 & 3 & 25 & 7 & 28 & 16 & 17 \\ 27 & 31 & 21 & 5 & 10 & 29 & 1 & 8 \\ 24 & 26 & 0 & 13 & 9 & 19 & 6 & 4 \end{pmatrix}.$$

$m = 20$:

$$\begin{pmatrix} 21 & 26 & 24 & 37 & 7 & 16 & 9 & 22 & 2 & 10 \\ 35 & 13 & 29 & 12 & 5 & 0 & 11 & 17 & 31 & 28 \\ 6 & 3 & 36 & 8 & 39 & 20 & 14 & 38 & 1 & 32 \\ 15 & 4 & 27 & 25 & 19 & 33 & 30 & 23 & 34 & 18 \end{pmatrix}.$$

$m = 24$:

$$\begin{pmatrix} 3 & 41 & 47 & 46 & 30 & 8 & 20 & 4 & 42 & 31 & 22 & 15 \\ 6 & 26 & 5 & 39 & 12 & 13 & 7 & 18 & 11 & 17 & 21 & 23 \\ 19 & 1 & 27 & 37 & 16 & 45 & 14 & 9 & 35 & 32 & 0 & 33 \\ 2 & 29 & 38 & 10 & 36 & 40 & 43 & 28 & 24 & 34 & 44 & 25 \end{pmatrix}.$$

$m = 28$:

$$\begin{pmatrix} 42 & 52 & 2 & 31 & 17 & 35 & 29 & 6 & 32 & 18 & 43 & 36 & 3 & 9 \\ 20 & 28 & 19 & 24 & 15 & 37 & 47 & 44 & 51 & 4 & 26 & 25 & 10 & 33 \\ 49 & 5 & 13 & 39 & 14 & 7 & 16 & 50 & 0 & 41 & 21 & 53 & 22 & 34 \\ 46 & 8 & 48 & 23 & 30 & 55 & 1 & 40 & 11 & 54 & 12 & 27 & 45 & 38 \end{pmatrix}.$$

$m = 32$:

$$\begin{pmatrix} 8 & 45 & 7 & 54 & 51 & 52 & 35 & 18 & 13 & 17 & 24 & 22 & 50 & 40 & 5 & 6 \\ 39 & 14 & 53 & 46 & 49 & 10 & 42 & 21 & 56 & 31 & 23 & 11 & 15 & 12 & 30 & 33 \\ 59 & 26 & 27 & 1 & 37 & 19 & 25 & 47 & 29 & 2 & 20 & 34 & 16 & 63 & 62 & 48 \\ 3 & 32 & 61 & 43 & 58 & 4 & 41 & 0 & 9 & 57 & 60 & 36 & 55 & 44 & 28 & 38 \end{pmatrix}.$$

□

Lemma 5.4. *There exist* 4 *MNOLS*$(2m)$ *for any* $m \geq 30$ *and* $m \equiv 0 \pmod 6$.

Proof. For $m = 30$, since there exist 4 MOLS(5), 4 MOLS(12) by Lemma 2.1 and 4 MNOLS(12) by Lemma 4.1, we get 4 MNOLS(60) by Construction 3.3.

For $m > 30$ and $m \equiv 0 \pmod 6$, we write $m = 6n$, where $n \geq 6$. Since there exist 4 HMOLS(12^n) and 4 MNOLS(12) coming from Lemma 2.4 and Lemma 1.2, respectively, we obtain 4 MNOLS($2m$) by Construction 3.4. □

Lemma 5.5. *There exist* 4 *MNOLS*$(2m)$ *for any* $m \in \{35, 40, 45, 50, 51\}$.

Proof. For each $m \in \{35, 40, 45, 50\}$, we can write $m = 5t$, where $t \in \{7, 8, 9, 10\}$. Since there exist 4 HMOLS(10^t) coming from Lemma 2.4 and 4 MNOLS(10) coming from Lemma 1.2, we get 4 MNOLS($2m$) by Construction 3.4.

For $m = 51$, since there exist 7 MOLS(8), 4 MOLS(12) coming from Lemma 2.1, 4 IMOLS(12+3,3) coming from Lemma 2.2 and 4 MNOLS (12) coming from Lemma 1.2, we get 4 MNOLS(102) by Construction 3.5. □

Lemma 5.6. *There exist* 4 *MNOLS*$(2m)$ *for any* $m \in \{47, 49, 52, 57\}$.

Proof. For each $m \in \{47, 49\}$, it can be written $m = 42 + x$, where $x \in \{5, 7\}$. Delete $7 - x$ points from the last group of a TD$(7, 7)$ from Lemma 2.1, we get a $\{6, 7\}$-GDD of type $7^6 x^1$.

Applying Construction 3.1 with 4 HMOLS(2^6) and 4 HMOLS(2^7) coming from Lemma 2.4, we get 4 HMOLS($(14)^6(2x)^1$). Since there exist 4 MNOLS(14) and 4 MNOLS($2x$) coming from Lemma 5.1 and Lemma 1.2, we obtain 4 MNOLS($2m$) by Construction 3.4.

For $m = 52$, delete 2 points from the last two groups of a TD$(8, 7)$, respectively, we get a $\{6, 7, 8\}$-GDD of type $7^6 5^2$. Applying Construction 3.1 with 4 HMOLS(2^t) for $t = 6, 7, 8$ coming from 2.4, we get 4 HMOLS($(14)^6(10)^2$). Since there exist 4 MNOLS(14) and 4 MNOLS(10) coming from Lemma 5.1 and Lemma 1.2, respectively, we obtain 4 MNOLS(104) by Construction 3.4.

For $m = 57$, delete 7 points which belong to a block from a TD$(8, 8)$, we get a $\{7, 8\}$-GDD of type $7^7 8^1$. Applying Construction 3.1 with 4 HMOLS(2^7) and 4 HMOLS(2^8) coming from Lemma 2.4, we get 4 HMOLS($(14)^7(16)^1$). Noting that there exist 4 MNOLS(14) and 4 MNOLS(16) coming from Lemma 5.1 and Lemma 5.3, respectively, we obtain 4 MNOLS(114) by Construction 3.4. □

Lemma 5.7. *There exist* 4 *MNOLS*($2m$) *for all integers* $m \in [53, 64] \setminus \{57\}$.

Proof. For each integer $m \in [53, 64] \setminus \{57\}$, we can write $m = 48 + (x + y)$, where $5 \le (x + y) \le 16$ and $x, y \in \{0, 5, 6, 7, 8\}$.

Delete $8 - x$ and $8 - y$ points from the last two groups of a TD$(8, 8)$, respectively, we get a $\{6, 7, 8\}$-GDD of type $8^6 x^1 y^1$. Applying Construction 3.1 with 4 HMOLS(2^6), 4 HMOLS(2^7) and 4 HMOLS(2^8) from Lemma 2.4, we get 4 HMOLS($(16)^6(2x)^1(2y)^1$). Since there exist 4 MNOLS(16), 4 MNOLS($2x$) and 4 MNOLS($2y$) coming from Lemma 5.1 and Lemma 1.2, we obtain 4 MNOLS($2m$) by Construction 3.4. □

Lemma 5.8. *There exist* 4 *MNOLS*($2m$) *for all integers* $m \in [65, 90]$.

Proof. For each integer $m \in [65, 90]$, we can write $m = 54 + (x + y + z + w)$, where $11 \le (x + y + z + w) \le 36$ and $x, y, z, w \in \{0, 5, 6, 7, 8, 9\}$.

Delete $9 - x$, $9 - y$, $9 - z$, $9 - w$ points from the last four groups of a TD$(10, 9)$, respectively, we get a $\{6, 7, 8, 9\}$-GDD of type $9^6 x^1 y^1 z^1 w^1$. Applying Construction 3.1 with 4 HMOLS(2^t), $t \in \{6, 7, 8, 9\}$, coming from 2.4, we get 4 HMOLS($(18)^6(2x)^1(2y)^1(2z)^1(2w)^1$).

Since there exist 4 MNOLS(18), 4 MNOLS($2x$), 4 MNOLS($2y$), 4 MNOLS($2z$), 4 MNOLS($2w$) coming from Lemma 5.1 and Lemma 1.2, we obtain 4 MNOLS($2m$) by Construction 3.4. □

Lemma 5.9. *There exist* 4 *MNOLS*($2m$) *for all integers* $m \in [91, 100]$.

Proof. For $m \in \{91, 98\}$, we can write $m = 7t$, where $t \in \{13, 14\}$. Since there exist 4 HMOLS(14^t) coming from Lemma 2.4 and 4 MNOLS(14) coming from Lemma 5.1, we get 4 MNOLS($2m$) by Construction 3.4.

For $m \in \{92, 94\}$, we can write $m = 84 + a$, where $a \in \{8, 10\}$. Delete $24 - 2a$ points from the last group of a TD$(8, 24)$ coming from [4, p.186]), we get a $\{7, 8\}$-GDD of type $(24)^7(2a)^1$. Applying Construction 3.1 with 4 HMOLS(1^7) and 4 HMOLS(1^8) coming from Lemma 2.4, we get 4 HMOLS($(24)^7(2a)^1$). Since there exist 4 MNOLS(24) and 4 MNOLS ($2a$) coming from Lemmas 2.4, we obtain 4 MNOLS ($2m$) by Construction 3.4.

For $m = 93$, delete two points from the last group of a TD$(8, 8)$, we get a $\{7, 8\}$-GDD of type $8^7 6^1$. Since there exist 4 HMOLS(3^7) and 4 HMOLS(3^8) coming from Lemma 2.4, by Construction 3.1, we get 4 HMOLS($(24)^7(18)^1$). Since there exist 4 MNOLS(18) and 4 MNOLS(24) coming from Lemmas 5.1–5.3, we obtain 4 MNOLS(186) by Construction 3.4.

For $m \in \{95, 100\}$, we can write $m = 5t$, where $t \in \{19, 20\}$. Since there exist 4 HMOLS(10^t) coming from Lemma 2.4 and 4 MNOLS(10) coming from Lemma 1.2, we get 4 MNOLS($2m$)

by Construction 3.4.

For $m = 96$, 4 MNOLS(192) is given in Lemma 5.4.

For $m = 97$, since there exist 7 MOLS(16), 4 MOLS(12) by Lemma 2.1, 4 IMOLS(12+1,1) by Lemma 2.2, 4 MNOLS(12) by Lemma 1.2 and 4 MNOLS(14) by Lemma 5.1, we get 4 MNOLS(194) by Construction 3.4.

For $m = 99$, there exists 4 HMOLS(18^{11}) by Lemma 2.4 and 4 MNOLS(18) by Lemma 5.1, we get 4 MNOLS(198) by Construction 3.5. □

Lemma 5.10. *There exist* 4 *MNOLS*$(2m)$ *for all integers* $m \in [101, 206]$.

Proof. For integer $m \in [101, 206]$, it can be written $m = 96 + \sum_{i=1}^{11} g_i$, where $5 \le \sum_{i=1}^{11} g_i \le 110$, $g_i \in \{0, 5, 6, 7, 8, 9, 10\}$, $1 \le i \le 11$.

Remove $16 - g_i$ points from i-th group of a TD(17, 16), $1 \le i \le 11$, respectively, we get a $\{6, 7, \cdots, 17\}$-GDD of type $(16)^6 (g_1)^1 \cdots (g_{11})^1$. Applying Construction 3.1 with 4 HMOLS(2^u), $6 \le u \le 17$ coming from Lemma 2.4, we get 4 HMOLS($(32)^6 (2g_1)^1 \cdots (2g_{11})^1$). Applying Construction 3.4, we obtain 4 MNOLS($2m$). Here, the input 4 MNOLS(32) and 4 MNOLS($2g_i$), $1 \le i \le 11$, come from Lemma 1.2 and Lemmas 5.1–5.3. □

Lemma 5.11. *There exist* 4 *MNOLS*$(2m)$ *for all integers* $m \in [207, 370]$.

Proof. For each integer $m \in [207, 370]$, it can be written $m = 192 + \sum_{i=1}^{18} g_i$, where $15 \le \sum_{i=1}^{18} g_i \le 178$, $g_i \in \{0, 5, 6, 7, 8, 9, 10\}$, $1 \le i \le 18$.

Remove $32 - g_i$ points from i-th group of a TD(24, 32), $1 \le i \le 18$, respectively, we get a $\{6, 7, \cdots, 24\}$-GDD of type $(32)^6 (g_1)^1 (g_2)^1 \cdots (g_{18})^1$. Applying Construction 3.1 with 4 HMOLS(2^u), $6 \le u \le 24$ coming from Lemma 2.4, we get 4 HMOLS($(64)^6 (2g_1)^1 \cdots (2g_{18})^1$). Applying Construction 3.4, we obtain 4 MNOLS($2m$). Here, the input 4 MNOLS(64) and 4 MNOLS($2g_i$), $1 \le i \le 18$, come from Lemma 1.2 and Lemmas 5.1–5.3. □

Lemma 5.12. *There exist* 4 *MNOLS*$(2m)$ *for all integers* $m \ge 371$.

Proof. For each integer $m \ge 371$, we can write $m = 6n + a$, where $n \ge 61$ and $5 \le a \le 10$.

Delete $n - a$ points from the last group of a TD$(7, n)$ coming from Lemma 2.1, we get a $\{6, 7, n\}$-GDD of type $6^n a^1$. Applying Construction 3.1 with 4 HMOLS(2^6), 4 HMOLS(2^7) and 4 HMOLS(2^n) coming from Lemma 2.4, we get 4 HMOLS(T), where $T = (12)^n (2a)^1$. Since there exist 4 MNOLS(12) and 4 MNOLS($2a$) coming from Lemma 1.2 and Lemmas 5.1–5.3, we obtain 4 MNOLS($2m$) by Construction 3.4. □

Combine Lemma 1.2 and Lemmas 5.1–5.12, we get the proof of the Theorem 1.4. □

6 Concluding Remarks

In this paper, we solved the existence of 3-MNOLS($2m$) completely and almost solved the existence of 4-MNOLS($2m$) by direct and recursive constructions. In a similar way, one can consider the existence of 5 MNOLS($2m$). For some small values m, to construct 5 MNOLS($2m$), by Lemma 4.1, we need only to find a $(5, 2m)$-difference set. For $m \equiv 0 \pmod 5$, similar to Construction 5.2, we have the following.

Construction 6.1. *Suppose that* $m \equiv 0 \pmod 5$*, let* $C_k = (a_{5k},\ a_{5k+1}, a_{5k+2}, a_{5k+3}, a_{5k+4})$,

$k = 0, 1, \cdots, \frac{2m}{5} - 1$ and $C_0, C_1, \cdots, C_{\frac{2m}{5}-1}$ form a partition of $\{0, 1, \cdots, 2m-1\}$. Let

$$M_1 = \bigcup_{k=1}^{\frac{2m}{5}-1} \{a_{5k+1} - a_{5k}, a_{5k+2} - a_{5k+1}, a_{5k+3} - a_{5k+2}, a_{5k+4} - a_{5k+3}, a_{5k} - a_{5k+4}\},$$

$$M_2 = \bigcup_{k=1}^{\frac{2m}{5}-1} \{a_{5k+2} - a_{5k}, a_{5k+3} - a_{5k+1}, a_{5k+4} - a_{5k+2}, a_{5k} - a_{5k+3}, a_{5k+1} - a_{5k+4}\},$$

here the operations are all taken modulo $2m$. If for each $i = 1, 2$, each number of $\{0, 1, \cdots, 2m-1\}$ appears exactly once in M_i except 0 and m, where $0 \notin M_i$ and m appears exactly twice, then there exist a $(5, 2m)$-difference set over $\{0, 1, \cdots, 2m-1\}$.

Proof. For each k, $0 \le k \le \frac{2m}{5} - 1$, let $\sigma = (0\ 4\ 3\ 2\ 1)$ be a component permutation of C_k. Let $A = (A_0, A_1, \cdots, A_{\frac{2m}{5}-1})$, where

$$A_k = \begin{pmatrix} C_k \\ \sigma(C_k) \\ \sigma^2(C_k) \\ \sigma^3(C_k) \\ \sigma^4(C_k) \end{pmatrix} = \begin{pmatrix} a_{5k} & a_{5k+1} & a_{5k+2} & a_{5k+3} & a_{5k+4} \\ a_{5k+1} & a_{5k+2} & a_{5k+3} & a_{5k+4} & a_{5k} \\ a_{5k+2} & a_{5k+3} & a_{5k+4} & a_{5k} & a_{5k+1} \\ a_{5k+3} & a_{5k+4} & a_{5k} & a_{5k+1} & a_{5k+2} \\ a_{5k+4} & a_{5k} & a_{5k+1} & a_{5k+2} & a_{5k+3} \end{pmatrix}, \qquad k = 0, 1, \cdots, \frac{2m}{5} - 1.$$

It is easy to check that A is a $(5, 2m)$-difference set. □

Example 1. There exists a 5 MNOLS(30).

Proof. With the help of a computer, we find 5-tuples $C_0, C_1, \cdots, C_{\frac{2m}{5}-1}$ satisfying the conditions described in Construction 6.1 which are list as the columns of the following array:

$$\begin{pmatrix} 5 & 22 & 13 & 16 & 19 & 0 \\ 20 & 3 & 27 & 23 & 9 & 29 \\ 17 & 21 & 2 & 1 & 26 & 24 \\ 18 & 25 & 28 & 11 & 15 & 10 \\ 12 & 7 & 4 & 14 & 6 & 8 \end{pmatrix}.$$

By Construction 6.1, there exits a $(5, 30)$-difference set. Consequently, there exists a 5 MNOLS(30) by Lemma 4.1. □

To determine the spectrum of 5 MNOLS($2m$) completely, more computation are needed for small values m.

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