Acta Mathematicae Applicatae Sinica, English Series
Vol. 31, No. 4 (2015) 863–878
DOI: 10.1007/s10255-015-0536-7
http://www.ApplMath.com.cn & www.SpringerLink.com

Acta Mathematicae Applicatae Sinica, English Series

A Class of Stable and Conservative Finite Difference Schemes for the Cahn-Hilliard Equation

Ting-chun WANG[1,2,*], **Li-mei ZHAO**[1], **Bo-ling GUO**[2]

[1]School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China (E-mail: wangtingchun@nuaa.edu.cn)

[2]Institute of Applied Physics and Computational Mathematics, Beijing 100088, China

Abstract In this paper, we propose a class of stable finite difference schemes for the initial-boundary value problem of the Cahn-Hilliard equation. These schemes are proved to inherit the total mass conservation and energy dissipation in the discrete level. The dissipation of the total energy implies boundness of the numerical solutions in the discrete H^1 norm. This in turn implies boundedness of the numerical solutions in the maximum norm and hence the stability of the difference schemes. Unique existence of the numerical solutions is proved by the fixed-point theorem. Convergence rate of the class of finite difference schemes is proved to be $O(h^2+\tau^2)$ with time step τ and mesh size h. An efficient iterative algorithm for solving these nonlinear schemes is proposed and discussed in detail.

Keywords Cahn-Hilliard equation; finite difference scheme; conservation of mass; dissipation of energy; convergence; iterative algorithm

2000 MR Subject Classification 65M06; 65M12

1 Introduction

The Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} - q\frac{\partial^4 u}{\partial x^4} = \frac{\partial^2}{\partial x^2}\phi(u), \qquad (x,t)\in(0,L)\times(0,T], \tag{1.1}$$

$$\phi(u) = pu + ru^3, \qquad (x,t)\in(0,L)\times(0,T], \tag{1.2}$$

where $p<0,\ q<0$ and $r>0$ are constants, arises in the study of phase separation in cooling binary solutions such as alloys, glasses and polymer mixtures[3,25,28] and the references cited therein. Here u(x,t) is a perturbation of the concentration of one of the phases. Initial and boundary conditions are

$$u(x,0) = u_0(x), \qquad x\in[0,L], \tag{1.3}$$

$$\frac{\partial}{\partial x}u(x,t) = \frac{\partial^3}{\partial x^3}u(x,t) = 0, \qquad (x,t)\in\{0,L\}\times(0,T]. \tag{1.4}$$

(1.4) leads to

$$\frac{\partial}{\partial x}\phi(u(x,t)) = 0, \qquad (x,t)\in\{0,L\}\times(0,T].$$

Manuscript received December 11, 2009.
Supported by National Natural Science Foundation of China (Nos. 11201239, 11571181).
*Corresponding author.

By simple calculation we can see that $\phi(u) + q\frac{\partial^2 u}{\partial x^2}$ is the variational derivative of

$$G(u(x,t)) = \frac{1}{2}pu^2 + \frac{1}{4}ru^4 - \frac{1}{2}q\Big(\frac{\partial u}{\partial x}\Big)^2$$

with respect to $u(x,t)$, i.e., $\phi(u) + q\frac{\partial^2 u}{\partial x^2} = \frac{\delta G}{\delta u}$, where the functional G means a local free energy called a Ginzburg-Landau free energy.

The important features of the Cahn-Hilliard equation are that the total mass $\int_0^L u(x,t)dx$ is conserved and the total free energy $\int_0^L G(u(x,t))dx$ decreases with time. Namely,

$$\mathcal{M} := \int_0^L u(x,t)dx \equiv \int_0^L u_0(x)dx, \qquad t > 0, \tag{1.5}$$

$$\mathcal{F}(u) := \int_0^L G(u(x,t))dx, \qquad \frac{d}{dt}\mathcal{F}(u) \le 0. \tag{1.6}$$

The conservation of mass (1.5) and the dissipation of the total energy (1.6) can be shown easily as follows:

$$\frac{d}{dt}\int_0^L u(x,t)dx = \int_0^L \frac{\partial u(x,t)}{\partial t}dx = \int_0^L \frac{\partial^2}{\partial x^2}\frac{\delta G}{\delta u}dx = \Big[\frac{\partial}{\partial x}\frac{\delta G}{\delta u}\Big]_0^L = 0, \tag{1.7}$$

$$\frac{d}{dt}\int_0^L G(u(x,t))dx = \int_0^L \frac{\delta G}{\delta u}\frac{\partial u}{\partial t}dx = -\int_0^L \Big[\frac{\partial}{\partial x}\frac{\delta G}{\delta u}\Big]^2 dx \le 0. \tag{1.8}$$

Remark 1.1. We see from (1.8) that the total energy can be employed as a Lyapunov functional of the system[11].

Since the pioneering work of Cahn and Hilliard[3], the Cahn-Hilliard equation has been extensively studied by Wang and Shi[30], Jabbari and Peppas[24], Puri and Binder[26] for the study of interfaces. Global existence and uniquiness of the solution have been shown by Elliott and Zheng[16]. Yin[36] has shown the existence of the continuous solution for the problem with degenerate mobility. Finite element Galerkin solutions have been obtained by Elliott and French[12,13] and French and Jensen[18]. Elliott et al.[14] have obtained optimal order bounds using a second order splitting method. Elliott and Larsson[15] have discussed the error estimates with smooth and nonsmooth data for a finite element method for the Cahn-Hilliard equation. Mixed finite element method has been applied by Dean et al.[9]. A finite difference scheme has been studied by Furihata et al.[20] who have examined the boundedness of the solution. Sun[27] has proposed an interesting linearized conservative finite difference scheme which is uniquely solvable and convergent with the convergence rate of order two in discrete L_2 norm. Choo et al.[6,7] have proposed a nonlinear difference scheme based on the Crank-Nicolson scheme for the Cahn-Hilliard equation. Their schemes are proved to be unconditional stable and conserve the total mass. Dehghan and Mirzae[10] have described a numerical method based on the boundary integral equation and dual reciprocity methods for solving the one-dimensional Cahn-Hilliard equation. A time-stepping method and a predictor-corrector scheme have been employed to deal with the time derivative and the nonlinearity respectively. In [23,32], a combined spectral and large time-stepping methods have been proposed and studied for the nonlinear diffusion equations for thin film epitaxy. In [21], the convergence of the spatial discretization of the Cahn-Hilliard has been considered. In resent study[5,17,29], the unconditionally stable algorithms have been developed for Cahn-Hillard equation. These algorithms allow for an increasing time step in Cahn-Hillard systems as time proceeds. He and Liu[22] have proposed a class of fully discrete dissipative Fourier spectral schemes for solving the two-dimensional Cahn-Hilliard equation, and presented semi-implicit prediction-correction schemes. Ye[33−35] has studied numerically the

Cahn-Hilliard equation by using the Fourier collocation method, Fourier spectral method and Legendre spectral method, respectively. Both the semi-discrete and the fully discrete schemes derived in [33–35] are uniquely solvable and inherit the energy dissipation property and the mass conservation property. The optimal error bounds of numerical solutions has also been obtained. Ceniceros and Roma[4] present a nonstiff, fully adaptive mesh refinement-based method for the Cahn-Hilliard equation. Yinhua Xia et al.[31] develop local discontinuous Galerkin methods for the fourth order nonlinear Cahn-Hilliard equation and system. In [19], Furihata has designed a difference scheme which inherits the conservation of the total mass and the decrease of the total energy, and proved that the designed scheme preserving characteristic properties of the original equation are numerically stable. Adopting the idea of Furihata[19], Choo, Chung and Lee[8] have proposed a nonlinear difference scheme for the viscous Cahn-Hilliard equations with nonconstant gradient energy coefficient q and showed the scheme preserves the energy dissipation property and mass conservation as for the classical solution.

In this paper, we mainly do three things. Firstly, we propose a class of finite difference schemes which are stable and preserve both of the two properties (1.5) and (1.6). Secondly, we prove the unique existence and convergence of the numerical solutions. Lastly, we construct and discuss in detail an iterative algorithm for solving the proposed nonlinear schemes.

The remainder of this paper is arranged as follows. In Section 2, we propose a class of finite difference schemes which are proved to inherit the properties (1.5) and (1.6) in the descrete level, and consequently the stability of them is obtained. In Section 3, the unique existence of the numerical solutions is discussed by *Brouwer* fixed-point theorem. In Section 4, the convergence of the class of finite difference schemes is proved. In Second 5, an iterative algorithm for solving the proposed nonlinear schemes is constructed and discussed in detail, and then a prediction-correction scheme is proposed based on the iterative algorithm.

2 Finite Difference Schemes

For a positive number N, let time step $\tau = T/N$ and denote time steps $t_n = n\tau,\ n = 0,1,2,\cdots,N$. Given a temporal discrete function $\{u^n \mid n = 0,1,2,\cdots,N\}$, we denote $\delta_t^+ u^n = (u^{n+1} - u^n)/\tau,\ \delta_t^- u^n = (u^n - u^{n-1})/\tau,\ \delta_t u^n = (u^{n+1} - u^{n-1})/2\tau$.

For a positive integer J, let mesh size $h = L/J$ and denote mesh points $x_j = jh,\ j = 0,1,2,\cdots,J$. To approximate the boundary conditions, let $x_{-2} = -2h,\ x_{-1} = -h,\ x_{J+1} = (J+1)h,\ x_{J+2} = (J+2)h$. Given a grid function $u = (u_0, u_1, \cdots, u_J)$, denote $\delta_x^+ u_j = \frac{1}{h}(u_{j+1} - u_j)$, $\delta_x^- u_j = \frac{1}{h}(u_j - u_{j-1}),\ \delta_x^{\langle 1\rangle} u_j = \frac{1}{2h}(u_{j+1} - u_{j-1}),\ \delta_x^{\langle 2\rangle} u_j = \frac{1}{h^2}(u_{j-1} - 2u_j + u_{j+1}),\ \delta_x^{\langle 3\rangle} u_j = \frac{1}{2h^3}(u_{j+2} - 2u_{j+1} + 2u_{j-1} - u_{j-2}),\ \delta_x^{\langle 4\rangle} u_j = \frac{1}{h^4}(u_{j+2} - 4u_{j+1} + 6u_j - 4u_{j-1} + u_{j-2})$,

For two grid functions $u = (u_0, u_1, \cdots, u_J)$ and $v = (v_0, v_1, \cdots, v_J)$, we define discrete inner product and semi-norms as

$$(u,v)_h = h\sum_{j=0}^{J}{}'' u_j v_j := h\Big[\frac{1}{2}u_0 v_0 + \sum_{j=1}^{J-1} u_j v_j + \frac{1}{2}u_J v_J\Big],$$

$$||v^n|| := \sqrt{(v,v)_h}, \qquad ||v||_\infty := \max_{0\le j\le J} |v_j|, \qquad ||\delta_x^+ v|| := \sqrt{h\sum_{j=0}^{J-1}\Big(\delta_x^+ v_j\Big)^2}$$

In this paper we denote $C_k, \widetilde{C}_k,\ k = 0,1,2,\cdots$, as general positive constants which may have different values in different occurrences but independent of discrete parameters. We denote U_j^n and u_j^n be the numerical approximation, and respectively the exact solution of $u(x,t)$ at the point (x_j, t_n) for $j = 0,1,2,\cdots,J$ and $n = 0,1,2,\cdots,N$.

For simplicity of notations, as [22], we denote

$$\varphi(u,v,\alpha)=\Big(\frac{u+v}{2}\Big)\Big(r(\alpha u^2+(1-\alpha)v^2)+p\Big).$$

Clearly, $\varphi(u,u,0)=\phi(u)$. In particular,

$$\varphi\Big(U_j^{n+1},U_j^n,\frac{1}{2}\Big)=\Big(\frac{U_j^{n+1}+U_j^n}{2}\Big)\Big(r\frac{(U_j^{n+1})^2+(U_j^n)^2}{2}+p\Big).$$

For the Problems (1.1)–(1.4), we propose the following finite difference scheme

$$\begin{aligned}
&\delta_t^+U_j^n+\beta h^2\delta_x^{\langle 2\rangle}\delta_t^+U_j^n-\frac{q}{2}\delta_x^{\langle 4\rangle}(U_j^{n+1}+U_j^n)=\delta_x^{\langle 2\rangle}\varphi(U_j^{n+1},U_j^n,\alpha),\\
&\qquad j=0,1,2,\cdots,J,\quad n=0,1,2,\cdots,N-1, &(2.1)\\
&U_j^0=u_0(x_j),\qquad j=0,1,2,\cdots,J, &(2.2)\\
&\delta_x^{\langle 1\rangle}U_0^n=\delta_x^{\langle 1\rangle}U_J^n=\delta_x^{\langle 3\rangle}U_0^n=\delta_x^{\langle 3\rangle}U_J^n=0,\qquad n=0,1,2,\cdots,N. &(2.3)
\end{aligned}$$

From the boundary Condition (2.3) and the definition of $\delta_x^{\langle 2\rangle}$ and $\delta_x^{\langle 4\rangle}$, we obtain

$$\delta_x^{\langle 2\rangle}u_j=\begin{cases}\dfrac{2}{h^2}(u_1-u_0), & \text{when } j=0,\\ \dfrac{1}{h^2}(u_{j-1}-2u_j+u_{j+1}), & \text{when } j=1,2,\cdots,J-1,\\ \dfrac{2}{h^2}(u_{J-1}-u_J), & \text{when } j=J,\end{cases} \tag{2.4}$$

and

$$\delta_x^{\langle 4\rangle}u_j=\begin{cases}\dfrac{2}{h^4}(u_2-4u_1+5u_0), & \text{when } j=0,\\ \dfrac{2}{h^4}(u_3-4u_2+7u_1-4u_0), & \text{when } j=1,\\ \dfrac{1}{h^4}(u_{j-2}-4u_{j-1}+6u_j-4u_{j+1}+u_{j+2}), & \text{when } j=2,\cdots,J-2,\\ \dfrac{2}{h^4}(u_{J-3}-4u_{J-2}+7u_{J-1}-4u_J), & \text{when } j=J-1,\\ \dfrac{2}{h^4}(u_{J2}-4u_{J-1}+5u_J), & \text{when } j=J.\end{cases} \tag{2.5}$$

Obviously, the scheme (2.1)–(2.3) is a nonlinear implicit one. In order to obtain the solution U_j^{n+1} at the level $n+1$, an outer nonlinear iteration for U_j^{n+1} needs to be done and the iterative values of U_j^{n+1} are solved by an inner linear system. Therefore, an efficient iterative algorithm is required to solve the scheme (2.1)–(2.3). For this purpose, we construct and analyze one in Section 5.

The boundary Condition (2.3), together with the definition of φ, gives

$$\big[\delta_x^{\langle 1\rangle}\varphi(U_j^{n+1},U_j^n,\alpha)\big]_0^J=0.$$

We now turn to establish the discrete analogues of (1.5), (1.6). Let $\alpha\geq\frac{1}{2}$ and $\beta\leq\frac{1}{4}$. If U_j^n is the numerical solution of the scheme (2.1)–(2.3), we can obtain the following Lemma.

Lemma 2.1. *The finite difference scheme (2.1)–(2.3) satisfies*

$$\mathcal{M}^n := h\sum_{j=0}^{J}{}'' U_j^n \equiv \mathcal{M}^0 = h\sum_{j=0}^{J}{}'' u_0(x_j), \qquad n = 0, 1, 2, \cdots, N, \tag{2.6}$$

$$\frac{1}{\tau}\Big[h\sum_{j=0}^{J}{}'' G_d(U_j^{n+1}) - h\sum_{j=0}^{J}{}'' G_d(U_j^n)\Big] \le 0, \qquad n = 0, 1, 2, \cdots, N-1, \tag{2.7}$$

where

$$G_d(U_j^n) = \frac{1}{2}p(U_j^n)^2 + \frac{1}{4}r(U_j^n)^4 - \frac{q}{4}\big[(\delta_x^+ U_j^n)^2 + (\delta_x^- U_j^n)^2\big].$$

Proof. Applying summation by parts and noticing the boundary Condition (2.3), we obtain

$$\begin{aligned}
&\frac{1}{\tau}\Big\{h\sum_{j=0}^{J}{}'' U_j^{n+1} - h\sum_{j=0}^{J}{}'' U_j^n\Big\} = h\sum_{j=0}^{J}{}'' \delta_t^+ U_j^n \\
=&h\sum_{j=0}^{J}{}'' \big[-\beta h^2 \delta_x^{\langle 2\rangle}\delta_t^+ U_j^n + \frac{q}{2}\delta_x^{\langle 4\rangle}(U_j^{n+1} + U_j^n) + \delta_x^{\langle 2\rangle}\varphi(U_j^{n+1}, U_j^n, \alpha)\big] \\
=&h\sum_{j=0}^{J}{}'' \delta_x^{\langle 2\rangle}\big[-\beta h^2 \delta_t^+ U_j^n + \frac{q}{2}\delta_x^{\langle 2\rangle}(U_j^{n+1} + U_j^n) + \varphi(U_j^{n+1}, U_j^n, \alpha)\big] \\
=&\big[\delta_x^{\langle 1\rangle}\big[-\beta h^2 \delta_t^+ U_j^n + \frac{q}{2}\delta_x^{\langle 2\rangle}(U_j^{n+1} + U_j^n) + \varphi(U_j^{n+1}, U_j^n, \alpha)\big]\big]_0^J = 0.
\end{aligned} \tag{2.8}$$

This gives (2.6). Denoting $\widetilde{\phi}(s) = \frac{1}{2}ps^2 + \frac{1}{4}rs^4$ and applying summation by parts, we obtain

$$\begin{aligned}
&\frac{1}{\tau}\Big[h\sum_{j=0}^{J}{}'' G_d(U_j^{n+1}) - h\sum_{j=0}^{J}{}'' G_d(U_j^n)\Big] \\
=&h\sum_{j=0}^{J}{}'' \Big[\widetilde{\phi}(U_j^{n+1}) - \frac{q}{2}\frac{(\delta_j^+ U_j^{n+1})^2 + (\delta_j^- U_j^{n+1})^2}{2} - \widetilde{\phi}(U_j^n) + \frac{q}{2}\frac{(\delta_j^+ U_j^n)^2 + (\delta_j^- U_j^n)^2}{2}\Big] \\
=&h\sum_{j=0}^{J}{}'' \Big[\varphi(U_j^{n+1}, U_j^n, \frac{1}{2}) + \frac{q}{2}\delta_x^{\langle 2\rangle}(U_j^{n+1} + U_j^n)\Big]\delta_t^+ U_j^n \\
=&h\sum_{j=0}^{J}{}'' \Big[\varphi(U_j^{n+1}, U_j^n, \alpha) + \frac{q}{2}\delta_x^{\langle 2\rangle}(U_j^{n+1} + U_j^n)\Big]\delta_t^+ U_j^n - \Big(\alpha - \frac{1}{2}\Big)\frac{1}{2\tau}h\sum_{j=0}^{J}{}'' \big[(U_j^{n+1})^2 - (U_j^n)^2\big]^2 \\
=&-\Big(\alpha - \frac{1}{2}\Big)\frac{1}{2\tau}h\sum_{j=0}^{J}{}'' \big[(U_j^{n+1})^2 - (U_j^n)^2\big]^2 + h\sum_{j=0}^{J}{}'' \Big[\varphi(U_j^{n+1}, U_j^n, \alpha) + \frac{q}{2}\delta_x^{\langle 2\rangle}(U_j^{n+1} + U_j^n)\Big] \\
&\times \delta_x^{\langle 2\rangle}\Big[-\beta h^2 \delta_t^+ U_j^n + \frac{q}{2}\delta_x^{\langle 2\rangle}(U_j^{n+1} + U_j^n) + \varphi(U_j^{n+1}, U_j^n, \alpha)\Big].
\end{aligned} \tag{2.9}$$

When $0 \le \beta \le \frac{1}{4}$, it follows from (2.9) that

$$\frac{1}{\tau}\Big[h\sum_{j=0}^{J}{}'' G_d(U_j^{n+1}) - h\sum_{j=0}^{J}{}'' G_d(U_j^n)\Big]$$

$$
\begin{aligned}
&= -\Big(\alpha-\frac{1}{2}\Big)\frac{1}{2\tau}h\sum_{j=0}^{J}{}''\big[(U_j^{n+1})^2-(U_j^n)^2\big]^2 - h\sum_{j=0}^{J}{}''\Big(\delta_x^+\Big[\varphi(U_j^{n+1},U_j^n,\alpha)+\frac{q}{2}\delta_x^{\langle 2\rangle}(U_j^{n+1}+U_j^n)\Big]\Big)^2 \\
&\quad - h\sum_{j=0}^{J}{}''\Big[\varphi(U_j^{n+1},U_j^n,\alpha)+\frac{q}{2}\delta_x^{\langle 2\rangle}(U_j^{n+1}+U_j^n)\Big]\times\beta h^2\delta_x^{\langle 2\rangle}\delta_t^+U_j^n \\
&= -\Big(\alpha-\frac{1}{2}\Big)\frac{1}{2\tau}h\sum_{j=0}^{J}{}''\big[(U_j^{n+1})^2-(U_j^n)^2\big]^2 - h\sum_{j=0}^{J}{}''\Big(\delta_x^+\Big[\varphi(U_j^{n+1},U_j^n,\alpha)+\frac{q}{2}\delta_x^{\langle 2\rangle}(U_j^{n+1}+U_j^n)\Big]\Big)^2 \\
&\quad - h\sum_{j=0}^{J}{}''\delta_x^{\langle 2\rangle}\Big[\varphi(U_j^{n+1},U_j^n,\alpha)+\frac{q}{2}\delta_x^{\langle 2\rangle}(U_j^{n+1}+U_j^n)\Big]\times\beta h^2\delta_t^+U_j^n \\
&= -\Big(\alpha-\frac{1}{2}\Big)\frac{1}{2\tau}h\sum_{j=0}^{J}{}''\big[(U_j^{n+1})^2-(U_j^n)^2\big]^2 - h\sum_{j=0}^{J}{}''(\delta_x^+\Big[\varphi(U_j^{n+1},U_j^n,\alpha)+\frac{q}{2}\delta_x^{\langle 2\rangle}(U_j^{n+1}+U_j^n)\Big]\Big)^2 \\
&\quad - h\sum_{j=0}^{J}{}''\big[\delta_t^+U_j^n+\beta h^2\delta_x^{\langle 2\rangle}\delta_t^+U_j^n\big]\times\beta h^2\delta_t^+U_j^n \\
&= -\Big(\alpha-\frac{1}{2}\Big)\frac{1}{2\tau}h\sum_{j=0}^{J}{}''\big[(U_j^{n+1})^2-(U_j^n)^2\big]^2 - h\sum_{j=0}^{J}{}''\Big(\delta_x^+\Big[\varphi(U_j^{n+1},U_j^n,\alpha)+\frac{q}{2}\delta_x^{\langle 2\rangle}(U_j^{n+1}+U_j^n)\Big]\Big)^2 \\
&\quad - h\sum_{j=0}^{J}{}''\big[\beta h^2(\delta_t^+U_j^n)^2-\beta^2h^4(\delta_x^+\delta_t^+U_j^n)^2\big] \\
&\le -\Big(\alpha-\frac{1}{2}\Big)\frac{1}{2\tau}h\sum_{j=0}^{J}{}''\big[(U_j^{n+1})^2-(U_j^n)^2\big]^2 - h\sum_{j=0}^{J}{}''\Big(\delta_x^+\Big[\varphi(U_j^{n+1},U_j^n,\alpha)+\frac{q}{2}\delta_x^{\langle 2\rangle}(U_j^{n+1}+U_j^n)\Big]\Big)^2 \\
&\quad - h\sum_{j=0}^{J}{}''\Big[\beta h^2(\delta_t^+U_j^n)^2-\beta^2h^4\frac{4}{h^2}(\delta_t^+U_j^n)^2\Big] \\
&\le -\Big(\alpha-\frac{1}{2}\Big)\frac{1}{2\tau}h\sum_{j=0}^{J}{}''\big[(U_j^{n+1})^2-(U_j^n)^2\big]^2 - h\sum_{j=0}^{J}{}''\Big(\delta_x^+\Big[\varphi(U_j^{n+1},U_j^n,\alpha)+\frac{q}{2}\delta_x^{\langle 2\rangle}(U_j^{n+1}+U_j^n)\Big]\Big)^2 \\
&\quad - h\sum_{j=0}^{J}{}''\beta h^2(1-4\beta)(\delta_t^+U_j^n)^2 \le 0. \qquad (2.10)
\end{aligned}
$$

When $\beta \le 0$, denote $\varphi_0(U_j^{n+1},U_j^n,\alpha) = -\beta h^2\delta_t^+U_j^n + \varphi(U_j^{n+1},U_j^n,\alpha)$, then it follows from (2.9) that

$$
\begin{aligned}
&\frac{1}{\tau}\Big[h\sum_{j=0}^{J}{}''G_d(U_j^{n+1}) - h\sum_{j=0}^{J}{}''G_d(U_j^n)\Big] \\
&= -\Big(\alpha-\frac{1}{2}\Big)\frac{1}{2\tau}h\sum_{j=0}^{J}{}''\big[(U_j^{n+1})^2-(U_j^n)^2\big]^2 + h\sum_{j=0}^{J}{}''\Big[\varphi_0(U_j^{n+1},U_j^n,\alpha)+\frac{q}{2}\delta_x^{\langle 2\rangle}(U_j^{n+1}+U_j^n)\Big] \\
&\quad \times\delta_x^{\langle 2\rangle}\Big[\frac{q}{2}\delta_x^{\langle 2\rangle}(U_j^{n+1}+U_j^n)+\varphi_0(U_j^{n+1},U_j^n,\alpha)\Big] - h\sum_{j=0}^{J}{}''\beta h^2\delta_x^{\langle 2\rangle}\delta_t^+U_j^n\delta_t^+U_j^n \\
&= -\Big(\alpha-\frac{1}{2}\Big)\frac{1}{2\tau}h\sum_{j=0}^{J}{}''\big[(U_j^{n+1})^2-(U_j^n)^2\big]^2 + h\sum_{j=0}^{J}{}''\beta h^2(\delta_x^+\delta_t^+U_j^n)^2
\end{aligned}
$$

$$- h\sum_{j=0}^{J}{}''\Big(\delta_x^+\Big[\frac{q}{2}\delta_x^{\langle 2\rangle}(U_j^{n+1}+U_j^n)+\varphi_0(U_j^{n+1},U_j^n,\alpha)\Big]\Big)^2 \le 0. \tag{2.11}$$

Then (2.7) is obtained from (2.10) and (2.11). □

Based on Lemma 2.1, we turn to estimate the numerical solution of the scheme (2.1)–(2.3) by using the similar method in [20].

Lemma 2.2. *The numerical solution of the scheme (2.1)–(2.3) satisfies*

$$||U^n||^2_{d(1,2)} \le \frac{1}{\min(\lambda,-\frac{q}{2})}\Big\{\frac{(p-2\lambda)^2}{4r}L + h\sum_{j=0}^{J}{}''G_d(U_j^0)\Big\}, \tag{2.12}$$

where λ *is any a positive number and* $||U^n||_{d(1,2)}$ *is a discrete first-order Sobolev-Hilbert norm which is defined as*

$$||f||_{d(1,2)} := \sqrt{h\sum_{j=0}^{J}{}''(f_j)^2 + h\sum_{j=0}^{J-1}{}''(\delta_x^+ f_j)^2}, \qquad f=(f_j)_{j=-l}^{J+l}\in\mathcal{R}^{J+1+2l}, \qquad l\ge 0.$$

Proof. It follows from the dissipation of the total energy (2.7) that

$$\begin{aligned}
& h\sum_{j=0}^{J}{}''G_d(U_j^0)\\
\ge & h\sum_{j=0}^{J}{}''G_d(U_j^n) \ge h\sum_{j=0}^{J}{}''\Big\{\lambda(U_j^n)^2 - \frac{(p-2\lambda)^2}{4r} - \frac{q}{2}\frac{(\delta_j^+U_j^n)^2+(\delta_j^-U_j^n)^2}{2}\Big\}\\
& \Big(\text{since } \frac{1}{2}pX^2+\frac{1}{4}rX^4 \ge \lambda X^2 - \frac{(p-2\lambda)^2}{4r}\Big)\\
\ge & \min\Big(\lambda,-\frac{q}{2}\Big)\sum_{j=0}^{J}{}''\Big\{(U_j^n)^2 + \frac{(\delta_j^+U_j^n)^2+(\delta_j^-U_j^n)^2}{2}\Big\} - \frac{(p-2\lambda)^2}{4r}L\\
= & \min\Big(\lambda,-\frac{q}{2}\Big)\,||U^n||^2_{d(1,2)} - \frac{(p-2\lambda)^2}{4r}L,
\end{aligned} \tag{2.13}$$

where the boundary condition (2.3) has been used. Then (2.12) is obtain from (2.13). □

Lemma 2.3[37]**.** *For any grid function f, there is*

$$||f||_\infty \le C_5\sqrt{||f||}\sqrt{||f||+||\delta_x^+ f||}.$$

Lemma 2.4[19]**.** *For any grid function f, there is*

$$||f||_\infty \le 2\max\Big(\frac{1}{\sqrt{L}},\sqrt{L}\Big)||f||_{d(1,2)}.$$

Applying Lemma 2.4 to (2.11) we obtain the following inequality.

Theorem 2.1. *The numerical solution of the scheme (2.1)–(2.3) satisfies*

$$||U^n||_\infty \le 2\sqrt{\frac{\max(\frac{1}{L},L)}{\min(\lambda,-\frac{q}{2})}\Big\{\frac{(p-2\lambda)^2}{4r}L + h\sum_{j=0}^{J}{}''G_d(U_j^0)\Big\}}. \tag{2.14}$$

Theorem 2.1 implies that the proposed difference scheme (2.1)–(2.3) is stable.

3 Unique Existence of the Numerical Solutions

To show the existence of the numerical solutions $U^1, U^2, \cdots, U^N$ for the scheme (2.1)–(2.3), we shall use the following Brouwer-type theorem[1,2].

Lemma 3.1. *Let $(H, (\cdot,\cdot))$ be a finite-dimensional inner product space, $||\cdot||$ be the associated norm, and $g : H \to H$ be continuous. Assume, moreover, that*

$$\exists\, \kappa > 0, \qquad \forall\, z \in H, \qquad ||z|| = \kappa, \qquad \mathrm{Re}\,(g(z), z) \geq 0.$$

Then, there exists a $z^* \in H$ such that $g(z^*) = 0$ and $||z^*|| \leq \kappa$.

For fixed j, we rewrite (2.1) in the form

$$\begin{aligned}&\frac{U_j^{n+1}+U_j^n}{2} - U_j^n + \beta h^2 \delta_x^{\langle 2\rangle}\Big(\frac{U_j^{n+1}+U_j^n}{2} - U_j^n\Big) - \frac{\tau}{2} q \delta_x^{\langle 4\rangle}\frac{U_j^{n+1}+U_j^n}{2}\\ &-\frac{\tau}{2}\delta_x^{\langle 2\rangle}\Big[p\frac{U_j^{n+1}+U_j^n}{2} + 4r\alpha(1-\alpha)\Big(\frac{U_j^{n+1}+U_j^n}{2}\Big)^3 + r\frac{U_j^{n+1}+U_j^n}{2}\Big(2\alpha\frac{U_j^{n+1}+U_j^n}{2} - U_j^n\Big)^2\Big]\\ =&0.\end{aligned}$$

The mapping $\mathcal{F} : \mathcal{R}^{J+1} \to \mathcal{R}^{J+1}$,

$$\begin{aligned}(\mathcal{F}(V))_j =&V_j - U_j^n + \beta h^2 C V_j - \beta h^2 \delta_x^{\langle 2\rangle} U_j^n - \frac{\tau}{2} q \delta_x^{\langle 4\rangle} V_j\\ &-\frac{\tau}{2}\delta_x^{\langle 2\rangle}\big[pV_j + 4r\alpha(1-\alpha)(V_j)^3 + rV_j(2\alpha V_j - U_j^n)^2\big], \qquad 0 \leq j \leq J\end{aligned} \tag{3.1}$$

is obviously continuous. In (3.1), $V = (V_0, V_1, \cdots, V_J)$ and operators $\delta_x^{\langle 2\rangle}, \delta_x^{\langle 4\rangle}$ are defined as (2.4) and (2.5). If the mapping $\mathcal{F}$ has a zero-point V^*, then $2V^* - U^n$ is the solution U^{n+1} of the proposed scheme (2.1)–(2.3). From Theorem 2.1 we know that the $||V^*||_\infty$ is bounded if there exists a numerical solution for the scheme (2.1)–(2.3). In the mapping (3.1), we assume $||V||_\infty \leq C(U^n)$ where $C(U^n) > \frac{10}{\sqrt{L}}||U^n||$.

Theorem 3.1. *If τ is sufficiently small, then the scheme (2.1)–(2.3) has an unique solution.*

Proof. Computing the inner product of (3.1) with V and noticing

$$(\delta_x^{\langle 2\rangle} V, U)_h = -h\sum_{j=0}^{J-1} \delta_x^+ U_j \delta_x^+ V_j = (V, \delta_x^{\langle 2\rangle} U)_h, \tag{3.2}$$

$$\begin{aligned}&-\frac{\tau}{2}(\delta_x^{\langle 2\rangle}[4r\alpha(1-\alpha)(V)^3], V)_h\\ =&2\tau r\alpha(1-\alpha)h\sum_{j=0}^{J-1}(\delta_j^+ V_j)^2[(V_{j+1})^2 + V_{j+1}V_j^n + (V_j^n)^2],\end{aligned} \tag{3.3}$$

$$\begin{aligned}&-\frac{\tau}{2}(\delta_x^{\langle 2\rangle}[rV_j(2\alpha V_j - U_j^n)^2], V)_h\\ \geq& -\frac{\tau|q|}{4}||\delta_x^{\langle 2\rangle} V||^2 - \frac{\tau r}{4|q|}||V(2\alpha V - U^n)^2||^2\\ \geq& -\frac{\tau|q|}{4}||\delta_x^{\langle 2\rangle} V||^2 - \frac{\tau r}{4|q|}||(2\alpha V - U^n)^2||_\infty^2 ||V||^2\end{aligned}$$

$$\geq -\frac{\tau|q|}{4}||\delta_x^{\langle 2\rangle}V||^2 - C_0\frac{r}{4|q|}\tau||V||^2, \tag{3.4}$$

where $((V)^3)_j = (V_j)^3$, $(\delta_x^+ V)_j = \delta_x^+ V_j$, $(\delta_x^{\langle 2\rangle}V)_j = \delta_x^{\langle 2\rangle}V_j$ and $C_0 = (2\alpha+1)^4[C(U^n)]^4$, we have

$$\begin{aligned}
(\mathcal{F}(V),V)_h =&(V,V)_h - (U^n,V)_h + \beta h^2(\delta_x^{\langle 2\rangle}V,V)_h - \beta h^2(\delta_x^{\langle 2\rangle}U^n,V)_h - \frac{\tau}{2}q(\delta_x^{\langle 4\rangle}V,V)_h \\
&- \frac{\tau}{2}(\delta_x^{\langle 2\rangle}[pV_j + 4r\alpha(1-\alpha)(V_j)^3 + rV_j(2\alpha V_j - U_j^n)^2],V)_h \\
=&||V||^2 - (U^n,V)_h - \beta h^2||\delta_x^+V||^2 \\
&+ \frac{\beta}{2}h^2(\delta_x^+U^n,\delta_x^+V)_h + \frac{\beta}{2}h^2(\delta_x^-U^n,\delta_x^-V)_h - \frac{\tau}{2}q||\delta_x^{\langle 2\rangle}V||^2 \\
&+ 2\tau r\alpha(1-\alpha)h\sum_{j=0}^{J-1}(\delta_x^+V_j)^2[(V_{j+1})^2 + V_{j+1}V_j^n + (V_j^n)^2] \\
&- \frac{\tau}{2}(pV_j + rV_j(2\alpha V_j - U_j^n)^2, \delta_x^{\langle 2\rangle}V)_h \\
\geq&||V||^2 - \frac{1}{2}||V||^2 - \frac{1}{2}||U^n||^2 - \frac{5|\beta|}{4}h^2||\delta_x^+V||^2 - |\beta|h^2||\delta_x^+U^n||^2 - \frac{\tau}{2}q||\delta_x^{\langle 2\rangle}V||^2 \\
&- \frac{\tau|q|}{4}||\delta_x^{\langle 2\rangle}V||^2 - C_0\frac{r}{4|q|}\tau||V||^2 - \frac{\tau|q|}{4}||\delta_x^{\langle 2\rangle}V||^2 - \frac{p^2}{4|q|}\tau||V||^2 \\
\geq&\frac{1}{2}||V||^2 - \frac{1}{2}||U^n||^2 - 5|\beta|h^2||V||^2 - 4|\beta|h^2||U^2||^2 - C_1\tau||V||^2 \\
\geq&\Big(\frac{1}{2} - 5|\beta| - C_1\tau\Big)||V||^2 - \Big(\frac{1}{2} + 4|\beta|\Big)||U^n||^2,
\end{aligned} \tag{3.5}$$

where $C_1 = C_0\frac{r}{4|q|} + \frac{p^2}{4|q|}$. Taking $\beta \leq \frac{1}{12}$, $\tau \leq \frac{1}{24C_1}$ and $C(U^n) > \frac{10}{\sqrt{L}}||U^n||$, we obtain $(\mathcal{F}(V),V)_h \geq 0$ for $||V|| \geq 5||U^n||$. The existence of $U^{n+\frac{1}{2}}$ satisfying $||U^{n+\frac{1}{2}}||_\infty \leq C(U^n)$ follows from Lemma 3.1 and consequently the existence of U^{n+1} is obtained.

Using the similar proof of Theorem 4.1 in the next section, we can obtain the uniqueness of the numerical solution. □

4 Error Estimate

The purpose of this section is to discuss the convergence of the numerical solutions. We denote $u_j^n = u(x_j,t_n)$ and define the error as

$$e_j^n \triangleq u_j^n - U_j^n, \quad j = -1,0,1\cdots,J,J+1, \tag{4.1}$$

where $u(x_j,t_n)$ is the solution to the Cahn-Hilliard equation at the point (x_j,t_n). We define an extension of u by

$$u(x,t) = \begin{cases} u(x-2mL,t), & \text{for } 2mL \leq x \leq (2m+1)L, \\ u(2mL-x,t), & \text{for } (2m-1)L \leq x \leq 2mL, \end{cases}$$

where $m \in \mathcal{Z}$. Define the truncation error of the scheme (2.1)–(2.3) as follows

$$r_j^{n+1/2} = \delta_t^+u_j^n + \beta h^2\delta_x^{\langle 2\rangle}\delta_t^+u_j^n - \delta_x^{\langle 2\rangle}v_j^{n+1/2}, \tag{4.2}$$

$$\eta_j^{n+1/2} = v_j^n - \varphi(u_j^{n+1},u_j^n,\alpha) - \frac{q}{2}\delta_x^{\langle 2\rangle}(u_j^{n+1} + u_j^n), \tag{4.3}$$

where

$$v_j^{n+\frac{1}{2}} = \Big\{pu + ru^3 + q\frac{\partial^2 u}{\partial x^2}\Big\}\Big|_{(x,t)=(x_j,t_{n+1/2})}, \qquad \text{for} \;\; j=0,1,\cdots,J.$$

Using Taylor expansion, we can obtain

$$r_j^{n+1/2} = O(h^2+\tau^2), \qquad \eta_j^{n+1/2} = O(h^4+\tau). \tag{4.4}$$

Especially for $\beta = \frac{1}{12}$ and $\alpha = \frac{1}{2}$, there is

$$r_j^{n+1/2} = O(h^4+\tau^2), \qquad \eta_j^{n+1/2} = O(h^4+\tau^2). \tag{4.5}$$

Lemma 4.1.

$$\begin{aligned}&\frac{1}{\tau}\big\{(||e^{n+1}||^2 - \beta h^2||\delta_x^+ e^{n+1}||^2) - (||e^n||^2 - \beta h^2||\delta_x^+ e^n||^2)\big\}\\ \le&\frac{1}{2}\big\{||e^{n+1}||^2 + ||e^n||^2\big\} + \frac{1}{q}||\varphi(u^{n+1},u^n,\alpha) - \varphi(U^{n+1},U^n,\alpha)||^2 + ||r^n||^2 + \frac{1}{q}||\eta^n||^2,\end{aligned} \tag{4.6}$$

where

$$\phi_j^{n+\frac{1}{2}} = \{pu + ru^3\}|_{(x,t)=(x_j,t_{n+1/2})}, \qquad \text{for} \;\; j=0,1,\cdots,J.$$

Proof. Denote

$$V_j^{n+\frac{1}{2}} = \varphi(U_j^{n+1},U_j^n,\alpha) + \frac{q}{2}\delta_x^{\langle 2\rangle}(U_j^{n+1}+U_j^n), \tag{4.7}$$

then (2.1) can be written as

$$\delta_t^+ U_j^n + \beta h^2 \delta_x^{\langle 2\rangle}\delta_t^+ U_j^n = \delta_x^{\langle 2\rangle} V_j^{n+\frac{1}{2}}. \tag{4.8}$$

Denote

$$\xi_j^n \triangleq v_j^n - V_j^n, \qquad j=-1,0,1\cdots,J,J+1, \tag{4.9}$$

then it follows from (4.7) and (4.1)–(4.3) that

$$\delta_t^+ e_j^n + \beta h^2 \delta_x^{\langle 2\rangle}\delta_t^+ e_j^n = \delta_x^{\langle 2\rangle}\xi_j^{n+1/2} + r_j^{n+1/2}, \tag{4.10}$$

$$\xi_j^{n+\frac{1}{2}} = \widetilde{\varphi}^{n+1/2} + \frac{q}{2}\delta_x^{\langle 2\rangle}(e_j^{n+1}+e_j^n) + \eta_j^{n+1/2}. \tag{4.11}$$

where $\widetilde{\varphi}^{n+1/2} = \varphi(u_j^{n+1},u_j^n,\alpha) - \varphi(U_j^{n+1},U_j^n,\alpha)$.

Taking the inner product of (4.10) and (4.11) with $\frac{1}{2}(e^{n+1}+e^n)$ and $\frac{1}{q}\xi^{n+\frac{1}{2}}$ respectively, then adding the results together, we obtain

$$\begin{aligned}&\frac{1}{2\tau}\big\{(||e^{n+1}||^2 - \beta h^2||\delta_x^+ e^{n+1}||^2) - (||e^n||^2 - \beta h^2||\delta_x^+ e^n||^2)\big\} - \frac{1}{q}||\xi^{n+\frac{1}{2}}||^2\\ =&\frac{1}{2}(\delta_x^{\langle 2\rangle}\xi^{n+1/2}, e^{n+1}+e^n)_h + \frac{1}{2}(r^{n+1/2}, e^{n+1}+e^n)_h - \frac{1}{q}(\widetilde{\varphi}^{n+1/2},\xi^{n+\frac{1}{2}})_h\\ &-\frac{1}{2}(\delta_x^{\langle 2\rangle}e^{n+1}+e^n,\xi^{n+1/2})_h - \frac{1}{q}(\eta^{n+1/2},\xi^{n+1/2})_h\\ =&-\frac{1}{q}(\widetilde{\varphi}^{n+1/2},\xi^{n+\frac{1}{2}})_h + \frac{1}{2}(r^{n+1/2}, e^{n+1}+e^n)_h - \frac{1}{q}(\eta^{n+1/2},\xi^{n+1/2})_h\\ \le&\frac{1}{2q}||\widetilde{\varphi}^{n+1/2}||^2 + \frac{1}{2q}||\xi^{n+\frac{1}{2}}||^2 + \frac{1}{2}||r^{n+1/2}||^2\\ &+\frac{1}{4}(||e^{n+1}||^2 + ||e^n||^2) + \frac{1}{2q}||\eta^{n+1/2}||^2 + \frac{1}{2q}||\xi^{n+1/2}||^2.\end{aligned} \tag{4.12}$$

Hence the inequality (4.6) is obtain from (4.12). □

Lemma 4.2.

$$\frac{1}{q}||\varphi(u^{n+1},u^n,\alpha)-\varphi(U^{n+1},U^n,\alpha)||^2 \le C_3\big(||e^{n+1}||^2+||e^n||^2\big), \tag{4.13}$$

where

$$C_3=-\frac{1}{2q}(-p+2r(C_2)^2)^2, \qquad C_2=\max_{0\le n\le N}\{||U^n||_\infty,||u^n||_{L^\infty},\}.$$

Proof.

$$\begin{aligned}&\varphi(u_j^{n+1},u_j^n,\alpha)-\varphi(U_j^{n+1},U_j^n,\alpha)\\ =&p\frac{u_j^{n+1}+u_j^n}{2}+r\frac{u_j^{n+1}+u_j^n}{2}\big[\alpha(u_j^{n+1})^2+(1-\alpha)(u_j^n)^2\big]\\ &-p\frac{U_j^{n+1}+U_j^n}{2}-r\frac{U_j^{n+1}+U_j^n}{2}\big[\alpha(U_j^{n+1})^2+(1-\alpha)(U_j^n)^2\big]\\ =&p\frac{e_j^{n+1}+e_j^n}{2}+r\frac{e_j^{n+1}+e_j^n}{2}\big[\alpha(u_j^{n+1})^2+(1-\alpha)(u_j^n)^2\big]\\ &+r\frac{U_j^{n+1}+U_j^n}{2}\big[\alpha(u_j^{n+1}+U_j^{n+1})e_j^{n+1}+(1-\alpha)(u_j^n+U_j^n)e_j^n\big].\end{aligned} \tag{4.14}$$

It follows from (4.13) that

$$|\varphi(u_j^{n+1},u_j^n,\alpha)-\varphi(U_j^{n+1},U_j^n,\alpha)|\le(-p+2r(C_2)^2)\left|\frac{e_j^{n+1}+e_j^n}{2}\right|. \tag{4.15}$$

Hence the inequality (4.13) is obtain from (4.15). □

Theorem 4.1. *If (1.1) has a solution such that $u(x,t)\in C^{6,3}([0,L]\times[0,T])$, $\beta\le\frac{1}{12}$ and τ is sufficiently small, then the solution of the difference scheme (2.1) converges to the solution of (1.1) in the sense of discrete L_2-norm, and the convergence rate is $O(h^2+\tau^2)$ for $\alpha=1/2$ and $O(h^2+\tau)$ for $\alpha\ne1/2$.*

Proof. It follows from Lemma 4.1 and Lemma 4.2 that

$$\begin{aligned}&\frac{1}{\tau}\big\{(||e^{n+1}||^2-\beta h^2||\delta_x^+e^{n+1}||^2)-(||e^n||^2-\beta h^2||\delta_x^+e^n||^2)\big\}\\ \le&\Big(\frac{1}{2}+C_3\Big)(||e^{n+1}||^2+||e^n||^2)+||r^n||^2+\frac{1}{q}||\eta^n||^2.\end{aligned} \tag{4.16}$$

Denote

$$B^{n+1}=||e^{n+1}||^2-\beta h^2||\delta_x^+e^{n+1}||^2. \tag{4.17}$$

Since

$$||\delta_x^+e^{n+1}||^2\le\frac{4}{h^2}||e^{n+1}||^2,$$

we obtain that $B^{n+1}\ge\frac{2}{3}||e^{n+1}||^2$ if $\beta\le\frac{1}{12}$. Hence, it follows from (4.16) we obtain

$$B^{n+1}-B^n\le\Big(\frac{3}{4}+\frac{3}{2}C_3\Big)\tau(B^{n+1}+B^n)+\tau\Big(||r^n||^2+\frac{1}{q}||\eta^n||^2\Big). \tag{4.18}$$

Using Gronwall inequality, we obtain

$$\max_{0\le n\le N} B^n \le \Big(B^0 + \tau\sum_{k=1}^{N}\Big(||r^n||^2 + \frac{1}{q}||\eta^n||^2\Big)\Big)e^{(3+6C_3)T}$$
$$= e^{(3+6C_3)T}\sum_{k=1}^{N}\Big(||r^n||^2 + \frac{1}{q}||\eta^n||^2\Big)\tau, \qquad \text{for } \ \tau \le \frac{N+1}{(3+6C_3)N}. \tag{4.19}$$

It follows from (4.2), (4.3), (4.17) and (4.19) that Theorem 4.1 is hold. □

5 Iterative Algorithm

Computing the numerical solutions $U^1, U^2, \cdots, U^n$ satisfying (2.1)–(2.3) requires solving at each level a nonlinear system with $J+1$ unknowns, so it is necessary to construct an effective iterative algorithm to solve it in implementation. In this section, we will construct two iterative algorithms based on which two predication-correction schemes will be proposed. The first iterative algorithm is as follows:

$$\delta_t^- U_j^{n+1(s+1)} + \beta h^2 \delta_x^{\langle 2\rangle}\delta_t^- U_j^{n+1(s+1)} - \frac{q}{2}\delta_x^{\langle 4\rangle}(U_j^{n+1(s+1)} + U_j^n)$$
$$= \delta_x^{\langle 2\rangle}\varphi(U_j^{n+1(s)}, U_j^n, \alpha), \qquad 0\le j\le J, \ \ n = 0,1,2,\cdots,N-1, \tag{5.1}$$
$$U_j^0 = u_0(x_j), \qquad j = -2,-1,0,\cdots,J,J+1,J+2, \tag{5.2}$$
$$\delta_x^{\langle 1\rangle}U_0^n = \delta_x^{\langle 1\rangle}U_J^n = \delta_x^{\langle 3\rangle}U_0^n = \delta_x^{\langle 3\rangle}U_J^n = 0, \qquad n = 0,1,2,\cdots,N, \tag{5.3}$$

with

$$U_j^{n+1(0)} = \begin{cases} U_j^n, & \text{for } \ n = 0,\\ 2U_j^n - U_j^{n-1}, & \text{for } \ n\ge 1,\end{cases}$$

where

$$\delta_t^- U_j^{n+1(s+1)} = \frac{U_j^{n+1(s+1)} - U_j^n}{\tau}.$$

Theorem 5.1. *Suppose that the solution $u(x,t)$ of (1.1)–(1.4) belongs to $C^{6,3}([0,L]\times[0,T])$, $\beta \le \frac{1}{12}$ and τ, h are sufficiently small, then the solution of the iterative method (5.1)–(5.3) converges to the numerical solution of the scheme (2.1)–(2.3).*

Proof. Denote $\theta_j^{n+1(s)} = U_j^{n+1(s)} - U_j^{n+1}$, $n = 0,1,2,\cdots,N-1$; $s = 0,1,2,\cdots$. Then subtracting (2.1) from (5.1), we obtain

$$\frac{1}{\tau}\theta_j^{n+1(s+1)} + \frac{1}{\tau}\beta h^2\delta_x^{\langle 2\rangle}\theta_j^{n+1(s+1)} - \frac{q}{2}\delta_x^{\langle 4\rangle}\theta_j^{n+1(s+1)}$$
$$= \delta_x^{\langle 2\rangle}\varphi(U_j^{n+1(s)}, U_j^n, \alpha) - \delta_x^{\langle 2\rangle}\varphi(U_j^{n+1}, U_j^n, \alpha)$$
$$= \frac{p}{2}\delta_x^{\langle 2\rangle}\theta_j^{n+1(s)} + \frac{r}{2}\delta_x^{\langle 2\rangle}\Big\{\theta_j^{n+1(s)}\big[\alpha(U_j^{n+1})^2 + (1-\alpha)(U_j^n)^2\big]$$
$$+ \frac{r}{2}\alpha(U_j^{n+1(s)} + U_j^n)(U_j^{n+1(s)} + U_j^{n+1})\theta_j^{n+1(s)}\Big\}. \tag{5.4}$$

Noticing, when $n = 0$,

$$\theta_j^{1(0)} = U_j^{1(0)} - U_j^1 = U_j^0 - U_j^1$$
$$= [(U_j^0 - u_j^0) + (u_j^0 - u_j^1) + (u_j^1 - U_j^1)]$$
$$= 0 + O(\tau) + O(h^2+\tau) = O(h^2+\tau), \tag{5.5}$$

and when $n \geq 1$,

$$\begin{aligned}\theta_j^{n+1(0)} =& U_j^{n+1(0)} - U_j^{n+1} = 2U_j^n - U_j^{n-1} - U_j^{n+1}\\ =& \big[2(U_j^n - u_j^n) + (u_j^{n-1} - U_j^{n-1}) + (u_j^{n+1} - U_j^{n+1}) + (2u_j^n - u_j^{n-1} - u_j^{n+1})\big]\\ =& O(h^2+\tau) + O(h^2+\tau) + O(h^2+\tau) = O(h^2+\tau). \end{aligned} \tag{5.6}$$

If taking $\alpha = 1/2$, we have

$$\theta_j^{n+1(0)} = O(h^2+\tau^2), \qquad \text{for} \ \ n \geq 1. \tag{5.7}$$

It follows from (5.5)–(5.7) that

$$||\theta^{n+1(0)}||_\infty \leq \widetilde{C}_0(h^2+\tau), \qquad \text{for}\, 1/2 < \alpha \leq 1 \tag{5.8}$$

and

$$||\theta^{n+1(0)}||_\infty \leq \begin{cases} \widetilde{C}_0(h^2+\tau), & \text{for} \ \ n=0,\\ \widetilde{C}_0(h^2+\tau^2), & \text{for} \ \ n \geq 1, \end{cases} \tag{5.9}$$

if we take $\alpha = 1/2$. In next study, we just only discuss the case $\alpha = 1/2$, the case $1/2 < \alpha \leq 1$ can be discussed by the similar method.

Now, suppose

$$||\theta^{n+1(s)}|| \leq \widetilde{C}_s(h^2+\tau^2), \qquad n = 1,2,\cdots,N-1; \ \ s = 0,1,\cdots. \tag{5.10}$$

It follows from *Sobolev* estimate, we obtain

$$\begin{aligned}||\theta^{n+1(s)}||_\infty \leq& C_5\sqrt{||\theta^{n+1(s)}||}\sqrt{||\delta_x^+\theta^{n+1(s)}|| + ||\theta^{n+1(s)}||}\\ \leq& C_5\sqrt{||\theta^{n+1(s)}||}\sqrt{\frac{2}{h}||\theta^{n+1(s)}|| + ||\theta^{n+1(s)}||}\\ \leq& C_5\sqrt{1+\frac{2}{h}}||\theta^{n+1(s)}|| \leq C_5\widetilde{C}_s(1+2h^{-1/2})(h^2+\tau^2)\\ & n = 1,2,\cdots,N-1; \qquad s = 0,1,2\cdots. \end{aligned} \tag{5.11}$$

Thus

$$\begin{aligned}||U^{n+1(s)}||_\infty \leq& ||U^{n+1}||_\infty + ||\theta^{n+1(s)}||_\infty \leq C_4 + C_5\widetilde{C}_s(1+2h^{-1/2})(h^2+\tau^2),\\ & n=1,2,\cdots,N-1; \ \ s=0,1,2\cdots, \end{aligned} \tag{5.12}$$

where

$$C_4 = 2\sqrt{\frac{\max(\frac{1}{L},L)}{\min(\lambda,-\frac{q}{2})}\Big\{\frac{(p-2\lambda)^2}{4r}L + h\sum_{j=0}^{J}{}''G_d(U_j^0)\Big\}}.$$

Computing the inner product of (5.4) with $\theta^{n+1(s+1)}$,

$$\begin{aligned} & ||\theta^{n+1(s+1)}||^2 - \beta h^2||\delta_x^+\theta^{n+1(s+1)}||^2 - \frac{q}{2}\tau||\delta_x^{\langle 2\rangle}\theta^{n+1(s+1)}||^2\\ =& \frac{p}{2}\tau(\theta^{n+1(s)},\delta_x^{\langle 2\rangle}\theta^{n+1(s+1)})_h + \frac{r}{2}\tau([\alpha(U^{n+1})^2 + (1-\alpha)(U^n)^2]\theta^{n+1(s)},\delta_x^{\langle 2\rangle}\theta^{n+1(s+1)})_h\\ &+ \frac{r}{2}\tau(\alpha(U^{n+1(s)}+U^n)(U^{n+1(s)}+U^{n+1})\theta^{n+1(s)}, +\delta_x^{\langle 2\rangle}\theta^{n+1(s+1)})_h\\ \leq& -\frac{q}{4}\tau||\delta_x^{\langle 2\rangle}\theta^{n+1(s+1)}||^2 - \frac{p^2}{4q}\tau||\theta^{n+1(s)}||^2 - \frac{q}{8}\tau||\delta_x^{\langle 2\rangle}\theta^{n+1(s+1)}||^2 - \frac{r^2}{2q}(C_4)^4\tau||\theta^{n+1(s)}||^2\\ &- \frac{q}{8}\tau||\delta_x^{\langle 2\rangle}\theta^{n+1(s+1)}||^2 - \frac{r^2}{2q}[2C_4 + C_5\widetilde{C}_s(1+2h^{-1/2})(h^2+\tau^2)]^4\tau||\theta^{n+1(s)}||^2. \end{aligned} \tag{5.13}$$

When $\beta \leq \frac{1}{12}$, we obtain from (5.13) that

$$||\theta^{n+1(s+1)}||^2 \leq C_6\tau||\theta^{n+1(s)}||^2, \tag{5.14}$$

where

$$C_6 = -\frac{3p^2}{8q} - \frac{3r^2}{4q}\big[(C_4)^4 + (2C_4 + C_5\widetilde{C}_s(1+2h^{-1/2})(h^2+\tau^2))^4\big].$$

Thus for sufficiently small τ and h such that $C_6\tau < 1$, then the solution of the iterative algorithm (5.1)–(5.3) converges to the solution of the nonlinear scheme (2.1)–(2.3). The convergence in the case of $n = 0$ can be proved by the similar method.

In the case of $s = 1$, we obtain from the iterative algorithm (5.1)–(5.3) the linearized implicit prediction-correction scheme

$$\begin{aligned}
&\delta_t^-\widetilde{U}_j^{n+1} + \beta h^2\delta_x^{\langle 2\rangle}\delta_t^-\widetilde{U}_j^{n+1} - \frac{q}{2}\delta_x^{\langle 4\rangle}(\widetilde{U}_j^{n+1} + U_j^n)\\
=&\delta_x^{\langle 2\rangle}\varphi(U_j^n, U_j^n, \alpha), \qquad 0 \leq j \leq J, \quad n = 0, \qquad (5.15)\\
&\delta_t^-\widetilde{U}_j^{n+1} + \beta h^2\delta_x^{\langle 2\rangle}\delta_t^-\widetilde{U}_j^{n+1} - \frac{q}{2}\delta_x^{\langle 4\rangle}(\widetilde{U}_j^{n+1} + U_j^n)\\
=&\delta_x^{\langle 2\rangle}\varphi(2U_j^n - U_j^{n-1}, U_j^n, \alpha), \qquad 0 \leq j \leq J, \quad n = 1, 2, \cdots, N, \qquad (5.16)\\
&\delta_t^- U_j^{n+1} + \beta h^2\delta_x^{\langle 2\rangle}\delta_t^- U_j^{n+1} - \frac{q}{2}\delta_x^{\langle 4\rangle}(U_j^{n+1} + U_j^n)\\
=&\delta_x^{\langle 2\rangle}\varphi(\widetilde{U}_j^n, U_j^n, \alpha), \qquad 0 \leq j \leq J, \quad n = 0, 1, 2, \cdots, N, \qquad (5.17)\\
&U_j^0 = u_0(x_j), \qquad j = -2, -1, 0, \cdots, J, J+1, J+2, \qquad (5.18)\\
&\delta_x^{\langle 1\rangle}U_0^n = \delta_x^{\langle 1\rangle}U_J^n = \delta_x^{\langle 3\rangle}U_0^n = \delta_x^{\langle 3\rangle}U_J^n = 0, \qquad n = 0, 1, 2, \cdots, N, \qquad (5.19)
\end{aligned}$$

where

$$\delta_t^-\widetilde{U}_j^{n+1} = \frac{\widetilde{U}_j^{n+1} - U_j^n}{\tau}.$$

Now, we give the second iterative algorithm as follows:

$$\begin{aligned}
&\delta_t^- U_j^{n+1(s+1)} + \beta h^2\delta_x^{\langle 2\rangle}\delta_t^- U_j^{n+1(s+1)} - \frac{q}{2}\delta_x^{\langle 4\rangle}(U_j^{n+1(s+1)} + U_j^n)\\
=&\delta_x^{\langle 2\rangle}\Big(\frac{p}{2}(U_j^{n+1(s+1)} + U_j^n) + \frac{r}{2}(U_j^{n+1(s+1)} + U_j^n)(\alpha(U_j^{n+1(s)})^2 + (1-\alpha)(U_j^n)^2)\Big),\\
&0 \leq j \leq J, \qquad n = 0, 1, 2, \cdots, N-1, \qquad (5.20)\\
&U_j^0 = u_0(x_j), \qquad j = -2, -1, 0, \cdots, J, J+1, J+2, \qquad (5.21)\\
&\delta_x^{\langle 1\rangle}U_0^n = \delta_x^{\langle 1\rangle}U_J^n = \delta_x^{\langle 3\rangle}U_0^n = \delta_x^{\langle 3\rangle}U_J^n = 0, \qquad n = 0, 1, 2, \cdots, N, \qquad (5.22)
\end{aligned}$$

with

$$U_j^{n+1(0)} = \begin{cases} U_j^n, & \text{for } n = 0,\\ 2U_j^n - U_j^{n-1}, & \text{for } n \geq 1. \end{cases}$$

□

By the similar proof, we can obtain the following theorem:

Theorem 5.2. *Suppose that the solution $u(x,t)$ of (1.1)–(1.4) belongs to $C^{6,3}([0,L]\times[0,T])$, $\beta \leq \frac{1}{12}$ and τ, h are sufficiently small, then the solution of the iterative method (5.20)–(5.22) converges to the numerical solution of the scheme (2.1)–(2.3).*

Similarly, in the case of $s = 1$, we obtain from the iterative algorithm (5.20)–(5.22) the linearized implicit prediction-correction scheme:

$$\begin{aligned}
&\delta_t^- \widetilde{U}_j^{n+1} + \beta h^2 \delta_x^{\langle 2\rangle} \delta_t^- \widetilde{U}_j^{n+1} - \frac{q}{2}\delta_x^{\langle 4\rangle}(\widetilde{U}_j^{n+1} + U_j^n)\\
=&\delta_x^{\langle 2\rangle}\Big(\frac{p}{2}(\widetilde{U}_j^{n+1} + U_j^n) + \frac{r}{2}(\widetilde{U}_j^{n+1} + U_j^n)(U_j^n)^2\Big), \qquad 0 \le j \le J, \quad n = 0, && (5.23)\\
&\delta_t^- \widetilde{U}_j^{n+1} + \beta h^2 \delta_x^{\langle 2\rangle} \delta_t^- \widetilde{U}_j^{n+1} - \frac{q}{2}\delta_x^{\langle 4\rangle}(\widetilde{U}_j^{n+1} + U_j^n)\\
=&\delta_x^{\langle 2\rangle}\Big(\frac{p}{2}(\widetilde{U}_j^{n+1} + U_j^n) + \frac{r}{2}(\widetilde{U}_j^{n+1} + U_j^n)(\alpha(2U_j^n - U_j^{n-1})^2 + (1-\alpha)(U_j^n)^2)\Big),\\
&0 \le j \le J, \qquad n = 1, 2, \cdots, N-1, && (5.24)\\
&\delta_t^- U_j^{n+1} + \beta h^2 \delta_x^{\langle 2\rangle} \delta_t^- U_j^{n+1} - \frac{q}{2}\delta_x^{\langle 4\rangle}(U_j^{n+1} + U_j^n)\\
=&\delta_x^{\langle 2\rangle}\Big(\frac{p}{2}(U_j^{n+1} + U_j^n) + \frac{r}{2}(U_j^{n+1} + U_j^n)(\alpha(\widetilde{U}_j^{n+1})^2 + (1-\alpha)(U_j^n)^2)\Big),\\
&0 \le j \le J, \qquad n = 0, 1, 2, \cdots, N-1, && (5.25)\\
&U_j^0 = u_0(x_j), \qquad j = -2, -1, 0, \cdots, J, J+1, J+2, && (5.26)\\
&\delta_x^{\langle 1\rangle} U_0^n = \delta_x^{\langle 1\rangle} U_J^n = \delta_x^{\langle 3\rangle} U_0^n = \delta_x^{\langle 3\rangle} U_J^n = 0, \qquad n = 0, 1, 2, \cdots, N. && (5.27)
\end{aligned}$$

Obviously, the above implicit scheme is linearized in the practical computation, i.e. at each time step, we just only use Thomas algorithm to solve two five-diagonal linear systems. Thus, the prediction-correction scheme can be expected to be more efficient in the practical computation.

References

[1] Akrives, G.D. Finite difference discretization of the cubic Schrödinger equation. *IMA J. Numer Anal.*, 13: 115–124 (1993)

[2] Akrives, G.D., Dogalis, V.A., karakashina, O.A. On fully discrete Galerkin methods of second-order temporal accuracy for the nonlinear Schrödinger equation. *Numer. Math.*, 59: 31–53 (1991)

[3] Cahn, J.W., Hilliard, J.E. Free energy of non-uniform system, I. Interfacial free erengy. *J. Chem. Phys.*, 28: 258–267 (1958)

[4] Ceniceros, H.D., Roma, A.M. A nonstiff, adaptive mesh refinement-based method for the Cahn-Hilliard equation. *J. Comput. Phys.*, 225: 1849–1862 (2007)

[5] Cheng, M., Warren, J.A. An efficient algorithm for solving the phase field crystal model. *J. Comput. Phys.*, 227: 6241–6248 (2008)

[6] Choo, S.M., Chung, S.K. Conservative nonlinear difference scheme for the Cahn-Hilliard equation. *Comput. Math. Appl.*, 36: 31–39 (1998)

[7] Choo, S.M., Chung, S.K., Kim, K.I. Conservative nonlinear difference scheme for the Cahn-Hilliard equation: II. *Comput. Math. Appl.*, 39: 229–243 (2000)

[8] Choo, S.M., Chung, S.K., Lee, Y.J. A conservative difference scheme for the viscous Cahn-Hilliard equation with a nonconstant gradient energy coefficient. *Applied Numerical Mathematics*, 51: 207–219 (2004)

[9] Dean, E.J., Glowinski, R., Trevas, D.A. An approximate factorization/least squares solution method for a mixed finite element approximation of the Cahn-Hilliard equation. *Japan J. Indust. Appl. Math.*, 13: 495–517 (1996)

[10] Dehghan, M., Mirzae, D. A numerical method based on the boundary integral equation and dual reciprocity methods for one-dimensional Cahn-Hilliard equation. *Engineering Analysis with Boundary Elements*, 33: 522–528 (2009)

[11] Du, Q., Nicolaides, R.A. Numerical analysis of a continuum Model of phase transition. *SIAM J. Numer. Anal.*, 28: 1310–1322 (1991)

[12] Elliott, C.M., French, D.A. Numerical studies of the Cahn-Hilliard equation for phase separation. *IMA J. Appl. Math.*, 38: 97–128 (1987)

[13] Elliott, C.M., French, D.A. A nonconforming finite-element method for the two-dimensional Cahn-Hilliard equation. *SIAM J. Numer. Anal.*, 26: 884–903 (1989)

[14] Elliott, C.M., French, D.A., Milner, F.A. A second-order splliting method for the Cahn-Hilliard equation. *Numer. Math.*, 54: 575–590 (1989)

[15] Elliott, C.M., Larsson, S. Error estimates with smooth and nonsmooth data for a finite element method for the Cahn-Hilliard equation. *Math. Comp.*, 58: 603–630 (1992)

[16] Elliott, C.M., Zheng, S. On the Cahn-Hilliard equation. *Arch. Rat. Mech. Anal.*, 96: 339–357 (1986)
[17] Eyre, D.J. http://www.math.utah.edu/eyre/research/methods/stable.ps
[18] French, D.A., Jensen, S. Long-time behaviour of arbitrary order continuous time Galerkin schemes for some onedimensional phase transition problems. *IMA J. Numer. Anal.*, 14: 421–442 (1994)
[19] Furihata, D. A stable and conservative finite difference scheme for the Cahn-Hilliard equation. *Numer. Math.*, 87: 675–699 (2001)
[20] Furihata, D., Onda, T., Mori, M. A finite difference scheme for the Cahn-Hilliard equation based on a Lyapunov functional. *GAKUTO Internat, Ser. Math. Sci. Appl.*, 2: 347–358 (1993)
[21] He, Y., Liu, Y. Stability and convergence of the spectral Galerkin method for the Cahn-Hilliard equation. *Numer. Meth. PDEs.*, 24: 1485–1500 (2008)
[22] He, L., Liu, Y. A class of stable spectral methods for the Cahn-Hilliard equation. *J. Comput. Phys.*, 228: 5101–5110 (2009)
[23] He, Y., Liu, Y., Tang, Tao. On large time-stepping methods for the Cahn-Hilliard equation. *Appl. Numer. Math.*, 57: 616–628 (2007)
[24] Jabbari, E., Peppas, N.A. A model for interdiffusion at interfaces of polymers with dissimilar physical properties. *Polymer*, 36: 575–586 (1995)
[25] Novick-Cohen, A., Segel, L.A. Nonlinear aspects of the Cahn-Hilliard equation. *Physica D.*, 10: 277–298 (1994)
[26] Puri, S., Binder, K. Phenomenological theory for the formation of interfaces via the interdiffusion of layers. *Phys. Rev. B.*, 44: 9735–9738 (1991)
[27] Sun, Z. A second-order accurate linearized difference scheme for the two-dimensional Cahn-Hilliard equation. *Math. Comp.*, 64: 1463–1471 (1995)
[28] Tanaka, H., Nishi, T. Direct Detemination of the Probability Distribution Function of Concentration in Polymer Mixtures Undergoing Phase Separation. *Phys. Rev. Lett.*, 59: 692–695 (1987)
[29] Vollmayr-Lee, B.P., Rutenberg, A.D. Fast and accurate coarsening simulation with an unconditionally stable time step. *Phys. Rev. E.*, 68: 066703 (2003)
[30] Wang, S., Shi, Q. Interdiffusion in binary polymer mixtures. *Macromolecules*, 26: 1091–1096 (1993)
[31] Xia, Y., Xu, Y., Shu, C. Local discontinuous Galerkin methods for the Cahn C Hilliard type equations. *J. Comput. Phys.*, 227: 472–491 (2007)
[32] Xu, C., Tang, T. Stability analysis of large time-stepping methods for epitaxial growth models. *SIAM J. Numer. Anal.*, 44: 1759–1779 (2006)
[33] Ye, X. The Fourier collocation method for the Cahn-Hilliard equation. *Comput. Math. Appl.*, 44: 213–229 (2002)
[34] Ye, X. The Fourier spectral method for the Cahn-Hilliard equation. *Appl. Math. Comput.*, 171: 345–357 (2005)
[35] Ye, X. Cheng, Xiaoliang. Legendre spectral approximation for the Cahn-Hilliard equation. *Math. Numer. Sinica*, 25: 157–170 (2003)
[36] Yin, J. On the existence of nonnegative continuous solutions of the Cahn.Hilliard equation. *J. Differential Equations*, 97: 310–327 (1992)
[37] Zhou, Y. Application of Discrete Functional Analysis to the Finite Difference Methods. International Academic Publishers, Beijing, 1990