Acta Mathematicae Applicatae Sinica, English Series
Vol. 31, No. 4 (2015) 977–990
DOI: 10.1007/s10255-015-0517-x
http://www.ApplMath.com.cn & www.SpringerLink.com

Acta Mathematicae Applicatae Sinica, English Series

The Applications of Vague Soft Sets and Generalized Vague Soft Sets

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Abstract The problem of decision making in an imprecise environment has found paramount importance in recent years. In this paper, we define vague soft relation and similarity measure of vague soft sets. Using these definitions, some novel methods of object recognition from an imprecise multiobserver data has been presented. Moreover, we introduce the notion of generalized vague soft sets and study some of its properties. The similarity measure of generalized vague soft sets is also presented and an application of this measure in decision making problems has been shown.

Keywords soft sets; vague sets; vague soft sets; vague soft relation; similarity measure; generalized vague soft sets

2000 MR Subject Classification 03E72; 28E10

1 Introduction

Dealing with uncertainties is a major problem in many areas such as economics, engineering, environmental science, medical science and social science. To solve these problems, classical mathematical tools may not be successfully used. While a wide range of theories such as probability theory, fuzzy sets theory[30], intuitionistic fuzzy sets theory[3], rough sets theory[24], vague sets theory[13] and the interval mathematics[14] are well know and often useful mathematical approaches to modeling vagueness. However, all of these theories have their own difficulties which have been pointed out in [23]. Molodtsov suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov[23] introduced the concept of soft sets as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches.

Up to the present, research on soft sets has been very active and many important results have been achieved in theoretical aspect. Maji et al. introduced several algebraic operations in soft sets theory and extended crisp soft sets to fuzzy soft sets[20,21]. Aktaş and Çağman compared soft sets to the related concepts of fuzzy sets and rough sets. They also defined the notion of soft groups and derived some related properties[1]. Feng et al. studied the combination of soft sets with other soft computing models[9,11]. Chen and Tsang studied the parameterization reduction of soft sets and its application[6]. Jun and Park proposed the notion of soft ideals and idealistic soft BCK/BCI-algebras, and constructed several examples[15]. Ali and Feng et al. corrected some errors of former studies and proposed some new operations on soft sets[2]. The

Manuscript received 22 August 2011. Revised 22 September 2012.
This work is supported by the National Natural Science Foundation of China under Grant Nos.11571276, 11501444; the Postdoctoral Science Foundation of China under Grant Nos. 2013M532079, 2014T70932; the Science Research Foundation of Education Department of Shanxi Provincial Government under Grant No.15JK1735; the Science Research Foundation of Northwest University under Grant No.12NW04.

practice of soft sets theory was also extended to data analysis under incomplete information[32], decision-making problems[8,10,12], normal parameter reduction[17] and d-algebras[16]. Majumdar and Samanta further generalized the concept of fuzzy soft sets and some of its properties were studied, the relation on generalized fuzzy soft sets has also been discussed in [22]. Xu et al. introduced the notion of vague soft sets, derived its basic properties and illustrated its potential applications[29]. Qin and Hong introduced the concept of soft equality and some related properties were derived, some equivalent conditions for soft sets being equality were given by [25]. Xiao et al.[27] proposed a combined forecasting approach and a new application of soft set theory, they have also introduced the notion of exclusive disjunctive soft sets and have given an application of these new sets[28].

In [29], Xu et al. introduced a new notion of vague soft sets, derived its basic properties and presented open questions for its potential applications. The purpose of this paper is to further extend the concept of soft sets and solve the open questions presented in [29]. We obtain some results on the applications of vague soft sets and present the generalized vague soft sets which is a generalization of vague soft sets, after that, some of its properties are discussed. We further study the similarity measure between two generalized vague soft sets and give an application of this measure in decision making problems.

The paper is organized as follows: The following section briefly reviews some definitions for vague sets, soft sets and vague soft sets. Section 3 introduces the notion of vague soft relation and give an application of this relation. Section 4 discusses similarity between two vague soft sets and an application of this similarity measure in decision making problems has been shown. Section 5 introduces the notion of generalized vague soft sets and studies some of its properties. Section 6 discusses similarity between two generalized vague soft sets and an application of this similarity measure has also been shown. In the finial section, some concluding comments are presented.

2 Preliminaries

In this section, we shall recall several definitions which are necessary for our paper. They are stated as follows:

Definition 2.1[13]**.** *A vague set A in the universe $U = \{x_1, x_2, ..., x_n\}$ can be expressed by the following notion, $A = \{(x_i, [t_A(x_i), 1 - f_A(x_i)])|x_i \in U\}$, i.e. $A(x_i) = [t_A(x_i), 1 - f_A(x_i)]$ and the condition $0 \leq t_A(x_i) \leq 1 - f_A(x_i)$ should hold for any $x_i \in U$, where $t_A(x_i)$ is called the membership degree (true membership) of element x_i to the vague set A, while $f_A(x_i)$ is the degree of nonmembership (false membership) of the element x_i to the set A.*

In fact, vague sets are intuitionistic fuzzy sets[4]. Moreover, some authors pointed out that there is a strong connection between intuitionistic fuzzy sets and interval valued fuzzy sets, for more details we refer the reader to [7, 26].

Definition 2.2[13]**.** *Let A, B be two vague sets in the universe $U = \{x_1, x_2, \cdots, x_n\}$, then the union, intersection and complement of vague sets are defined as follows:*

$A \cup B = \big\{(x_i, [\max(t_A(x_i), t_B(x_i)), \max(1 - f_A(x_i), 1 - f_B(x_i))])|x_i \in U\big\}$,

$A \cap B = \big\{(x_i, [\min(t_A(x_i), t_B(x_i)), \min(1 - f_A(x_i), 1 - f_B(x_i))])|x_i \in U\big\}$,

$A^c = \{(x_i, [f_A(x_i), 1 - t_A(x_i)])|x_i \in U\}$.

Definition 2.3[13]**.** *Let A, B be two vague sets in the universe $U = \{x_1, x_2, \cdots, x_n\}$. If $\forall x_i \in U,\ t_A(x_i) \leq t_B(x_i), 1 - f_A(x_i) \leq 1 - f_B(x_i)$, then A is called a vague subset of B, denoted* *by $A \subseteq B$, where $1 \le i \le n$.*

Definition 2.4[23]. *Let U be an initial universal set and E be a set of parameters, $P(U)$ denote the power set of U, and $A \subseteq E$. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.*

Definition 2.5[29]. *Let U be an initial universal set, $V(U)$ the set of all vague sets on U, E a set of parameters and $A \subseteq E$. A pair (F, A) is called a vague soft set over U, where F is a mapping given by $F : A \to V(U)$.*

In other words, a vague soft set over U is a parameterized family of vague set of the universe U. For $\varepsilon \in A, \mu_{F(\varepsilon)} : U \to [0,1]^2$ is regard as the set of $\varepsilon-$approximate elements of the vague soft set (F, A).

Definition 2.6[29]. *For two vague soft sets (F, A) and (G, B) over a universe U, we say that (F, A) is a vague soft subset of (G, B), if $A \subseteq B$ and $\forall\, \varepsilon \in A, F(\varepsilon)$ is a vague subset of $G(\varepsilon)$. This relation is denoted by $(F, A)\widetilde{\subseteq}(G, B)$.*

Definition 2.7[29]. *Two vague soft sets (F, A) and (G, B) over a universe U are said to be vague soft equal if (F, A) is a vague soft subset of (G, B) and (G, B) is a vague soft subset of (F, A). This relation is denoted by $(F, A) = (G, B)$.*

Definition 2.8[29]. *Let $E = \{e_1, e_2, \cdots, e_m\}$ be a parameter set. The not set of E denoted by $\neg E$ is defined by $\neg E = \{\neg e_1, \neg e_2, \cdots, \neg e_m\}$, where $\neg e_i =$ not e_i.*

Definition 2.9[29]. *The complement of a vague soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, \neg A)$, where $F^c : \neg A \to V(U)$ is a mapping given by $t_{F^c(\neg\alpha)}(x) = f_{F(\alpha)}(x), 1 - f_{F^c(\neg\alpha)}(x) = 1 - t_{F(\alpha)}(x),\ \forall\, \neg\alpha \in \neg A,\ x \in U$.*

Definition 2.10[29]. *A vague soft set (F, A) over U is said to be a null vague soft set denoted by $\widehat{\emptyset}$, if $\forall\, \varepsilon \in A,\ t_{F(\varepsilon)}(x) = 0, 1 - f_{F(\varepsilon)}(x) = 0,\ x \in U$.*

Definition 2.11[29]. *A vague soft set (F, A) over U is said to be an absolute vague soft set denoted by $\widehat{A}$, if $\forall \varepsilon \in A,\ t_{F(\varepsilon)}(x) = 1,\ 1 - f_{F(\varepsilon)}(x) = 1,\ x \in U$.*

Definition 2.12[29]. *The union of two vague soft sets of (F, A) and (G, B) over a universe U is a vague soft set (H, C), where $C = A \cup B$ and $\forall\, e \in C,\ \ H : C \to V(U)$.*

$$t_{H(e)}(x) = \begin{cases} t_{F(e)}(x), & \text{if } e \in A - B,\ x \in U, \\ t_{G(e)}(x), & \text{if } e \in B - A,\ x \in U, \\ \max\left(t_{F(e)}(x), t_{G(e)}(x)\right), & \text{if } e \in A \cap B,\ x \in U. \end{cases}$$

$$1 - f_{H(e)}(x) = \begin{cases} 1 - f_{F(e)}(x), & \text{if } e \in A - B,\ x \in U, \\ 1 - f_{G(e)}(x), & \text{if } e \in B - A,\ x \in U, \\ \max(1 - f_{F(e)}(x), 1 - f_{G(e)}(x)), & \text{if } e \in A \cap B,\ x \in U. \end{cases}$$

We denote it by $(F, A)\widetilde{\cup}(G, B) = (H, C)$.

Definition 2.13[29]. *The intersection of two vague soft sets of (F, A) and (G, B) over a universe U is a vague soft set (H, C), where $C = A \cup B$ and $\forall e \in C,\ \ H : C \to V(U)$.*

$$t_{H(e)}(x) = \begin{cases} t_{F(e)}(x), & \text{if } e \in A - B,\ x \in U, \\ t_{G(e)}(x), & \text{if } e \in B - A,\ x \in U, \\ \min(t_{F(e)}(x), t_{G(e)}(x)), & \text{if } e \in A \cap B,\ x \in U, \end{cases}$$

$$1 - f_{H(e)}(x) = \begin{cases} 1 - f_{F(e)}(x), & \text{if } e \in A - B, \ x \in U, \\ 1 - f_{G(e)}(x), & \text{if } e \in B - A, \ x \in U, \\ \min(1 - f_{F(e)}(x), 1 - f_{G(e)}(x)), & \text{if } e \in A \cap B, \ x \in U. \end{cases}$$

We denote it by $(F, A)\widetilde{\cap}(G, B) = (H, C)$.

3 Relation on Vague Soft Sets

In this section, relation on vague soft sets are defined and a decision making problem has been solved using this relation.

Definition 3.1. *Let (F, A) and (G, B) be two vague soft sets over a universe U and $C \subseteq E^2$. Then a vague soft relation R from (F, A) to (G, B) is a function $R : C \to V(U)$, defined by $R((F, A), (G, B)) = (R, C)$, where $R(\alpha, \beta) = F(\alpha) \cap G(\beta)$, $\forall \alpha \in A$, $\beta \in B, (\alpha, \beta) \in C$.*

A generalization of this may be:

Definition 3.2. *Let $F = \{(F_i, A_i),\ i \in \Delta\}$, where Δ is the index set, be any collection of vague soft sets over U and $C \subseteq E^n$. Then an n-array vague soft relation R on F is the mapping $R : C \to V(U)$, defined by $R((F_1, A_1), (F_2, A_2), \cdots, (F_n, A_n)) = (R, C)$, where $R(e_{1_j}, e_{2_j}, \cdots, e_{n_j}) = \bigcap_{i=1}^{n} F_i(e_{i_j})$, $\forall e_{i_j} \in A_i, (e_{1_j}, e_{2_j}, \cdots, e_{n_j}) \in C$.*

Let (F, A) and (G, B) be two vague soft sets over a universe U, then a relation table can be introduced as follows: rows are labelled by the $R((F, A), (G, B))$, columns are labelled by object names $x_1, x_2, \cdots, x_n$ of the universe, and entries are $c_{ij}, i = 1, 2, \cdots, p \times q, j = 1, 2, \cdots, n$. where p is the total number of parameter set A and q is the total number of parameter set B. Hence $c_{ij} = [t_{c_{ij}}, 1 - f_{c_{ij}}]$ is a vague set, where $t_{c_{ij}} \in [0, 1], f_{c_{ij}} \in [0, 1], t_{c_{ij}} + f_{c_{ij}} \leq 1$.

The degree of accuracy of c_{ij} can be evaluated by the accuracy function J, shown as follows: $J(c_{ij}) = t_{c_{ij}} + f_{c_{ij}}$, where $J(c_{ij}) \in [0, 1]$.

Clearly, every decision maker want to know more information in order to reduce the influence produced by the uncertainties, so the larger the value of $J(c_{ij})$, the more the degree of accuracy of the grade of membership of vague value. Hence the comparison table can be formulated by $r_{ij} = J(c_{ij})$, where is its entries, i represents the ith row vector and j represents the jth column vector.

The score of an object x_i is $S(x_i)$ may be given by $S(x_i) =$ the sum of the products of numerical grades which are the highest numerical in each row. Then the one with the highest score is the desired one.

The problem here is to choose an object from the set of given objects with respect to a set of choice parameters. We now present an algorithm for identification of an object, it can be formulated by the following steps:

1. Input the vague soft sets (F, A) and (G, B).
2. Compute the corresponding vague soft relation R from the vague soft set (F, A) to (G, B) and place it in tabular form.
3. Construct the comparison table of vague soft relation using the accuracy function J.
4. Compute the score of x_i.
5. The decision is S_k if $S_k = \max_i S(x_i)$.
6. If k has more than one value then any one of x_i may be chosen.

An application of this vague soft relation in a decision making problem is shown below.

Example 3.1. Assume that (F, A) and (G, B) are two vague soft sets, (F, A) describe the "objects having price space" and (G, B) describe the "objects having color space", where $A = \{\alpha_1, \alpha_2, \alpha_3\} = \{$expensive, average, cheap$\}$ and $B = \{\beta_1, \beta_2, \beta_3\} = \{$greenish, blackish, reddish$\}$. U is a set of four shirts under consideration of a decision maker to purchase, which is denoted by $U = \{x_1, x_2, x_3, x_4\}$. Suppose the person wants to buy one such shirt depending on price and color only. Let two experts P and Q give two observations (F, A) and (G, B) respectively, their corresponding membership matrices be as follows:

$$(F, A) = \begin{pmatrix} [0.4,0.8] & [0.4, 0.7] & [0.3, 0.6] & [0.5, 0.7] \\ [0.6,0.7] & [0.5, 0.8] & [0.7, 0.9] & [0.3, 0.9] \\ [0.4,0.6] & [0.2, 0.7] & [0.6, 0.8] & [0.5, 0.6] \end{pmatrix},$$

$$(G, B) = \begin{pmatrix} [0.3,0.9] & [0.4, 0.6] & [0.2, 0.8] & [0.2, 0.6] \\ [0.7,0.8] & [0.6, 0.9] & [0.4, 0.7] & [0.6, 0.8] \\ [0.1,0.5] & [0.3, 0.8] & [0.1, 0.8] & [0.4, 0.9] \end{pmatrix}.$$

Let $R : C \to V(U)$ be the vague soft relation from (F, A) to (G, B), then the relation table can be expressed as follows:

R	x_1	x_2	x_3	x_4
(α_1, β_1)	[0.3, 0.8]	[0.4, 0.6]	[0.2, 0.6]	[0.2, 0.6]
(α_1, β_2)	[0.4, 0.8]	[0.4, 0.7]	[0.3, 0.6]	[0.5, 0.7]
(α_1, β_3)	[0.1, 0.5]	[0.3, 0.7]	[0.1, 0.6]	[0.4, 0.7]
(α_2, β_1)	[0.3, 0.7]	[0.4, 0.6]	[0.2, 0.8]	[0.2, 0.6]
(α_2, β_2)	[0.6, 0.7]	[0.5, 0.8]	[0.4, 0.7]	[0.3, 0.8]
(α_2, β_3)	[0.1, 0.5]	[0.3, 0.8]	[0.1, 0.8]	[0.3, 0.9]
(α_3, β_1)	[0.3, 0.6]	[0.2, 0.6]	[0.2, 0.8]	[0.2, 0.6]
(α_3, β_2)	[0.4, 0.6]	[0.2, 0.7]	[0.4, 0.7]	[0.5, 0.6]
(α_3, β_3)	[0.1, 0.5]	[0.2, 0.7]	[0.1, 0.8]	[0.4, 0.6]

By the relation table and the accuracy function J, we can get the comparison table as follows:

R	x_1	x_2	x_3	x_4
(α_1, β_1)	0.5	0.8	0.6	0.6
(α_1, β_2)	0.6	0.7	0.7	0.8
(α_1, β_3)	0.6	0.6	0.5	0.7
(α_2, β_1)	0.6	0.8	0.4	0.6
(α_2, β_2)	0.9	0.7	0.7	0.5
(α_2, β_3)	0.6	0.5	0.3	0.4
(α_3, β_1)	0.7	0.6	0.4	0.6
(α_3, β_2)	0.8	0.5	0.7	0.9
(α_3, β_3)	0.6	0.5	0.3	0.8

Now to determine the best shirt, we first mark the highest numerical in each row. The score of each of such shirts is calculated by taking the sum of the products of these numerical grades. The shirt with the highest score is the desired one. Hence the grade table can be expressed as follows:

R	(α_1, β_1)	(α_1, β_2)	(α_1, β_3)	(α_2, β_1)	(α_2, β_2)	(α_2, β_3)	(α_3, β_1)	(α_3, β_2)	(α_3, β_3)
x_i	x_2	x_4	x_4	x_2	x_1	x_1	x_1	x_4	x_4
grade	0.8	0.8	0.7	0.8	0.9	0.6	0.7	0.9	0.8

Score $(x_1) = 0.9 + 0.6 + 0.7 = 2.2$, Score $(x_2) = 0.8 + 0.8 = 1.6$,
Score $(x_3) = 0$, Score $(x_4) = 0.8 + 0.7 + 0.9 + 0.8 = 3.2$.

Then the person will select the shirt with highest score. Hence he will buy shirt x_4.

Clearly, after performing the operation of vague soft relation, we obtain another vague soft set. The newly obtained vague soft set can also perform the operation of vague soft relation with the other vague soft sets, so the algorithm may be used to solve the problems which involve many different parameter sets.

4 Similarity Measure of Vague Soft Sets

A similarity measure is used for estimating the degree of similarity between two sets, which have been or could be applied in areas such as data preprocessing, for identifying the functional dependency relationships between concepts in data mining systems, for approximate reasoning, and for other purposes to include pattern recognition. Several researchers have studied the problem of similarity measurement between different sets, such as fuzzy sets, vague sets and soft sets. For example, Zeng et al.[31] investigated the relationship among the normalized distance, similarity measure, inclusion measure and entropy of interval-valued fuzzy sets. Chen[5] first put forward the concept of similarity measure for vague sets and gave a computation formula. As [19] makes clear, many scholars have presented formulae to calculate the similarity measure of vague sets from different viewpoints.

In this section we introduce the notion of similarity measure between vague soft sets, and an example is given to illustrate the application of this measure in decision making problems.

Definition 4.1. *Let $U = \{x_1, x_{2,} \cdots, x_n\}$ be the universal set of elements and $E = \{e_1, e_{2,} \cdots, e_m\}$ be the universal set of parameters. A mapping $M : VSS(U) \times VSS(U) \to [0,1]$. $VSS(U)$ denotes the set of all vague soft sets in U. $M((F,E),(G,E))$ is said to be the degree of similarity between $(F,E) \in VSS(U)$ and $(G,E) \in VSS(U)$, if $M((F,E),(G,E))$ satisfies the following properties condition:*

(M1) $M((F,E),(G,E)) = M((G,E),(F,E))$,

(M2) $M((F,E),(G,E)) \in [0,1]$,

(M3) $M((F,E),(G,E)) = 1 \Leftrightarrow (F,E) = (G,E)$,

(M4) $(F,E)\widetilde{\subseteq}(G,E)\widetilde{\subseteq}(H,E) \Rightarrow$,

$M((F,E),(H,E)) \leq M((F,E),(G,E)), M((F,E),(H,E)) \leq M((G,E),(H,E)), (H,E) \in VSS(U)$.

According to the definition of similarity measure of vague soft sets, one should note that the similarity measure is used for estimating the degree of similarity between two vague soft sets. Clearly, the value of similarity measure of vague soft sets is larger, the two vague soft sets are more similar.

In this definition, it is assume that the parameter set is fixed as E when defining the similarity measure, if two vague soft sets have different parameter sets, we only compare the part of the same in E.

In [18], Li and Xu proposed a kind of measures of similarity between vague sets and gave some explanations to illustrate the rationality and practicability of their formula. Benefitting from their idea, we introduce the formula to calculate the similarity between two vague soft sets as follows:

Theorem 4.1. *Let $U = \{x_1, x_2, \cdots, x_n\}$ be the universal set of elements and $E = \{e_1, e_2, \cdots,$*

$e_m\}$ be the universal set of parameters. Hence $(F,E) = \{F(e_i),\ i = 1,2,\cdots,m\}$ and $(G,E) = \{G(e_i),\ i = 1,2,\cdots,m\}$ are two vague soft sets. Let

$$M((F,E),(G,E)) = \frac{\sum\limits_{i=1}^{m} M_i((F,E),(G,E))}{m},$$

where

$$\begin{aligned} M_i((F,E),(G,E)) =& 1 - \frac{1}{4n}\sum_{j=1}^{n}\big[|S_{F(e_i)}(x_j) - S_{G(e_i)}(x_j)| \\ &+ |t_{F(e_i)}(x_j) - t_{G(e_i)}(x_j)| + |f_{F(e_i)}(x_j) - f_{G(e_i)}(x_j)|\big], \end{aligned}$$

$S_{F(e_i)}(x_j) = t_{F(e_i)}(x_j) - f_{F(e_i)}(x_j)$ and $S_{G(e_i)}(x_j) = t_{G(e_i)}(x_j) - f_{G(e_i)}(x_j)$ be called core of $F(e_i)$ and $G(e_i)$ or degree of support of $F(e_i)$ and $G(e_i)$ respectively, $S_{F(e_i)}(x_j) \in [-1,1]$, $S_{G(e_i)}(x_j) \in [-1,1]$. Then $M((F,E),(G,E))$ is the similarity measure between two vague soft sets (F,E) and (G,E).

Proof.

(M1) $M((F,E),(G,E))$

$$= \frac{\sum\limits_{i=1}^{m} M_i((F,E),(G,E))}{m} = \frac{\sum\limits_{i=1}^{m} M_i((G,E),(F,E))}{m} = M((G,E),(F,E)),$$

(M2) $0 \le t_{F(e_i)}(x_j) \le 1,\ \ 0 \le f_{F(e_i)}(x_j) \le 1,\ \ 0 \le t_{G(e_i)}(x_j) \le 1,\ \ 0 \le f_{G(e_i)}(x_j) \le 1$

$$\begin{aligned} &\Longrightarrow |t_{F(e_i)}(x_j) - t_{G(e_i)}(x_j)| \le 1, \quad |f_{F(e_i)}(x_j) - f_{G(e_i)}(x_j)| \le 1, \\ &|S_{F(e_i)}(x_j) - S_{G(e_i)}(x_j)| \le 2 \\ &\Longrightarrow M_i((F,E),(G,E)) \in [0,1] \\ &\Longrightarrow M((F,E),(G,E)) \in [0,1], \end{aligned}$$

(M3) $M((F,E),(G,E)) = 1 \Longleftrightarrow M_i((F,E),(G,E)) = 1$

$$\begin{aligned} &\Longleftrightarrow S_{F(e_i)}(x_j) = S_{G(e_i)}(x_j), \quad t_{F(e_i)}(x_j) = t_{G(e_i)}(x_j), \quad f_{F(e_i)}(x_j) = f_{G(e_i)}(x_j) \\ &\Longleftrightarrow (F,E) = (G,E), \end{aligned}$$

(M4) $(F,E)\widetilde{\subseteq}(G,E)\widetilde{\subseteq}(H,E)$

$$\begin{aligned} &\Longrightarrow 0 \le t_{F(e_i)}(x_j) \le t_{G(e_i)}(x_j) \le t_{H(e_i)}(x_j) \le 1, \\ &1 \ge f_{F(e_i)}(x_j) \ge f_{G(e_i)}(x_j) \ge f_{H(e_i)}(x_j) \ge 0 \\ &\Longrightarrow |t_{F(e_i)}(x_j) - t_{G(e_i)}(x_j)| \le |t_{F(e_i)}(x_j) - t_{H(e_i)}(x_j)|, \\ &|f_{F(e_i)}(x_j) - f_{G(e_i)}(x_j)| \le |f_{F(e_i)}(x_j) - f_{H(e_i)}(x_j)|, \\ &|S_{F(e_i)}(x_j) - S_{G(e_i)}(x_j)| \\ =&|(t_{F(e_i)}(x_j) - f_{F(e_i)}(x_j)) - (t_{G(e_i)}(x_j) - f_{G(e_i)}(x_j))| \\ =&|(t_{F(e_i)}(x_j) - t_{G(e_i)}(x_j)) + (f_{G(e_i)}(x_j) - f_{F(e_i)}(x_j))| \\ \le&|(t_{F(e_i)}(x_j) - t_{H(e_i)}(x_j)) + (f_{H(e_i)}(x_j) - f_{F(e_i)}(x_j))| \\ =&|(t_{F(e_i)}(x_j) - f_{F(e_i)}(x_j)) - (t_{H(e_i)}(x_j) - f_{H(e_i)}(x_j))| \\ =&|S_{F(e_i)}(x_j) - S_{H(e_i)}(x_j)| \end{aligned}$$

$$\Longrightarrow M_i((F,E),(H,E)) \le M_i((F,E),(G,E))$$
$$\Longrightarrow M((F,E),(H,E)) \le M((F,E),(G,E)).$$

Similarly, we have $M((F,E),(H,E)) \le M((G,E),(H,E))$.

Hence, we complete the proof of Theorem 4.1. □

For more details about the motivation to introduce the formula for calculating the similarity measure $M_i((F,E),(G,E))$, we refer the readers to [18].

Example 4.1. Assume that a real estate agent has three types of houses, which may be characterized by a set of parameters $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. For $j = 1, 2, 3, 4, 5, 6$, the parameters e_j stand for "beautiful", "large", "expensive", "modern", "wooden", "in green surrounding". Let our universal set contain only two elements "yes" and "no", denoted by $U = \{y, n\}$.

Suppose that a person comes to the real estate agent to buy a house, our model VSS for attractiveness of the housesis given in Table 1 and this can be prepared with the help of the choice. The VSS for the other three types of houses are given in Table 2, Table 3 and Table 4.

Table 1. Model VSS for "Attractiveness of the Houses"

(F,E)	e_1	e_2	e_3	e_4	e_5	e_6
y	$[1,1]$	$[1,1]$	$[0,0]$	$[1,1]$	$[0,0]$	$[1,1]$
n	$[0,0]$	$[0,0]$	$[1,1]$	$[0,0]$	$[1,1]$	$[0,0]$

Table 2. VSS for the First Type of House

(G,E)	e_1	e_2	e_3	e_4	e_5	e_6
y	$[0.4,0.5]$	$[0.5,0.7]$	$[0.3,0.6]$	$[0.8,0.9]$	$[0.4,0.6]$	$[0.5,0.7]$
n	$[0.2,0.6]$	$[0.4,0.5]$	$[0.6,0.7]$	$[0.4,0.8]$	$[0.5,0.6]$	$[0.3,0.5]$

Table 3. VSS for the Second Type of House

(H,E)	e_1	e_2	e_3	e_4	e_5	e_6
y	$[0.7,0.8]$	$[0.8,0.9]$	$[0.1,0.2]$	$[0.7,0.9]$	$[0.2,0.3]$	$[0.6,0.8]$
n	$[0.1,0.3]$	$[0.2,0.4]$	$[0.5,0.7]$	$[0.3,0.6]$	$[0.6,0.9]$	$[0.1,0.2]$

Table 4. VSS for the Third Type of House

(P,E)	e_1	e_2	e_3	e_4	e_5	e_6
y	$[0.3,0.9]$	$[0.4,0.8]$	$[0.5,0.6]$	$[0.3,0.7]$	$[0.4,0.6]$	$[0.5,0.8]$
n	$[0.2,0.5]$	$[0.4,0.6]$	$[0.3,0.5]$	$[0.5,0.8]$	$[0.4,0.5]$	$[0.3,0.6]$

Now here $M((F,E),(G,E)) \cong 0.575$, $M((F,E),(H,E)) \cong 0.738$, $M((F,E),(P,E)) \cong 0.517$.

The higher the value of M, the more similar between two vague soft sets. Then the person will select the house which is the most similar to the model VSS for "attractiveness of the houses", hence he will buy the second type of house.

5 Generalized Vague Soft Sets

In [22], Majumdar and Samanta defined generalised fuzzy soft sets and studied some of their properties. Application of generalised fuzzy soft sets in decision making problem and medical diagnosis problem has been shown. Benefitting from their idea, in this section we formulate the concept of generalized vague soft sets and study some of its properties.

Definition 5.1. *Let $U = \{x_1, x_2, \cdots, x_n\}$ be the universal set of elements and $E = \{e_1, e_2, \cdots, e_m\}$ be the universal set of parameters. The pair (U, E) will be called a soft universe. Let $F : E \to V^U$ and μ be a vague subset of E, i.e. $\mu : E \to V$, where V^U is the collection of all vague subsets of U, Let F_μ be the mapping $F_u : E \to V^U \times V$ be a function defined as follows: $F_\mu(e) = (F(e), \mu(e))$, where $F(e) \in V^U$. Then F_μ is called a generalized vague soft set (GVSS in short) over the soft universe (U, E).*

Here for each parameter e_i, $F_\mu(e_i) = (F(e_i), \mu(e_i))$ indicates not only the degree of belongingness of the elements of U in $F(e_i)$ but also the degree of possibility of such belongingness which is represented by $\mu(e_i)$.

Example 5.1. Let $\mu : E \to V$ be defined as follows: $\mu(e_1) = [0.2, 0.4]$, $\mu(e_2) = [0.3, 0.5]$, $\mu(e_3) = [0.3, 0.6]$. So a function $F_\mu : E \to V^U \times V$ can be defined as follows:

$$F_\mu(e_1) = \Big(\Big\{\frac{[0.4, 0.6]}{x_1}, \frac{[0.6, 0.9]}{x_2}, \frac{[0.9, 1]}{x_3}\Big\}, [0.2, 0.4]\Big),$$
$$F_\mu(e_2) = \Big(\Big\{\frac{[0.3, 0.5]}{x_1}, \frac{[0.5, 0.8]}{x_2}, \frac{[0.7, 0.9]}{x_3}\Big\}, [0.3, 0.5]\Big),$$
$$F_\mu(e_3) = \Big(\Big\{\frac{[0.5, 0.8]}{x_1}, \frac{[0.7, 0.8]}{x_2}, \frac{[0.6, 0.8]}{x_3}\Big]\}, [0.3, 0.6]\Big).$$

Then F_μ is a GVSS over (U, E), which can be expressed as

$$F_\mu = \begin{pmatrix} [0.4,0.6] & [0.6, 0.9] & [0.9, 1] & [0.2, 0.4] \\ [0.3,0.5] & [0.5, 0.8] & [0.7, 0.9] & [0.3, 0.5] \\ [0.5,0.8] & [0.7, 0.8] & [0.6, 0.8] & [0.3, 0.6] \end{pmatrix}.$$

In this matrix form, the ith row vector represents $F_\mu(e_i)$, the ith column vector represents x_i, the last column represents the value of μ.

Definition 5.2. *Let F_μ and G_δ be two GVSS over (U, E). If $\forall\, e \in E$, μ is a vague subset of δ and $F(e)$ is also a vague subset of $G(e)$, then F_μ is said to be a generalized vague soft subset of G_δ, denoted by $F_\mu \widetilde{\subseteq} G_\delta$.*

Example 5.2. Consider the GVSS F_μ over (U, E) given in Example 5.1. Let G_δ be another GVSS over (U, E) defined as follows:

$$G_\delta(e_1) = \Big(\Big\{\frac{[0.2, 0.4]}{x_1}, \frac{[0.4, 0.7]}{x_2}, \frac{[0.6, 0.8]}{x_3}\Big\}, [0.1, 0.3]\Big),$$
$$G_\delta(e_2) = \Big(\Big\{\frac{[0.1, 0.4]}{x_1}, \frac{[0.3, 0.6]}{x_2}, \frac{[0.2, 0.7]}{x_3}\Big\}, [0.2, 0.4]\Big),$$
$$G_\delta(e_3) = \Big(\Big\{\frac{[0.3, 0.6]}{x_1}, \frac{[0.2, 0.7]}{x_2}, \frac{[0.3, 0.7]}{x_3}\Big\}, [0.2, 0.5]\Big).$$

where δ is defined as above.

Then $G_\delta \widetilde{\subseteq} F_\mu$.

Definition 5.3. *Two GVSS F_μ and G_δ over (U,E) are said to be generalized vague soft equal if F_μ is a generalized vague soft subset of G_δ and G_δ is a generalized vague soft subset of F_μ. This relation is denoted by $F_\mu = G_\delta$.*

Definition 5.4. *Let F_μ be a GVSS over (U,E). Then the complement of F_μ, denoted by F_μ^c, is defined as $F_\mu^c = G_\delta$, if $\forall\, e \in E, \delta(e) = \mu^c(e)$ and $G(e) = F^c(e)$.*

Note 5.1. Obviously $(F_\mu^c)^c = F_\mu$ as the vague complement c is involutive in nature.

Definition 5.5. *The union of two GVSS F_μ and G_δ over (U,E) is a GVSS H_ν, defined as $H_\nu : E \to V^U \times V$ such that $H_\nu(e) = (H(e), \nu(e))$, where $\forall e \in E, H(e) = F(e) \cup G(e)$ and $\nu(e) = \mu(e) \cup \delta(e)$, we denoted it by $F_\mu \widetilde{\cup} G_\delta = H_\nu$.*

Definition 5.6. *The intersection of two GVSS F_μ and G_δ over (U,E) is a GVSS H_ν, defined as $H_\nu : E \to V^U \times V$ such that $H_\nu(e) = (H(e), \nu(e))$, where $\forall e \in E, H(e) = F(e) \cap G(e)$ and $\nu(e) = \mu(e) \cap \delta(e)$, we denoted it by $F_\mu \widetilde{\cap} G_\delta = H_\nu$.*

Example 5.3. Let us consider the GVSS F_μ and G_δ defined in Example 5.1 and 5.2 respectively, then

$$F_\mu \widetilde{\cup} G_\delta = F_\mu, \quad F_\mu \widetilde{\cap} G_\delta = G_\delta, \quad F_\mu^c = \begin{pmatrix} [0.4,0.6] & [0.1,0.4] & [0,0.1] & [0.6,0.8] \\ [0.5,0.7] & [0.2,0.5] & [0.1,0.3] & [0.5,0.7] \\ [0.2,0.5] & [0.2,0.3] & [0.2,0.4] & [0.4,0.7] \end{pmatrix}.$$

Definition 5.7. *A GVSS is said to be a generalized null vague soft set denoted by $\emptyset_\theta$, if $\emptyset_\theta : E \to V^U \times V$ such that $\emptyset_\theta(e) = (\emptyset(e), \theta(e))$, where $\forall e \in E, \emptyset(e) = \widehat{\emptyset}$ and $\theta(e) = \widehat{\emptyset}$.*

Definition 5.8. *A GVSS is said to be a generalized absolute vague soft set denoted by A_α, if $A_\alpha : E \to V^U \times V$ such that $A_\alpha(e) = (A(e), \alpha(e))$, where $\forall e \in E,\ A(e) = \widehat{A}$ and $\alpha(e) = \widehat{A}$.*

The following propositions can be obtained based on the definitions introduced above:

Proposition 5.1. *Let F_μ be a GVSS over (U,E), $\emptyset_\theta$ be a generalized null vague soft set, A_α be a generalized absolute vague soft set, then:*

(i) $F_\mu \widetilde{\cup} \emptyset_\theta = F_\mu$.
(ii) $F_\mu \widetilde{\cap} \emptyset_\theta = \emptyset_\theta$.
(iii) $F_\mu \widetilde{\cup} A_\alpha = A_\alpha$.
(iv) $F_\mu \widetilde{\cap} A_\alpha = F_\mu$.

Proposition 5.2. *Let F_μ, G_δ and H_λ are three GVSS over (U,E), then:*

(i) $F_\mu \widetilde{\cup} G_\delta = G_\delta \widetilde{\cup} F_\mu$.
(ii) $F_\mu \widetilde{\cap} G_\delta = G_\delta \widetilde{\cap} F_\mu$.
(iii) $F_\mu \widetilde{\cup} (G_\delta \widetilde{\cup} H_\lambda) = (F_\mu \widetilde{\cup} G_\delta) \widetilde{\cup} H_\lambda$.
(iv) $F_\mu \widetilde{\cap} (G_\delta \widetilde{\cap} H_\lambda) = (F_\mu \widetilde{\cap} G_\delta) \widetilde{\cap} H_\lambda$.
(v) $F_\mu \widetilde{\cap} (G_\delta \widetilde{\cup} H_\lambda) = (F_\mu \widetilde{\cap} G_\delta) \widetilde{\cup} (F_\mu \widetilde{\cap} H_\lambda)$.
(vi) $F_\mu \widetilde{\cup} (G_\delta \widetilde{\cap} H_\lambda) = (F_\mu \widetilde{\cup} G_\delta) \widetilde{\cap} (F_\mu \widetilde{\cup} H_\lambda)$.

Proposition 5.3. *Let F_μ and G_δ are two GVSS over (U,E), then:*

(i) $(F_\mu \widetilde{\cap} G_\delta)^c = F_\mu^c \widetilde{\cup} G_\delta^c$.
(ii) $(F_\mu \widetilde{\cup} G_\delta)^c = F_\mu^c \widetilde{\cap} G_\delta^c$.

6. Similarity Between Two Generalized Vague Soft Sets

We have studied the similarity measure between two vague soft sets in section 4, in this section, we will introduce the similarity between two GVSS, and give an example to illustrate the application of this measure in decision making problems.

Definition 6.1. *Let $U = \{x_1, x_2, \cdots, x_n\}$ be the universal set of elements and $E = \{e_1, e_2, \cdots, e_m\}$ be the universal set of parameters. Hence $F_\mu = \{(F(e_i), \mu(e_i)),\ i = 1, 2, \cdots, m\}$ and $G_\delta = \{(G(e_i), \delta(e_i)),\ i = 1, 2, \cdots, m\}$ are two GVSS over (U, E). A mapping $S : GVSS(U) \times GVSS(U) \to [0, 1]$. $GVSS(U)$ denotes the set of all generalized vague soft sets over (U, E). $S(F_\mu, G_\delta)$ is said to be the degree of similarity between $F_\mu \in GVSS(U)$ and $G_\delta \in GVSS(U)$, if $S(F_\mu, G_\delta)$ satisfies the following properties condition:*

(S1) $S(F_\mu, G_\delta) = S(G_\delta, F_\mu)$,

(S2) $S(F_\mu, G_\delta) \in [0, 1]$,

(S3) $S(F_\mu, G_\delta) = 1 \Leftrightarrow F_\mu = G_\delta$,

(S4) $F_\mu \widetilde{\subseteq} G_\delta \widetilde{\subseteq} H_\nu \Rightarrow S(F_\mu, H_\nu) \le S(G_\delta, H_\nu), S(F_\mu, H_\nu) \le S(F_\mu, G_\delta), H_\nu \in GVSS(U)$.

Based on Theorem 4.1, we can introduce the formula to calculate the similarity of generalized vague soft sets as follows:

Theorem 6.1. *Let F_μ and G_δ be two GVSS over (U, E). Hence $F_\mu = \{(F(e_i), \mu(e_i)), i = 1, 2, \cdots, m\}$ and $G_\delta = \{(G(e_i), \delta(e_i)), i = 1, 2, \cdots, m\}$.*

$$S(F_\mu, G_\delta) = M((F, E), (G, E)) \cdot m(\mu, \delta),$$

where $M((F, E), (G, E))$ is defined as Theorem 4.1 and

$$m(\mu, \delta) = 1 - \frac{1}{4m} \sum_{i=1}^{m} \left[|S_{\mu(e_i)} - S_{\delta(e_i)}| + |t_{\mu(e_i)} - t_{\delta(e_i)}| + |f_{\mu(e_i)} - f_{\delta(e_i)}| \right],$$

in which $S_{\mu(e_i)} = t_{\mu(e_i)} - f_{\mu(e_i)}$ and $S_{\delta(e_i)} = t_{\delta(e_i)} - f_{\delta(e_i)}$. Then $S(F_\mu, G_\delta)$ is the similarity measure between two GVSS F_μ and G_δ.

Proof. The proof follows from definition. □

Example 6.1. Consider the two GVSS F_μ and G_δ over (U, E) given as follows, where U be an initial universal set denoted by $U = \{x_1, x_2, x_3, x_4\}$ and E be a set of parameters denoted by $E = \{e_1, e_2, e_3\}$. Let

$$F_\mu = \begin{pmatrix} [0.4,0.7] & [0.2, 0.9] & [0.3, 0.7] & [0.5, 0.6] & [0.8, 0.9] \\ [0.3,0.8] & [0.6, 0.7] & [0.3, 0.6] & [0.4, 0.9] & [0.2, 0.4] \\ [0.2,0.7] & [0.7, 0.9] & [0.3, 0.6] & [0.8, 0.8] & [0.5, 0.8] \end{pmatrix},$$

$$G_\delta = \begin{pmatrix} [0.5,0.8] & [0.3, 0.7] & [0.2, 0.8] & [0.4, 0.7] & [0.6, 0.9] \\ [0.2,0.6] & [0.4, 0.9] & [0.2, 0.7] & [0.1, 0.9] & [0.5, 0.8] \\ [0.2,0.8] & [0.4, 0.7] & [0.5, 0.8] & [0.3, 0.7] & [0.6, 0.7] \end{pmatrix}.$$

Here $m(\mu, \delta) \cong 0.833$.

From the Theorem 4.1. we know $M((F, E), (G, E))) \cong 0.867$.

$\therefore S(F_\mu, G_\delta) = 0.867 \times 0.833 \cong 0.722$.

Example 6.2. Assume that a real estate agent has three types of houses, which may be characterized by a set of parameters $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. For $j = 1, 2, 3, 4, 5, 6$, the parameters

e_j stand for "beautiful", "large", "expensive", "modern", "wooden", "in green surrounding". Let our universal set contain only two elements "yes" and "no", denoted by $U = \{y, n\}$.

Suppose that a person comes to the real estate agent to buy a house, our model GVSS for attractiveness of the houses is given in Table 5 and this can be prepared with the help of the choice. The GVSS for the other three types of houses are given in Table 6, Table 7 and Table 8.

Table 5. Model GVSS for "Attractiveness of the Houses"

M_μ	e_1	e_2	e_3	e_4	e_5	e_6
y	$[1,1]$	$[1,1]$	$[0,0]$	$[1,1]$	$[0,0]$	$[1,1]$
n	$[0,0]$	$[0,0]$	$[1,1]$	$[0,0]$	$[1,1]$	$[0,0]$
μ	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$

Table 6. GVSS for the First Type of House

G_δ	e_1	e_2	e_3	e_4	e_5	e_6
y	$[0.4, 0.5]$	$[0.5, 0.7]$	$[0.3, 0.6]$	$[0.8, 0.9]$	$[0.4, 0.6]$	$[0.5, 0.7]$
n	$[0.2, 0.6]$	$[0.4, 0.5]$	$[0.6, 0.7]$	$[0.4, 0.8]$	$[0.5, 0.6]$	$[0.3, 0.5]$
δ	$[0.4, 0.5]$	$[0.6, 0.8]$	$[0.5, 0.6]$	$[0.6, 0.6]$	$[0.5, 0.7]$	$[0.5, 0.6]$

Table 7. GVSS for the Second Type of House

H_ν	e_1	e_2	e_3	e_4	e_5	e_6
y	$[0.7, 0.8]$	$[0.8, 0.9]$	$[0.1, 0.2]$	$[0.7, 0.9]$	$[0.2, 0.3]$	$[0.6, 0.8]$
n	$[0.1, 0.3]$	$[0.2, 0.4]$	$[0.5, 0.7]$	$[0.3, 0.6]$	$[0.6, 0.9]$	$[0.1, 0.2]$
ν	$[0.8, 0.9]$	$[0.7, 0.9]$	$[0.6, 0.8]$	$[0.8, 0.9]$	$[0.9, 1]$	$[0.8, 0.8]$

Table 8. GVSS for the Third Type of House

P_λ	e_1	e_2	e_3	e_4	e_5	e_6
y	$[0.7, 0.8]$	$[0.8, 0.9]$	$[0.1, 0.2]$	$[0.7, 0.9]$	$[0.2, 0.3]$	$[0.6, 0.8]$
n	$[0.1, 0.3]$	$[0.2, 0.4]$	$[0.5, 0.7]$	$[0.3, 0.6]$	$[0.6, 0.9]$	$[0.1, 0.2]$
λ	$[0.2, 0.3]$	$[0.1, 0.3]$	$[0.1, 0.2]$	$[0.2, 0.4]$	$[0.3, 0.3]$	$[0.1, 0.5]$

Now here $S(M_\mu, G_\delta) \cong 0.331$, $S(M_\mu, H_\nu) \cong 0.608$, $S(M_\mu, P_\lambda) \cong 0.184$.

Decision: The person can buy the second type of house.

We can see that $t_{P(e_i)}(x_j) = t_{H(e_i)}(x_j)$, $f_{P(e_i)}(x_j) = f_{H(e_i)}(x_j)$, $\forall\, i, j$, but $S(M_\mu, H_\nu) \neq S(M_\mu, P_\lambda)$. This is because the values of $\nu(e_i) \neq \lambda(e_i)$.

From the definition of the similarity measure of GVSS and the Example 6.2, one should clearly note that the results of similarity measure between two GVSS depends not only the degree of belongingness of the elements of U but also the degree of possibility of such belongingness.

7 Conclusion

The soft set theory of Molodtsov[23] offers a general mathematical tool for dealing with uncertain, fuzzy, or vague objects. Xu et al. introduced the notion of vague soft sets and presented open questions for its potential applications. In the present paper, we define vague soft relations and similarity measure of vague soft sets. Some applications of the new theory of vague soft sets in decision making problems has been shown. Moreover, we define generalized vague soft sets and study some of its properties. We then present the similarity measure of generalized vague soft sets and give an application for real estate agent to choose an optimal house. These works can be successfully applied to many other convenient problems that contain uncertainties.

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