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On Frankl and Füredi's Conjecture for 3-uniform Hypergraphs

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Abstract Frankl and Füredi in [1] conjectured that the r-graph with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$ has the largest Lagrangian of all r-graphs with m edges. Denote this r-graph by $C_{r,m}$ and the Lagrangian of a hypergraph by $\lambda(G)$. In this paper, we first show that if $\binom{t-1}{3} \le m < \binom{t}{3}$, G is a left-compressed 3-graph with m edges and on vertex set $[t]$, the triple with minimum colex ordering in G^c is $(t-2-i)(t-2)t$, then $\lambda(G) \le \lambda(C_{3,m})$. As an implication, the conjecture of Frankl and Füredi is true for $\binom{t}{3} - 6 \le m \le \binom{t}{3}$.

Keywords Colex ordering; Lagrangians of r-graphs; extremal problems in combinatorics
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1 Introduction and the Main Results

For a set V and a positive integer r we denote by $V^{(r)}$ the family of all r-subsets of V. An r-uniform graph or r-graph G consists of a set $V(G)$ of vertices and a set $E(G) \subseteq V(G)^{(r)}$ of edges. An edge $e = \{a_1, a_2, \ldots, a_r\}$ will be simply denoted by $a_1a_2\cdots a_r$. An r-graph H is a *subgraph* of an r-graph G, denoted by $H \subseteq G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The complement of an r-graph G is denoted by G^c. Let $K_t^{(r)}$ denote the complete r-graph on t vertices, that is the r-graph on t vertices containing all possible edges. A complete r-graph on t vertices is also called a clique with order t. Let $\mathbb{N}$ be the set of all positive integers. For an integer $n \in \mathbb{N}$, we denote the set $\{1, 2, 3, \ldots, n\}$ by $[n]$. Let $[n]^{(r)}$ represent the complete r-graph on the vertex set $[n]$. When $r = 2$, an r-graph is a simple graph. When $r \ge 3$, an r-graph is often called a hypergraph.

Definition 1.1. *For an r-graph G with the vertex set $[n]$, edge set $E(G)$ and a vector $\vec{x} = (x_1, \cdots, x_n) \in R^n$, define*

$$\lambda(G, \vec{x}) = \sum_{i_1 i_2 \cdots i_r \in E(G)} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

Definition 1.2. *Let $S = \big\{\vec{x} = (x_1, x_2, \cdots, x_n) : \sum\limits_{i=1}^{n} x_i = 1,\ x_i \ge 0$ for $i = 1, 2, \cdots, n\big\}$. The*

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Lagrangian of G*, denoted by* $\lambda(G)$*, is defined as*

$$\lambda(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \in S\}.$$

The value x_i *is called the* weight *of the vertex* i*. We call* $\vec{x} = (x_1, x_2, \cdots, x_n) \in R^n$ *a legal weighting for* G *if* $\vec{x} \in S$*. A vector* $\vec{y} \in S$ *is called an* optimal weighting *for* G *if* $\lambda(G, \vec{y}) = \lambda(G)$*.*

The following fact is easily implied by the definition of the Lagrangian.

Fact 1.1. Let G_1, G_2 be r-graphs and $G_1 \subseteq G_2$. Then $\lambda(G_1) \leq \lambda(G_2)$.

In [4], Motzkin and Straus provided the following simple expression for the Lagrangian of a 2-graph.

Theorem 1.2[4]**.** *If* G *is a 2-graph in which a largest clique has order* t *then* $\lambda(G) = \lambda(K_t^{(2)}) = \frac{1}{2}(1 - \frac{1}{t})$.

Based on this connection, they provided a new proof of the classical result of Turán on Turán densities of complete graphs. This new proof aroused interests in the study of Lagrangians of r-graphs. Since then the Lagrangian of a hypergraph has been a useful tool in hypergraph extremal problems. However, the obvious generalization of Motzkin and Straus' result to hypergraphs is false because there are many examples of hypergraphs that do not achieve their Lagrangian on any proper subhypergraph. Lagrangians of hypergraphs has been proved to be a useful tool in hypergraph extremal problems. In most applications, an upper bound is needed. Frankl and Füredi[1] asked the following question. Given $r \geq 3$ and $m \in \mathbb{N}$ how large can the Lagrangian of an r-graph with m edges be? For distinct $A, B \in \mathbb{N}^{(r)}$ we say that A is less than B in the *colex ordering* if $max(A \triangle B) \in B$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$. For example we have $246 < 156$ in $\mathbb{N}$ since $max(\{2, 4, 6\} \triangle \{1, 5, 6\}) \in \{1, 5, 6\}$. In colex ordering, $123 < 124 < 134 < 234 < 125 < 135 < 235 < 145 < 245 < 345 < 126 < 136 < 236 < 146 < 246 < 346 < 156 < 256 < 356 < 456 < 127 < \cdots$. Note that the first $\binom{t}{r}$ r-tuples in the colex ordering of $\mathbb{N}^{(r)}$ are the edges of $[t]^{(r)}$.

The following conjecture of Frankl and Füredi (if it is true) proposes a solution to the question mentioned above.

Conjecture 1.3[1]**.** *The* r*-graph with* m *edges formed by taking the first* m *sets in the colex ordering of* $\mathbb{N}^{(r)}$ *has the largest Lagrangian of all* r*-graphs with* m *edges. In particular, the* r*-graph with* $\binom{t}{r}$ *edges and the largest Lagrangian is* $[t]^{(r)}$*.*

This conjecture is true when $r = 2$ by Theorem 1.2. For the case $r = 3$, Talbot in [5] proved the following.

Theorem 1.4[5]**.** *Let* m *and* t *be integers satisfying* $\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2} - (t-1)$*. Then Conjecture 1.3 is true for* $r = 3$ *and this value of* m*. Conjecture 1.3 is also true for* $r = 3$ *and* $m = \binom{t}{3} - 1$ *or* $m = \binom{t}{3} - 2$*.*

For the case $r = 3$, Tang, Peng, Zhang, and Zhao in [6] proved the following.

Theorem 1.5[6]**.** *Let* m *and* t *be positive integers satisfying* $\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2} - (t-4)$*. Then Conjecture 1.3 is true for* $r = 3$ *and this value of* m*.*

It turns out this natural conjecture seems to be very challenging to verify. The truth of Frankl and Füredi's conjecture is not known in general for $r \geq 4$. In the case $r = 3$, the case when $\binom{t-1}{3} + \binom{t-2}{2} - (t-5) \leq m \leq \binom{t}{3} - 3$ is still open in this conjecture. Before we state our main results below, let us introduce some further notations and terminologies.

Let $C_{r,m}$ denote the r-graph with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$. Denote

$$\lambda_m^r = \max\{\lambda(G) : G \text{ is an } r-\text{graph with } m \text{ edges}\}.$$

Definition 1.3. *An* r*-graph* G *with* m *edges is called an extremal* r*-graph if* $\lambda(G) = \lambda_m^r$*.*

Frankl and Füredi's conjecture says that $\lambda_m^r = \lambda(C_{r,m})$. To verify the truth of Conjecture 1.3, it is sufficient to show that $\lambda(G) \leq \lambda(C_{r,m})$ holds for every extremal r-graph G with m edges.

Definition 1.4. *An r-graph $G = ([n], E)$ is left-compressed if $j_1 j_2 \cdots j_r \in E$ implies $i_1 i_2 \cdots i_r \in E$ provided $i_p \leq j_p$ for every $p, 1 \leq p \leq r$.*

The following lemma implies that we only need to consider left-compressed extremal r-graphs to verify that $\lambda(G) \leq \lambda(C_{r,m})$ holds for every extremal r-graph G with m edges.

Lemma 1.5[5]**.** *There exists a left-compressed r-graph G with m edges such that $\lambda(G) = \lambda_m^r$.*

To emphasize this, let us make a remark.

Remark 1.6. To verify Conjecture 1.3, it is sufficient to show that for a left-compressed extremal r-graphs G with m edges, $\lambda(G) \leq \lambda(C_{r,m})$ holds.

Applying the following result showed in [5], we can further reduce the classes of 3-graphs to verify in order to verify Conjecture 1.3.

Lemma 1.7[5]**.** *Let m be a positive integer. Let $G = ([n], E)$ be a left-compressed extremal 3-graph with m edges. If $\vec{x} = (x_1, x_2, \ldots, x_n)$ is an optimal weighting for G satisfying $x_1 \geq x_2 \geq \cdots \geq x_k > x_{k+1} = \ldots = x_n = 0$. Then*

$$|E| \geq \binom{k-1}{3} + \binom{k-2}{2} - (k-2).$$

Remark 1.8. Let G be a left-compressed extremal 3-graph with m edges. Let t be a positive integer such that $\binom{t-1}{3} \leq m < \binom{t}{3}$. To show $\lambda(G) \leq \lambda(C_{3,m})$, we can assume G is on vertex set $[t]$.

Proof. Let $G = ([n], E)$ and t satisfy the conditions in this remark. Let $\vec{x} = (x_1, x_2, \cdots, x_n)$ be an optimal weighting for G satisfying $x_1 \geq x_2 \geq \cdots \geq x_k > x_{k+1} = \cdots = x_n = 0$. We claim that $k \leq t$. Otherwise $k \geq t+1$ and Lemma 1.8 implies that

$$\begin{aligned} m = |E| \geq & \binom{k-1}{3} + \binom{k-2}{2} - (k-2) \\ \geq & \binom{t}{3} + \binom{t-1}{2} - (t-1) \geq \binom{t}{3} \end{aligned}$$

which contradicts to the assumption that $\binom{t-1}{3} \leq m < \binom{t}{3}$. So to show $\lambda(G) \leq \lambda(C_{3,m})$, we can assume G is on $[t]$. □

In this paper, we first give the following partial result.

Theorem 1.9. *Let m, t, a and i be positive integers satisfying $m = \binom{t}{3} - a$, $3 \leq a \leq t-2$ and $i \geq 1$. Let $G = ([t], E)$ be a left-compressed 3-graph with m edges. If the triple with minimum colex ordering in G^c is $(t-2-i)(t-2)t$. Then $\lambda(G) \leq \lambda(C_{3,m})$.*

In [5], the following result was proved.

Theorem 1.10[5]**.** *Let m, t and a satisfy $-(t-2) \leq a \leq (t-5)$ and*

$$m = \binom{t-1}{3} + \binom{t-2}{2} + a.$$

Suppose G is a left-compressed extremal 3-graph with m edges. Then G and $C_{3,m}$ differ in at most $2(t-a-2)$ edges, i.e., $|E(G) \Delta E(C_{3,m})| \leq 2(t-a-2)$.

We show that

Theorem 1.11. *Let m be any positive integer. Let G be a left-compressed extremal 3-graph with m edges satisfying $|E(G)\Delta E(C_{3,m})| \le 6$. Then $\lambda(G) \le \lambda(C_{3,m})$.*

In the proof of Theorem 1.12, we will prove several lemmas in Section 3. These lemmas themselves provide partial results to Conjecture 1.3 as well.

Using Theorem 1.12, we can prove Conjecture 1.3 holds for $\binom{t}{3} - 6 \le m \le \binom{t}{3} - 3$ when $r = 3$.

Corollary 1.12. *Let m and t be positive integers satisfying $\binom{t}{3} - 6 \le m \le \binom{t}{3} - 3$. Let G be a 3-graph with m edges, then $\lambda(G) \le \lambda(C_{3,m})$.*

All proofs will be given in Section 3.

2 Useful Results

We will impose one additional condition on any optimal weighting $\vec{x} = (x_1, x_2, \dots, x_n)$ for an r-graph G:

$$\begin{aligned} &|\{i : x_i > 0\}| \text{ is minimal, i.e. if } \vec{y} \text{ is a legal weighting for } G \text{ satisfying} \\ &|\{i : y_i > 0\}| < |\{i : x_i > 0\}|, \text{ then } \lambda(G, \vec{y}) < \lambda(G). \end{aligned} \tag{1}$$

For an r-graph $G = (V, E)$ we denote the $(r-1)$-neighborhood of a vertex $i \in V$ by $E_i = \{A \in V^{(r-1)} : A \cup \{i\} \in E\}$. Similarly, we will denote the $(r-2)$-neighborhood of a pair of vertices $i, j \in V$ by $E_{ij} = \{B \in V^{(r-2)} : B \cup \{i, j\} \in E\}$. We denote the complement of E_i by $E_i^c = \{A \in V^{(r-1)} : A \cup \{i\} \in V^{(r)} \backslash E\}$. Also, we denote the complement of E_{ij} by $E_{ij}^c = \{B \in V^{(r-2)} : B \cup \{i, j\} \in V^{(r)} \backslash E\}$. Denote

$$E_{i \backslash j} = E_i \cap E_j^c.$$

Clearly, an r-graph $G = ([n], E)$ is *left-compressed* if and only if $E_{j \backslash i} = \emptyset$ for any $1 \le i < j \le n$.

When the theory of Lagrange multipliers is applied to find the optimum of $\lambda(G, \vec{x})$, subject to $\sum\limits_{i=1}^{n} x_i = 1$, notice that $\lambda(E_i, \vec{x})$ corresponds to the partial derivative of $\lambda(G, \vec{x})$ with respect to x_i. The following lemma gives some necessary conditions of an optimal weighting for G.

Lemma 2.1[2]**.** *Let $G = ([n], E)$ be an r-graph and $\vec{x} = (x_1, x_2, \dots, x_n)$ be an optimal weighting for G with k positive weights $x_1, x_2, \dots, x_k$ satisfying Condition (1). Then for every $\{i, j\} \in [k]^{(2)}$, (a) $\lambda(E_i, \vec{x}) = \lambda(E_j, \vec{x}) = r\lambda(G)$, (b) there is an edge in E containing both i and j.*

Remark. (a) In Lemma 2.1, part(a) implies that

$$x_j \lambda(E_{ij}, \vec{x}) + \lambda(E_{i \backslash j}, \vec{x}) = x_i \lambda(E_{ij}, \vec{x}) + \lambda(E_{j \backslash i}, \vec{x}).$$

In particular, if G is left-compressed, then

$$(x_i - x_j)\lambda(E_{ij}, \vec{x}) = \lambda(E_{i \backslash j}, \vec{x})$$

for any i, j satisfying $1 \le i < j \le k$ since $E_{j \backslash i} = \emptyset$.

(b) If G is left-compressed, then for any i, j satisfying $1 \le i < j \le k$,

$$x_i - x_j = \frac{\lambda(E_{i \backslash j}, \vec{x})}{\lambda(E_{ij}, \vec{x})} \tag{2}$$

holds. If G is left-compressed and $E_{i \backslash j} = \emptyset$ for i, j satisfying $1 \le i < j \le k$, then $x_i = x_j$.

(c) By (2), if G is left-compressed, then an optimal legal weighting $\vec{x}=(x_1,x_2,\cdots,x_n)$ for G must satisfy

$$x_1 \ge x_2 \ge \cdots \ge x_n \ge 0. \tag{3}$$

We also need the following lemma from [5] in the proof of our main results.

Lemma 2.3[5]**.** *For integers m,t, and r satisfying $\binom{t-1}{r} \le m \le \binom{t-1}{r}+\binom{t-2}{r-1}$, we have $\lambda(C_{r,m})=\lambda([t-1]^{(r)})$.*

3 Proof of Main Results

The partial ordered diagram (Figure 1) on all triples on $[t]$ as described below is useful to help us to analyze all possible left-compressed 3-graphs on $[t]$ systematically.

An r-tuple $i_1i_2\cdots i_r$ is called a *descendant* of an r-tuple $j_1j_2\cdots j_r$ if $i_s \le j_s$ for each $1\le s\le r$, and $i_1+i_2+\cdots+i_r < j_1+j_2+\cdots+j_r$. In this case, $j_1j_2\cdots j_r$ is called an *ancestor* of $i_1i_2\cdots i_r$. The r-tuple $i_1i_2\cdots i_r$ is called a *direct descendant* of $j_1j_2\cdots j_r$ if $i_1i_2\cdots i_r$ is a descendant of $j_1j_2\cdots j_r$ and $j_1+j_2+\cdots+j_r = i_1+i_2+\cdots+i_r+1$. We say that $j_1j_2\cdots j_r$ has lower hierarchy than $i_1i_2\cdots i_r$ if $j_1j_2\cdots j_r$ is an ancestor of $i_1i_2\cdots i_r$. This is a partial order on the set of all r-tuples. Figure 1 is a Hessian diagram on all triples on vertex set $[t]$. In this diagram, $i_1i_2i_3$ and $j_1j_2j_3$ are connected by an edge if and only if $i_1i_2i_3$ is a direct descendant of $j_1j_2j_3$.

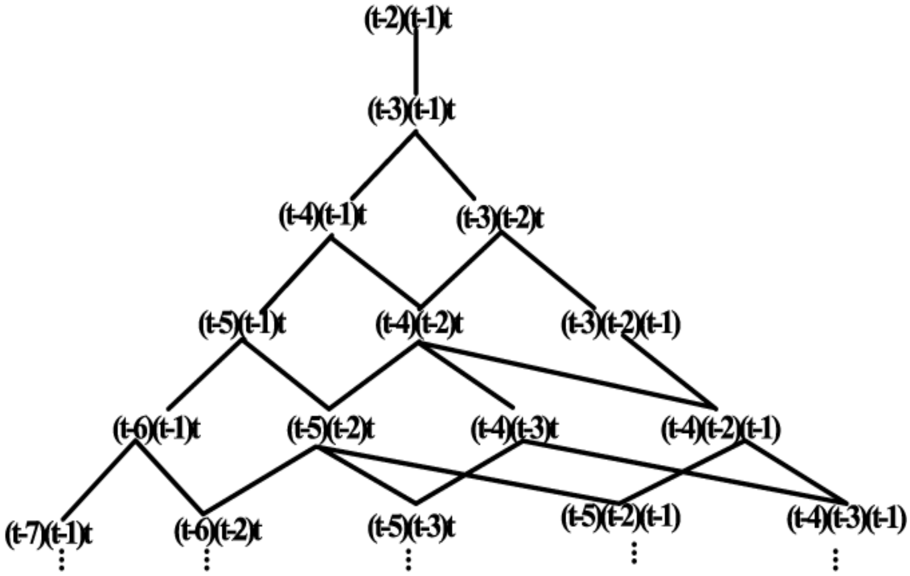

Figure 1. Hessian Diagram on $[t]^{(3)}$

3.1 Proof of Theorem 1.10

Proof. Since G is left-compressed, in view of Figure 1, then we have $a \ge 2i+1$.

To show that $\lambda(G)\le\lambda(C_{3,m})$, we will take an optimal weighting $\vec{x}$ for G, then we take a legal weighting, say $\vec{z}$ for $C_{3,m}$ by replacing a few coordinators of $\vec{x}$ and show that $\lambda(G,\vec{x})\le\lambda(C_{3,m},\vec{z})$. This would imply that

$$\lambda(G)=\lambda(G,\vec{x})\le\lambda(C_{3,m},\vec{z})\le\lambda(C_{3,m}).$$

Let us go into the details. Let $\vec{x}=(x_1,x_2,\cdots,x_t)$ be an optimal weighting for G satisfying $x_1\geq x_2\geq\cdots\geq x_t\geq 0$. First we point out that

$$\lambda(E_{1(t-2-i)},\vec{x})-\lambda(E_{(t-2)(t-1)},\vec{x})=x_{t-2}+x_{t-1}+x_t-x_1-x_{t-2-i}\geq 0. \tag{4}$$

To verify (4), by Remark 2.2 (b), we have

$$x_1 =x_{t-1}+\frac{\lambda(E_{1\backslash(t-1)},\vec{x})}{\lambda(E_{1(t-1)},\vec{x})}\leq x_{t-1}+\frac{(x_2+\cdots+x_{t-2})x_t}{x_2+\cdots+x_{t-2}+x_t}\leq x_{t-1}+x_t; \tag{5}$$

$$\begin{aligned}x_1 =&x_{t-2}+\frac{\lambda(E_{1\backslash(t-2)},\vec{x})}{\lambda(E_{1(t-2)},\vec{x})}\\ =&x_{t-2}+\frac{x_{t-2-i}+\cdots+x_{t-3}+x_{t-1}}{1-x_1-x_{t-2}}x_t\\ \leq&x_{t-2}+\frac{x_{t-2-i}+\cdots+x_{t-3}+x_{t-1}}{1-x_{t-2}-x_{t-1}-x_t}x_t \qquad \text{(By (5))}\\ \leq&x_{t-2}+\frac{x_1+x_2+\cdots+x_i+x_{t-2}}{1-x_{t-2-i}-x_{t-1}-x_t}x_t;\end{aligned} \tag{6}$$

and

$$\begin{aligned}x_{t-2-i} =&x_{t-1}+\frac{\lambda(E_{(t-2-i)\backslash(t-1)},\vec{x})}{\lambda(E_{(t-2-i)(t-1)},\vec{x})}\\ =&x_{t-1}+\frac{(x_{t-3-(a-i-2)}+x_{t-3-(a-i-1)}+\cdots+x_{t-3})-x_{t-2-i}}{1-x_{t-2-i}-x_{t-1}-x_t}x_t\\ \leq&x_{t-1}+\frac{(x_{i+1}+x_{i+2}+\cdots+x_{a-1})-x_{t-2-i}}{1-x_{t-2-i}-x_{t-1}-x_t}x_t\end{aligned} \tag{7}$$

since $a\leq t-2$. Adding (6) and (7), we obtain that

$$\begin{aligned}x_1+x_{t-2-i} \leq&x_{t-2}+x_{t-1}+\frac{(x_1+\cdots+x_{a+1})-x_{t-2-i}}{1-x_{t-2-i}-x_{t-1}-x_t}x_t\\ \leq&x_{t-2}+x_{t-1}+\frac{(x_1+\cdots+x_{t-2})-x_{t-2-i}}{1-x_{t-2-i}-x_{t-1}-x_t}x_t\\ =&x_{t-2}+x_{t-1}+\frac{1-x_{t-1}-x_t-x_{t-2-i}}{1-x_{t-2-i}-x_{t-1}-x_t}x_t\\ =&x_{t-2}+x_{t-1}+x_t.\end{aligned}$$

So, (4) is true. This implies that $\lambda(E_{(t-2)(t-1)},\vec{x})\leq\lambda(E_{1(t-2-i)},\vec{x})$. In what follows, we divide the rest of the proof into three cases: $a=2i+1$, $a=2i+2$, and $a\geq 2i+3$.

We first consider the case that $a\geq 2i+3$. By Remark 2.2 (b), we have $x_1=x_2=\cdots=x_{t-a-2+i}$ and $x_{t-2-i}=\cdots=x_{t-3}$. Hence $\lambda(C_{3,m},\vec{x})-\lambda(G,\vec{x})=i(x_{t-2-i}x_{t-2}x_t-x_1x_{t-1}x_t)$. Also by Remark 2.2 (b), we have

$$x_1=x_{t-2-i}+\frac{\lambda(E_{1\backslash(t-2-i)},\vec{x})}{\lambda(E_{1(t-2-i)},\vec{x})}=x_{t-2-i}+\frac{(x_{t-1}+x_{t-2})x_t}{\lambda(E_{1(t-2-i)},\vec{x})}$$

and

$$\begin{aligned}x_{t-2} =&x_{t-1}+\frac{\lambda(E_{(t-2)\backslash(t-1)},\vec{x})}{\lambda(E_{(t-2)(t-1)},\vec{x})}\\ =&x_{t-1}+\frac{(x_{t-3-i}+\cdots+x_{t-a-1+i})x_t}{\lambda(E_{(t-2)(t-1)},\vec{x})}.\end{aligned}$$

Recall that $a \geq 2i+3$ and $\lambda(E_{(t-2)(t-1)}, \vec{x}) \leq \lambda(E_{1(t-2-i)}, \vec{x})$. We have $x_{t-2} - x_{t-1} \geq x_1 - x_{t-2-i}$. Hence

$$\begin{aligned}\lambda(C_{3,m}, \vec{x}) - \lambda(G, \vec{x}) =& i(x_{t-2-i}x_{t-2}x_t - x_1x_{t-1}x_t) \\ =& i[x_{t-2-i}(x_{t-2} + x_{t-1} - x_{t-1})x_t - x_1x_{t-1}x_t] \\ \geq& i[x_{t-2-i}(x_{t-1} + x_1 - x_{t-2-i})x_t - x_1x_{t-1}x_t] \\ =& i(x_{t-2-i} - x_{t-1})(x_1 - x_{t-2-i})x_t \geq 0. \end{aligned} \tag{8}$$

Therefore $\lambda(C_{3,m}) \geq \lambda(C_{3,m}, \vec{x}) \geq \lambda(G, \vec{x}) = \lambda(G)$ in this case.

Next, we consider the case that $a = 2i+2$. Let $G' = ([t], E'), E' = E \cup \{(t-2-i)(t-2)t\} \backslash \{(t-4-i)(t-1)t\}$, then $\lambda(G') \leq \lambda(C_{3,m})$ by the case $a = 2(i-1)+4$. (Note that $G' = C_{3,m}$ when $i - 1 = 0$.) So it is sufficient to prove that $\lambda(G) \leq \lambda(G')$. Clearly,

$$\lambda(G', \vec{x}) - \lambda(G, \vec{x}) = x_{t-2-i}x_{t-2}x_t - x_{t-4-i}x_{t-1}x_t = x_{t-2-i}x_{t-2}x_t - x_1x_{t-1}x_t. \tag{9}$$

Consider a new weighting $\vec{y} = (y_1, y_2, \cdots, y_t)$ given by $y_j = x_j$ for $j \neq t-4-i$, $j \neq t-2-i$ and $y_{t-4-i} = x_{t-4-i} - \delta$, $y_{t-2-i} = x_{t-2-i} + \delta$. Then

$$\begin{aligned}\lambda(G', \vec{y}) - \lambda(G', \vec{x}) =& \delta[\lambda(E'_{t-2-i}, \vec{x}) - \lambda(E'_{t-4-i}, \vec{x})] - \delta^2\lambda(E'_{(t-4-i)(t-2-i)}, \vec{x}) \\ =& \delta(x_{t-4-i} - x_{t-2-i})\lambda(E'_{(t-4-i)(t-2-i)}, \vec{x}) - \delta^2\lambda(E'_{(t-4-i)(t-2-i)}, \vec{x}).\end{aligned}$$

Let $\delta = \frac{x_{t-4-i} - x_{t-2-i}}{2}$. Clearly, $\vec{y} = (y_1, y_2, \cdots, y_t)$ is also a legal weighting and

$$\begin{aligned}\lambda(G', \vec{y}) - \lambda(G', \vec{x}) =& \frac{(x_{t-4-i} - x_{t-2-i})^2}{4}\lambda(E'_{(t-4-i)(t-2-i)}, \vec{x}) \\ =& \frac{(x_1 - x_{t-2-i})^2}{4}\lambda(E_{1(t-2-i)}, \vec{x}).\end{aligned} \tag{10}$$

Let $\vec{z} = (z_1, z_2, \cdots, z_t)$ given by $z_j = y_j$ for $j \neq t-2$, $j \neq t-1$ and $z_{t-2} = y_{t-2} + \eta$, $z_{t-1} = y_{t-1} - \eta$. Then

$$\begin{aligned}\lambda(G', \vec{z}) - \lambda(G', \vec{y}) =& \eta[\lambda(E'_{t-2}, \vec{y}) - \lambda(E'_{t-1}, \vec{y})] - \eta^2\lambda('E_{(t-2)(t-1)}, \vec{y}) \\ =& \eta[(y_{t-2-i}y_t + y_{t-3-i}y_t + y_{t-4-i}y_t) - (y_{t-2} - y_{t-1})\lambda(E'_{(t-2)(t-1)}, \vec{y})] \\ &- \eta^2\lambda(E'_{(t-2)(t-1)}, \vec{y}).\end{aligned} \tag{11}$$

Let

$$\begin{aligned}\eta =& \frac{(y_{t-2-i} + y_{t-3-i} + y_{t-4-i})y_t - (y_{t-2} - y_{t-1})\lambda(E'_{(t-2)(t-1)}, \vec{y})}{2\lambda(E'_{(t-2)(t-1)}, \vec{y})} \\ =& \frac{(x_{t-2-i} + x_{t-3-i} + x_{t-4-i})x_t - (x_{t-2} - x_{t-1})\lambda(E_{(t-2)(t-1)}, \vec{x})}{2\lambda(E_{(t-2)(t-1)}, \vec{x})}.\end{aligned}$$

By Remark 2.2 (b), we have

$$x_{t-2} = x_{t-1} + \frac{x_{t-3-i}x_t}{\lambda(E_{(t-2)(t-1)}, \vec{x})}. \tag{12}$$

Hence, $\eta = \frac{(x_{t-2-i} + x_{t-4-i})x_t}{2\lambda(E_{(t-2)(t-1)}, \vec{x})}$ and $\vec{z} = (z_1, z_2, \cdots, z_t)$ is also a legal weighting and

$$\lambda(G', \vec{z}) - \lambda(G', \vec{y}) = \frac{(x_{t-2-i} + x_{t-4-i})^2x_t^2}{4\lambda(E_{(t-2)(t-1)}, \vec{x})}. \tag{13}$$

By Remark 2.2 (b), we have

$$x_1 = x_{t-i-2} + \frac{x_{t-2}x_t + x_{t-1}x_t}{\lambda(E_{1(t-i-2)}, \vec{x})}. \tag{14}$$

Combing (9), (10), (13) and (14), we have

$$\begin{aligned}\lambda(G', \vec{z}) - \lambda(G, \vec{x}) =& x_{t-2-i}x_{t-2}x_t - x_1x_{t-1}x_t + \frac{(x_{t-2}+x_{t-1})^2x_t^2}{4\lambda(E_{1(t-i-2)}, \vec{x})} + \frac{(x_{t-2-i}+x_{t-4-i})^2x_t^2}{4\lambda(E_{(t-2)(t-1)}, \vec{x})}\\ =&[x_1 - \frac{x_{t-2}x_t + x_{t-1}x_t}{\lambda(E_{1(t-i-2)}, \vec{x})}]x_{t-2}x_t - x_1x_{t-1}x_t\\ &+ \frac{(x_{t-2}+x_{t-1})^2x_t^2}{4\lambda(E_{1(t-i-2)}, \vec{x})} + \frac{(x_{t-2-i}+x_{t-4-i})^2x_t^2}{4\lambda(E_{(t-2)(t-1)}, \vec{x})}\\ \geq& \frac{x_t^2}{4\lambda(E_{1(t-i-2)}, \vec{x})}[-4(x_{t-2}+x_{t-1})x_{t-2} + (x_{t-2}+x_{t-1})^2 + 4x_{t-2}^2)]\\ =& \frac{x_t^2}{4\lambda(E_{1(t-i-2)}, \vec{x})}(x_{t-2}-x_{t-1})^2 \geq 0.\end{aligned}$$

Hence $\lambda(G') \geq \lambda(G', \vec{z}) \geq \lambda(G, \vec{x}) = \lambda(G)$ in this case.

What remains is the case that $a = 2i+1$. Let $G'' = ([t], E''), E'' = E\bigcup\{(t-2-i)(t-2)t\}\backslash\{(t-3-i)(t-1)t\}$, then $\lambda(G'') \leq \lambda(C_{3,m})$ by the case $a = 2(i-1)+3$. (Note that $G'' = C_{3,m}$ when $i-1=0$.) So it is sufficient to prove that $\lambda(G) \leq \lambda(G'')$. Clearly,

$$\lambda(G'', \vec{x}) - \lambda(G, \vec{x}) = x_{t-2-i}x_{t-2}x_t - x_{t-3-i}x_{t-1}x_t = x_{t-2-i}x_{t-2}x_t - x_1x_{t-1}x_t. \tag{15}$$

Consider a new weighting $\vec{u} = (u_1, u_2, \cdots, u_t)$ given by $u_j = x_j$ for $j \neq t-2-i,\ j \neq t-3-i$ and $u_{t-2-i} = x_{t-2-i} + \alpha,\ u_{t-3-i} = x_{t-3-i} - \alpha$. Then

$$\lambda(G'', \vec{u}) - \lambda(G'', \vec{x}) = \alpha(x_{t-3-i} - x_{t-2-i})\lambda(E''_{(t-3-i)(t-2-i)}, \vec{x}) - \alpha^2\lambda(E''_{(t-3-i)(t-2-i)}, \vec{x}).$$

Let $\alpha = \frac{x_{t-3-i}-x_{t-2-i}}{2}$. Clearly, $\vec{u} = (u_1, u_2, \cdots, u_t)$ is also a legal weighting and

$$\begin{aligned}\lambda(G'', \vec{u}) - \lambda(G'', \vec{x}) =& \frac{(x_{t-i-3}-x_{t-2-i})^2}{4}\lambda(E''_{(t-3-i)(t-2-i)}, \vec{x})\\ =& \frac{(x_1-x_{t-2-i})^2}{4}\lambda(E_{1(t-2-i)}, \vec{x}).\end{aligned} \tag{16}$$

Let $\vec{v} = (v_1, v_2, \cdots, v_t)$ given by $v_j = u_j$ for $j \neq t-2,\ j \neq t-1$ and $v_{t-2} = u_{t-2} + \beta$, $v_{t-1} = u_{t-1} - \beta$. Then

$$\begin{aligned}\lambda(G'', \vec{v}) - \lambda(G'', \vec{u}) =& \beta[\lambda(E''_{t-2}, \vec{u}) - \lambda(E''_{t-1}, \vec{u})] - \beta^2\lambda(E''_{(t-2)(t-1)}, \vec{u})\\ =& \beta(u_{t-2-i}u_t + u_{t-3-i}u_t) - \beta^2\lambda(E''_{(t-2)(t-1)}, \vec{u}).\end{aligned} \tag{17}$$

Let $\beta = \frac{u_{t-2-i}u_t + u_{t-3-i}u_t}{2\lambda(E''_{(t-2)(t-1)}, \vec{u})}$. Clearly, $\beta < u_t$. Hence, $\vec{v} = (v_1, v_2, \ldots, v_t)$ is also a legal weighting and

$$\lambda(G'', \vec{v}) - \lambda(G'', \vec{u}) = \frac{(u_{t-2-i}+u_{t-3-i})^2u_t^2}{4\lambda(E''_{(t-2)(t-1)}, \vec{u})} = \frac{(x_{t-2-i}+x_{t-3-i})^2x_t^2}{4\lambda(E_{(t-2)(t-1)}, \vec{x})}. \tag{18}$$

By Remark 2.2 (b), we have $x_{t-2} = x_{t-1}$ and

$$x_1 = x_{t-2-i} + \frac{2x_{t-1}x_t}{\lambda(E_{1(t-2-i)}, \vec{x})}. \tag{19}$$

Combing (15), (16), (18) and (19), we have

$$\begin{aligned}\lambda(G'',\vec{v})-\lambda(G,\vec{x}) =&x_{t-2-i}x_{t-2}x_t - x_1x_{t-1}x_t + \frac{(x_{t-2-i}+x_{t-3-i})^2x_t^2}{4\lambda(E_{(t-2)(t-1)},\vec{x})} + \frac{x_{t-1}^2x_t^2}{\lambda(E_{1(t-2-i)},\vec{x})}\\ =&-\frac{2x_{t-1}^2x_t^2}{\lambda(E_{1(t-2-i)},\vec{x})} + \frac{(x_{t-2-i}+x_{t-3-i})^2x_t^2}{4\lambda(E_{(t-2)(t-1)},\vec{x})} + \frac{x_{t-1}^2x_t^2}{\lambda(E_{1(t-2-i)},\vec{x})}\\ \geq&0\end{aligned}$$

since $\lambda(E_{(t-2)(t-1)},\vec{x}) \leq \lambda(E_{1(t-2-i)},\vec{x})$. Hence $\lambda(G'') \geq \lambda(G'',\vec{z}) \geq \lambda(G,\vec{x}) = \lambda(G)$. This completes the proof of Theorem 1.10. □

3.2 Proof of Theorem 1.12

Remark 3.1. Let G be a left-compressed extremal 3-graph with m edges satisfying

$$|E(G)\Delta E(C_{3,m})| \leq 6.$$

Let t be a positive integer such that $\binom{t-1}{3} \leq m < \binom{t}{3}$. To show $\lambda(G) \leq \lambda(C_{3,m})$, we can assume G is on vertex set $[t]$.

Proof. The proof is excatly the same as the proof of Remark 1.9. □

Let us be aware of the following simple observation for left-compressed r-graphs.

Remark 3.2. An r-graph G is left-compressed if and only if all descendants of an edge of G are edges of G. Equivalently, if an r-tuple is not an edge of G, then none of its ancestors will be an edge of G.

By analyzing possible cases under the assumption that G is a left-compressed extremal graph on $[t]$ satisfying $|E(G)\triangle E(C_{3,m})| \leq 6$, we give several lemmas to cover the possible cases below.

Using Theorem 1.10, we deal with the case when the graph G contains a clique of order $t-1$ in Lemma 3.3.

Lemma 3.3. *Let m, a and t be positive integers satisfying $m = \binom{t}{3} - a$, where $3 \leq a \leq t-2$. Let $G = ([t], E)$ be a left-compressed 3-graph with m edges and a clique of order $t-1$. If $|E\triangle E(C_{3,m})| \leq 6$, then $\lambda(G) \leq \lambda(C_{3,m})$.*

Proof. If the triple with the minimum colex ordering in G^c is $(t-2-i)(t-2)t$, where $i = 1, 2, 3$. Then $\lambda(G) \leq \lambda(C_{3,m})$ by Theorem 1.10. So we can assume that the triple with the minimum colex ordering in G^c is $(t-4)(t-3)t$. Let $\vec{x} = (x_1, x_2, \ldots, x_t)$ be an optimal weighting for G satisfying $x_1 \geq x_2 \geq \ldots \geq x_t \geq 0$.

First we consider the case $a \geq 7$. Let $G' = ([t], E')$, $E' = E \bigcup \{(t-4)(t-3)t\} \backslash \{(t-5)(t-2)t\}$, then $\lambda(G') \leq \lambda(C_{3,m})$ by Theorem 1.10. So it is sufficient to prove that $\lambda(G) \leq \lambda(G')$. Note that

$$\lambda(G',\vec{x}) - \lambda(G,\vec{x}) = x_{t-4}x_{t-3}x_t - x_{t-5}x_{t-2}x_t. \quad (20)$$

Consider a new weighting $\vec{y} = (y_1, y_2, \ldots, y_t)$ given by $y_j = x_j$ for $j \neq t-5$, $j \neq t-4$ and $y_{t-5} = x_{t-5} - \delta$, $y_{t-4} = x_{t-4} + \delta$. Therefore

$$\begin{aligned}\lambda(G',\vec{y}) - \lambda(G',\vec{x}) =&\delta[\lambda(E'_{t-4},\vec{x}) - \lambda(E'_{t-5},\vec{x})] - \delta^2\lambda(E'_{(t-5)(t-4)},\vec{x})\\ =&\delta(x_{t-5}-x_{t-4})\lambda(E'_{(t-5)(t-4)},\vec{x}) - \delta^2\lambda(E'_{(t-5)(t-4)},\vec{x}).\end{aligned}$$

Let $\delta = \frac{x_{t-5}-x_{t-4}}{2}$. Clearly, $\vec{y} = (y_1, y_2, \cdots, y_t)$ is also a legal weighting. Note that

$$\lambda(E'_{(t-5)(t-4)},\vec{x}) = \lambda(E_{(t-5)(t-4)},\vec{x}),$$

so

$$\lambda(G',\vec{y})-\lambda(G',\vec{x})=\frac{(x_{t-5}-x_{t-4})^2}{4}\lambda(E'_{(t-5)(t-4)},\vec{x})=\frac{(x_{t-5}-x_{t-4})^2}{4}\lambda(E_{(t-5)(t-4)},\vec{x}). \quad (21)$$

Let $\vec{z}=(z_1,z_2,\cdots,z_t)$ given by $z_j=y_j$ for $j\neq t-3$, $j\neq t-2$ and $z_{t-3}=y_{t-3}+\eta$, $z_{t-2}=y_{t-2}-\eta$. Then

$$\begin{aligned}\lambda(G',\vec{z})-\lambda(G',\vec{y})=&\eta[\lambda(E'_{t-3},\vec{y})-\lambda(E'_{t-2},\vec{y})]-\eta^2\lambda(E'_{(t-3)(t-2)},\vec{y})\\=&\eta(y_{t-4}y_t+y_{t-5}y_t)-\eta^2\lambda(E'_{(t-3)(t-2)},\vec{y}).\end{aligned} \quad (22)$$

Let $\eta=\frac{y_{t-5}y_t+y_{t-4}y_t}{2\lambda(E'_{(t-3)(t-2)},\vec{y})}$. Clearly, $\eta<y_t$. Hence, $\vec{z}=(z_1,z_2,\cdots,z_t)$ is also a legal weighting and

$$\lambda(G',\vec{z})-\lambda(G',\vec{y})=\frac{(y_{t-5}+y_{t-4})^2y_t^2}{4\lambda(E'_{(t-3)(t-2)},\vec{y})}=\frac{(x_{t-5}+x_{t-4})^2x_t^2}{4\lambda(E_{(t-3)(t-2)},\vec{x})}. \quad (23)$$

By Remark 2.2 (b), we have $x_{t-4}=x_{t-3}=x_{t-2}$ and

$$x_{t-5}=x_{t-4}+\frac{2x_{t-2}x_t}{\lambda(E_{(t-5)(t-4)},\vec{x})}. \quad (24)$$

Also,

$$\lambda(E_{(t-5)(t-4)},\vec{x})-\lambda(E_{(t-3)(t-2)},\vec{x})\geq x_{t-2}+x_t-x_{t-5}\geq 0 \quad (25)$$

since

$$x_{t-5}\leq x_1=x_{t-1}+\frac{\lambda(E_{1\setminus(t-1)},\vec{x})}{\lambda(E_{1(t-1)},\vec{x})}\leq x_{t-1}+\frac{(x_2+\cdots+x_{t-2})x_t}{x_2+\cdots+x_{t-2}+x_t}\leq x_{t-1}+x_t\leq x_{t-2}+x_t.$$

Combining (20), (21), (23) and (24), we have

$$\begin{aligned}\lambda(G',\vec{z})-\lambda(G,\vec{x})\geq&x_{t-4}x_{t-3}x_t-x_{t-5}x_{t-2}x_t+\frac{(x_{t-5}+x_{t-4})^2x_t^2}{4\lambda(E_{(t-3)(t-2)},\vec{x})}+\frac{x_{t-2}^2x_t^2}{\lambda(E_{(t-5)(t-4)},\vec{x})}\\=&-\frac{2x_{t-2}^2x_t^2}{\lambda(E_{(t-5)(t-4)},\vec{x})}+\frac{(x_{t-5}+x_{t-4})^2x_t^2}{4\lambda(E_{(t-3)(t-2)},\vec{x})}+\frac{x_{t-2}^2x_t^2}{\lambda(E_{(t-5)(t-4)},\vec{x})}\\\geq&0\end{aligned}$$

since $\lambda(E_{(t-5)(t-4)},\vec{x})\geq\lambda(E_{(t-3)(t-2)},\vec{x})$ in view of (25). Hence $\lambda(G')\geq\lambda(G',\vec{z})\geq\lambda(G,\vec{x})=\lambda(G)$.

Next, we consider the case when $a\leq 6$. Clearly the lemma holds if $a=3$, 4 and 5 in view of Theorem 1.10. So the only remaining case is that $a=6$. In view of Figure 1, we have $E(G)=E=[t]^{(3)}\setminus\{(t-2)(t-1)t,(t-3)(t-1)t,(t-4)(t-1)t,(t-3)(t-2)t,(t-4)(t-2)t,(t-4)(t-3)t\}$. In this case, $E(C_{3,m})=E''=[t]^{(3)}\setminus\{(t-2)(t-1)t,(t-3)(t-1)t,(t-4)(t-1)t,(t-5)(t-1)t,(t-6)(t-1)t,(t-7)(t-1)t\}$. By Remark 2.2 (b), we have

$$x_1=x_2=\cdots=x_{t-5}=\alpha;\quad x_{t-4}=x_{t-3}=x_{t-2}=x_{t-1}=\beta;\quad x_t=\gamma;$$

and

$$\begin{aligned}\alpha-\beta=&x_{t-5}-x_{t-4}=\frac{\lambda(E_{(t-5)\setminus(t-4)},\vec{x})}{\lambda(E_{(t-5)(t-4)},\vec{x})}=\frac{x_{t-2}x_t+x_{t-3}x_t+x_{t-4}x_t}{\lambda(E_{(t-5)(t-4)},\vec{x})}\\=&\frac{3\beta\gamma}{\lambda(E_{(t-5)(t-4)},\vec{x})}\leq\gamma.\end{aligned}$$

Then

$$\lambda(C_{3,m},\vec{x})-\lambda(G,\vec{x}) =3\beta^2\gamma-3\alpha\beta\gamma=-\frac{9\beta^2\gamma^2}{\lambda(E_{(t-5)(t-4)},\vec{x})}=-\frac{9\beta^2\gamma^2}{\lambda(E_{(t-6)(t-4)},\vec{x})}$$
$$=-\frac{9\beta^2\gamma^2}{\lambda(E_{(t-7)(t-4)},\vec{x})}. \tag{27}$$

Let $\vec{u}=(u_1,u_2,\cdots,u_t)$ given by $u_j=x_j$ for $j\neq t-7$, $j\neq t-4$ and $u_{t-7}=x_{t-7}-\delta,\ \ u_{t-4}=x_{t-4}+\delta$. Then

$$\lambda(C_{3,m},\vec{u})-\lambda(C_{3,m},\vec{x}) =\delta[\lambda(E''_{t-4},\vec{x})-\lambda(E''_{t-7},\vec{x})]-\delta^2\lambda(E''_{(t-7)(t-4)},\vec{x})$$
$$=\delta(x_{t-7}-x_{t-4})\lambda(E''_{(t-7)(t-4)},\vec{x})-\delta^2\lambda(E''_{(t-7)(t-4)},\vec{x}).$$

Let $\delta=\frac{x_{t-7}-x_{t-4}}{2}=\frac{\alpha-\beta}{2}$. Clearly, $\vec{u}=(u_1,u_2,\cdots,u_t)$ is also a legal weighting, $\lambda(E''_{(t-7)(t-4)},\vec{x})=\lambda(E_{(t-7)(t-4)},\vec{x})$ and

$$\lambda(C_{3,m},\vec{u})-\lambda(C_{3,m},\vec{x}) =\frac{(\alpha-\beta)^2}{4}\lambda(E''_{(t-7)(t-4)},\vec{x})$$
$$=\frac{9\beta^2\gamma^2}{4[\lambda(E_{(t-7)(t-4)},\vec{x})]^2}\lambda(E''_{(t-7)(t-4)},\vec{x})$$
$$=\frac{9\beta^2\gamma^2}{4\lambda(E_{(t-7)(t-4)},\vec{x})}. \tag{28}$$

Similarly, let $\vec{v}=(v_1,v_2,\ldots,v_t)$ given by $v_j=u_j$ for $j\neq t-6$, $j\neq t-3$ and $v_{t-6}=u_{t-6}-\frac{\alpha-\beta}{2}$, $v_{t-3}=u_{t-3}+\frac{\alpha-\beta}{2}$. Then

$$\lambda(C_{3,m},\vec{v})-\lambda(C_{3,m},\vec{u})=\frac{9\beta^2\gamma^2}{4\lambda(E_{(t-7)(t-4)},\vec{x})}. \tag{29}$$

Let $\vec{w}=(w_1,w_2,\cdots,w_t)$ given by $w_j=v_j$ for $j\neq t-5,\ \ j\neq t-2$ and $w_{t-5}=v_{t-5}-\frac{\alpha-\beta}{2}$, $w_{t-2}=v_{t-2}+\frac{\alpha-\beta}{2}$. Then

$$\lambda(C_{3,m},\vec{w})-\lambda(C_{3,m},\vec{v})=\frac{9\beta^2\gamma^2}{4\lambda(E_{(t-7)(t-4)},\vec{x})}. \tag{30}$$

Let $\vec{p}=(p_1,p_2,\cdots,p_t)$ given by $p_j=w_j$ for $j\neq t-1,\ \ j\neq t$ and $p_{t-1}=w_{t-1}-\frac{\beta-\gamma}{2}$, $p_t=w_t+\frac{\beta-\gamma}{2}$. Then

$$\lambda(C_{3,m},\vec{p})-\lambda(C_{3,m},\vec{w})=\frac{9\beta^4\lambda(E''_{(t-1)t},\vec{x})}{4[\lambda(E_{(t-1)t},\vec{x})]^2}. \tag{31}$$

Frankl and Füredi's conjecture holds for $t\leq 8$ by [3], so we assume $t\geq 9$ next. By Remark 2.2 (b),

$$\alpha-\beta=\frac{3\beta\gamma}{\lambda(E_{(t-5)(t-4)},\vec{x})}=\frac{3\beta\gamma}{(t-6)\alpha+3\beta+\gamma}\leq\frac{3\beta}{7}, \tag{32}$$

so $\alpha\leq\frac{10}{7}\beta$. Also,

$$\beta-\gamma=x_{t-1}-x_t=\frac{\lambda(E_{(t-1)\setminus t},\vec{x})}{\lambda(E_{(t-1)t},\vec{x})}=\frac{3\beta^2}{(t-5)\alpha}. \tag{33}$$

Hence

$$\frac{\gamma}{\beta} = 1 - \frac{3\beta}{(t-5)\alpha} \le \frac{t - \frac{71}{10}}{t-5}. \tag{34}$$

So

$$\begin{aligned}
&\frac{\lambda(E''_{(t-1)t}, \vec{x})\lambda(E_{(t-5)(t-4)}, \vec{x})}{[\lambda(E_{(t-1)t}, \vec{x})]^2} \\
=&\frac{(t-8)\alpha[(t-6)\alpha + 3\beta + \gamma]}{(t-5)^2\alpha^2} \ge \frac{(t-8)(t-4)\alpha^2}{(t-5)^2\alpha^2} = \frac{(t-8)(t-4)}{(t-5)^2} \\
\ge&\frac{(t-\frac{71}{10})^2}{(t-5)^2} \ge \frac{\gamma^2}{\beta^2}.
\end{aligned} \tag{35}$$

Combining (27)–(31) and (35), we have

$$\lambda(C_{3,m}, \vec{p}) - \lambda(G, \vec{x}) = \frac{9\beta^4\lambda(E''_{(t-1)t})}{4[\lambda(E_{(t-1)t}, \vec{x})]^2} - \frac{9\beta^2\gamma^2}{4\lambda(E_{(t-5)(t-4)}, \vec{x})} \ge 0. \tag{36}$$

Hence $\lambda(C_{3,m}) \ge \lambda(C_{3,m}, \vec{p}) \ge \lambda(G, \vec{x}) = \lambda(G)$. This completes the proof of Lemma 3.3. □

Using Lemma 3.3, we prove the next four lemmas which cover the cases when the 3-graph G does not contain a clique of order $t-1$.

Lemma 3.4. *Let $G = ([t], E)$ and $G' = ([t], E')$ be left-compressed 3-graphs with $m = \binom{t}{3} - a$ edges, where $5 \le a \le t-2$, satisfying $|E \triangle E(C_{3,m})| = |E' \triangle E(C_{3,m})| = 4$ and the triples with the minimum colex ordering in G^c and G'^c are $(t-3)(t-2)(t-1)$ and $(t-4)(t-2)t$ respectively. Then $\lambda(G) \le \lambda(G') \le \lambda(C_{3,m})$.*

Proof. By Lemma 3.3, $\lambda(G') \le \lambda(C_{3,m})$. So it is sufficient to show $\lambda(G) \le \lambda(G')$. Let $\vec{x} = (x_1, x_2, \cdots, x_t)$ be an optimal weighting for G satisfying $x_1 \ge x_2 \ge \cdots \ge x_t \ge 0$. By Remark 2.2 (b), $x_{t-2} = x_{t-3}$ and $x_{t-1} = x_t$. Hence

$$\lambda(G', \vec{x}) - \lambda(G, \vec{x}) = (x_{t-3} - x_{t-4})x_{t-2}x_{t-1}. \tag{37}$$

Consider a new weighting $\vec{y} = (y_1, y_2, \cdots, y_t)$ given by $y_j = x_j$ for $j \ne t-4,\ j \ne t-3$ and $y_{t-4} = x_{t-4} - \delta,\ y_{t-3} = x_{t-3} + \delta$. Then

$$\begin{aligned}
\lambda(G', \vec{y}) - \lambda(G', \vec{x}) =&\delta[\lambda(E'_{t-3}, \vec{x}) - \lambda(E'_{t-4}, \vec{x})] - \delta^2\lambda(E'_{(t-4)(t-3)}, \vec{y}) \\
=&\delta(x_{t-4} - x_{t-3})\lambda(E'_{(t-4)(t-3)}, \vec{x}) - \delta^2\lambda(E'_{(t-4)(t-3)}, \vec{x}).
\end{aligned} \tag{38}$$

Let $\delta = \frac{x_{t-4}-x_{t-3}}{2}$. Clearly, $\vec{y} = (y_1, y_2, \cdots, y_t)$ is also a legal weighting. Also note that $\lambda(E'_{(t-4)(t-3)}, \vec{x}) = \lambda(E_{(t-4)(t-3)}, \vec{x})$. Hence

$$\lambda(G', \vec{y}) - \lambda(G', \vec{x}) = \frac{(x_{t-4} - x_{t-3})^2}{4}\lambda(E'_{(t-4)(t-3)}, \vec{x}) = \frac{(x_{t-4} - x_{t-3})^2}{4}\lambda(E_{(t-4)(t-3)}, \vec{x}). \tag{39}$$

Let $\vec{z} = (z_1, z_2, \cdots, z_t)$ be given by $z_i = y_i$ for $i \ne t-1,\ i \ne t$ and $z_{t-1} = y_{t-1} + \eta,\ z_t = y_t - \eta$. Then

$$\begin{aligned}
\lambda(G', \vec{z}) - \lambda(G', \vec{y}) =&\eta[\lambda(E'_{t-1}, \vec{y}) - \lambda(E'_t, \vec{y})] - \eta^2\lambda(E'_{(t-1)t}, \vec{y}) \\
=&\eta(y_{t-3}y_{t-2} + y_{t-4}y_{t-2}) - \eta^2\lambda(E'_{(t-1)t}, \vec{y}) \\
=&\eta(x_{t-3}x_{t-2} + x_{t-4}x_{t-2}) - \eta^2\lambda(E_{(t-1)t}, \vec{x})
\end{aligned} \tag{40}$$

in view of $y_{t-4}+y_{t-3}=x_{t-4}+x_{t-3}$, $y_{t-2}=x_{t-2}$ and $\lambda(E'_{(t-1)t},\vec{y})=\lambda(E_{(t-1)t},\vec{x})$. Let $\eta=\frac{x_{t-3}x_{t-2}+x_{t-4}x_{t-2}}{2\lambda(E_{(t-1)t},\vec{x})}$. By the condition of $|E\triangle E(C_{3,m})|=4$ we have $\{1,2\}\subseteq E_{(t-1)t}$, so

$$\eta\le\frac{x_{t-2}}{2}.$$

Applying Remark 2.2 (b), we have

$$\begin{aligned}x_{t-2}=&x_t+\frac{\lambda(E_{(t-2)\backslash t},\vec{x})}{\lambda(E_{(t-2)t},\vec{x})}\\ \le&x_t+\frac{(x_{t-4}+\cdots+x_3)x_{t-1}}{1-x_t-x_{t-1}-x_{t-2}-x_{t-3}}\\ \le&x_t+x_{t-1}=2x_t.\end{aligned}\tag{41}$$

So $\frac{x_{t-2}}{2}\le x_t$. Recall that $\eta\le\frac{x_{t-2}}{2}$. Therefore, $\eta\le x_t$. Hence, $\vec{z}=(z_1,z_2,\cdots,z_t)$ is also a legal weighting, and

$$\lambda(G',\vec{z})-\lambda(G',\vec{y})=\frac{(x_{t-4}+x_{t-3})^2x_{t-2}^2}{4\lambda(E_{(t-1)t},\vec{x})}.\tag{42}$$

By Remark 2.2 (b), we have

$$x_{t-4}=x_{t-3}+\frac{2x_{t-2}x_t}{\lambda(E_{(t-4)(t-3)},\vec{x})}.\tag{43}$$

In addition,

$$\begin{aligned}&\lambda(E_{(t-4)(t-3)},\vec{x})-\lambda(E_{(t-1)t},\vec{x})\\ \ge&(1-x_{t-4}-x_{t-3})-(1-x_{t-4}-x_{t-3}-x_{t-2}-x_{t-1}-x_t)>0.\end{aligned}\tag{44}$$

Combing (37), (39), (42), (43) and (44), we have

$$\begin{aligned}\lambda(G',\vec{z})-\lambda(G,\vec{x})=&\frac{(x_{t-4}-x_{t-3})^2\lambda(E_{(t-4)(t-3)},\vec{x})}{4}+\frac{(x_{t-4}+x_{t-3})^2x_{t-2}^2}{4\lambda(E_{(t-1)t},\vec{x})}\\ &-\frac{2x_{t-2}^2x_t^2}{\lambda(E_{(t-4)(t-3)},\vec{x})}\\ =&\frac{x_{t-2}^2x_t^2}{\lambda(E_{(t-4)(t-3)},\vec{x})}+\frac{(x_{t-4}+x_{t-3})^2x_{t-2}^2}{4\lambda(E_{(t-1)t},\vec{x})}-\frac{2x_{t-2}^2x_t^2}{\lambda(E_{(t-4)(t-3)},\vec{x})}\\ \ge&0.\end{aligned}\tag{45}$$

Hence $\lambda(G)=\lambda(G,\vec{x})\le\lambda(G',\vec{z})\le\lambda(G')$. □

Lemma 3.5. *Let $G=([t],E)$ be the left-compressed 3-graphs with $m=\binom{t}{3}-4$ edges (where $t\ge5$) and $G^c=\{(t-2)(t-1)t,(t-3)(t-1)t,(t-3)(t-2)t,(t-3)(t-2)(t-1)\}$. Then $\lambda(G)\le\lambda(C_{3,m})$.*

Proof. The lemma holds if $t=5$ from the result of [4], so we can assume $t\ge6$. Let $\vec{x}=(x_1,x_2,\dots,x_t)$ be an optimal weighting for G satisfying $x_1\ge x_2\ge\dots\ge x_t\ge0$. By Remark 2.2 (b), we have

$$x_1=x_2=\cdots=x_{t-5}=x_{t-4}=\alpha,\quad x_{t-3}=x_{t-2}=x_{t-1}=x_t=\beta,$$

and

$$\alpha-\beta=x_{t-4}-x_{t-3}=\frac{\lambda(E_{(t-4)\backslash(t-3)},\vec{x})}{\lambda(E_{(t-4)(t-3)},\vec{x})}=\frac{3\beta^2}{\lambda(E_{(t-4)(t-3)},\vec{x})}\le\beta.$$

Therefore,

$$\alpha \le 2\beta. \tag{46}$$

Note that $E(C_{3,m}) = E' = [t]^{(3)} \backslash \{(t-2)(t-1)t, (t-3)(t-1)t, (t-4)(t-1)t, (t-5)(t-1)t\}$, so

$$\lambda(C_{3,m}, \vec{x}) - \lambda(G, \vec{x}) = (x_{t-3}x_{t-2}x_{t-1} + x_{t-3}x_{t-2}x_t) - (x_{t-4}x_{t-1}x_t + x_{t-5}x_{t-1}x_t) = 2(\beta - \alpha)\beta^2.$$

Let $\vec{y} = (y_1, y_2, \cdots, y_t)$ be given by $y_i = x_i$ for $i \neq t-3,\ i \neq t$ and $y_{t-3} = x_{t-3} + \frac{\alpha-\beta}{3} = \frac{\alpha+2\beta}{3}$, $y_t = x_t - \frac{\alpha-\beta}{3} = \frac{4\beta-\alpha}{3}$. Clearly, $\vec{y} = (y_1, y_2, \dots, y_t)$ is a legal weighting, and

$$\begin{aligned}
\lambda(C_{3,m}, \vec{y}) - \lambda(C_{3,m}, \vec{x}) =& \frac{\alpha-\beta}{3}[\lambda(E'_{t-3}, \vec{x}) - \lambda(E'_t, \vec{x})] - \Big(\frac{\alpha-\beta}{3}\Big)^2 \lambda(E'_{(t-3)t}, \vec{x}) \\
=& \frac{\alpha-\beta}{3}(x_{t-2}x_{t-1} + x_{t-4}x_{t-1} + x_{t-5}x_{t-1}) - \Big(\frac{\alpha-\beta}{3}\Big)^2 \lambda(E'_{(t-3)t}, \vec{x}) \\
=& \frac{\alpha-\beta}{3}(\beta^2 + 2\alpha\beta) - \frac{(\alpha-\beta)^2}{9}\lambda(E'_{(t-3)t}, \vec{x}).
\end{aligned}$$

Let $\vec{z} = (z_1, z_2, \dots, z_t)$ be given by $z_i = y_i$ for $i \neq t-2, i \neq t-1$ and $z_{t-2} = y_{t-2} + \frac{\alpha-\beta}{3} = \frac{\alpha+2\beta}{3}$, $z_{t-1} = y_{t-1} - \frac{\alpha-\beta}{3} = \frac{4\beta-\alpha}{3}$. Clearly $\vec{z} = (z_1, z_2, \dots, z_t)$ is also a legal weighting, and

$$\begin{aligned}
\lambda(C_{3,m}, \vec{z}) - \lambda(C_{3,m}, \vec{y}) =& \frac{\alpha-\beta}{3}[\lambda(E'_{t-2}, \vec{y}) - \lambda(E'_{t-1}, \vec{y})] - \Big(\frac{\alpha-\beta}{3}\Big)^2 \lambda(E'_{(t-2)(t-1)}, \vec{y}) \\
=& \frac{\alpha-\beta}{3}(y_{t-3}y_t + y_{t-4}y_t + y_{t-5}y_t) - \Big(\frac{\alpha-\beta}{3})^2 \lambda(E'_{(t-2)(t-1)}, \vec{y}\Big) \\
=& \frac{\alpha-\beta}{3}\Big(\frac{\alpha+2\beta}{3} + 2\alpha\Big)\frac{4\beta-\alpha}{3} - \frac{(\alpha-\beta)^2}{9}\lambda(E'_{(t-2)(t-1)}, \vec{y}).
\end{aligned}$$

Let $\vec{w} = (w_1, w_2, \cdots, w_t)$ be given by $w_i = z_i$ for $i \neq t-4,\ i \neq t-3$ and $w_{t-4} = z_{t-4} - \frac{\alpha-\beta}{3} = \frac{2\alpha+\beta}{3},\ w_{t-3} = z_{t-3} + \frac{\alpha-\beta}{3} = \frac{2\alpha+\beta}{3}$. Clearly, $\vec{w} = (w_1, w_2, \cdots, w_t)$ is also a legal weighting, and

$$\begin{aligned}
\lambda(C_{3,m}, \vec{w}) - \lambda(C_{3,m}, \vec{z}) =& \frac{\alpha-\beta}{3}[\lambda(E'_{t-3}, \vec{z}) - \lambda(E'_{t-4}, \vec{z})] - \Big(\frac{\alpha-\beta}{3}\Big)^2 \lambda(E'_{(t-4)(t-3)}, \vec{z}) \\
=& \frac{\alpha-\beta}{3}(z_{t-4} - z_{t-3})\lambda(E'_{(t-4)(t-3)}, \vec{z}) - \Big(\frac{\alpha-\beta}{3}\Big)^2 \lambda(E'_{(t-4)(t-3)}, \vec{z}) \\
=& \Big[\frac{\alpha-\beta}{3}\Big(\alpha - \frac{\alpha+2\beta}{3}\Big) - \Big(\frac{\alpha-\beta}{3}\Big)^2\Big]\lambda(E'_{(t-4)(t-3)}, \vec{z}) \\
=& \frac{(\alpha-\beta)^2}{9}\lambda(E'_{(t-4)(t-3)}, \vec{z}).
\end{aligned} \tag{50}$$

Let $\vec{u} = (u_1, u_2, \cdots, u_t)$ be given by $u_i = w_i$ for $i \neq t-5,\ i \neq t-2$ and $u_{t-5} = w_{t-5} - \frac{\alpha-\beta}{3} = \frac{2\alpha+\beta}{3},\ u_{t-2} = w_{t-2} + \frac{\alpha-\beta}{3} = \frac{2\alpha+\beta}{3}$. Clearly, $\vec{u} = (u_1, u_2, \cdots, u_t)$ is also a legal weighting, and

$$\begin{aligned}
\lambda(C_{3,m}, \vec{u}) - \lambda(C_{3,m}, \vec{w}) =& \frac{\alpha-\beta}{3}[\lambda(E'_{t-2}, \vec{w}) - \lambda(E'_{t-5}, \vec{w})] - \Big(\frac{\alpha-\beta}{3}\Big)^2 \lambda(E'_{(t-5)(t-2)}, \vec{w}) \\
=& \frac{\alpha-\beta}{3}(w_{t-5} - w_{t-2})\lambda(E'_{(t-5)(t-2)}, \vec{w}) - \Big(\frac{\alpha-\beta}{3}\Big)^2 \lambda(E'_{(t-5)(t-2)}, \vec{w}) \\
=& \Big[\frac{\alpha-\beta}{3}\Big(\alpha - \frac{\alpha+2\beta}{3}\Big) - \Big(\frac{\alpha-\beta}{3}\Big)^2\Big] \lambda(E'_{(t-5)(t-2)}, \vec{w}) \\
=& \frac{(\alpha-\beta)^2}{9}\lambda(E'_{(t-5)(t-2)}, \vec{w}).
\end{aligned} \tag{51}$$

Adding (47)–(51), we have

$$\begin{aligned}
&\lambda(C_{3,m},\vec{u})-\lambda(G,\vec{x})\\
=&\frac{(\alpha-\beta)^2}{9}[\lambda(E'_{(t-4)(t-3)},\vec{z})+\lambda(E'_{(t-5)(t-2)},\vec{w})-\lambda(E'_{(t-3)t},\vec{x})-\lambda(E'_{(t-2)(t-1)},\vec{y})]\\
&+\frac{\alpha-\beta}{3}(\beta^2+2\alpha\beta)+\frac{\alpha-\beta}{3}\Big(\frac{\alpha+2\beta}{3}+2\alpha\Big)\frac{4\beta-\alpha}{3}+2(\beta-\alpha)\beta^2\\
=&\frac{(\alpha-\beta)^2}{9}(y_{t-2}+y_{t-1}+y_t+x_{t-3}+x_t+x_{t-1}-w_{t-2}-w_{t-5}-z_{t-3}-z_{t-4})\\
&+\frac{(\alpha-\beta)^2}{27}(37\beta-7\alpha)\\
=&\frac{(\alpha-\beta)^2}{9}\Big(2\beta+\frac{4\beta-\alpha}{3}+3\beta-\frac{\alpha+2\beta}{3}-\alpha-\frac{\alpha+2\beta}{3}-\alpha\Big)+\frac{(\alpha-\beta)^2}{27}(37\beta-7\alpha)\\
=&\frac{(\alpha-\beta)^2}{27}(52\beta-16\alpha)\\
\geq&0 \qquad (52)
\end{aligned}$$

since $\alpha\leq 2\beta$. Hence $\lambda(C_{3,m})\geq\lambda(C_{3,m},\vec{u})\geq\lambda(G,\vec{x})=\lambda(G)$. □

Lemma 3.6. *Let $G=([t],E)$ and $G'=([t],E')$ be left-compressed 3-graphs with $m=\binom{t}{3}-a$ edges (where $7\leq a\leq t-2$) satisfying $|E\triangle E(C_{3,m})|=|E'\triangle E(C_{3,m})|=6$ and the triples with the minimum colex ordering in G^c and G'^c are $(t-3)(t-2)(t-1)$ and $(t-5)(t-2)t$ respectively. Then $\lambda(G)\leq\lambda(G')\leq\lambda(C_{3,m})$.*

Proof. By Lemma 3.3, $\lambda(G')\leq\lambda(C_{3,m})$. So it is sufficient to show $\lambda(G)\leq\lambda(G')$. Let $\vec{x}=(x_1,x_2,\ldots,x_t)$ be an optimal weighting for G satisfying $x_1\geq x_2\geq\ldots\geq x_t\geq 0$. We have

$$\lambda(G',\vec{x})-\lambda(G,\vec{x})=x_{t-3}x_{t-2}x_{t-1}-x_{t-5}x_{t-2}x_t. \qquad (53)$$

Consider a new weighting $\vec{y}=(y_1,y_2,\ldots,y_t)$ given by $y_i=x_i$ for $i\neq t-5,\ \ i\neq t-3$ and $y_{t-5}=x_{t-5}-\delta$, $y_{t-3}=x_{t-3}+\delta$. Then

$$\begin{aligned}
\lambda(G',\vec{y})-\lambda(G',\vec{x})=&\delta[\lambda(E'_{t-3},\vec{x})-\lambda(E'_{t-5},\vec{x})]-\delta^2\lambda(E'_{(t-5)(t-3)},\vec{y})\\
=&\delta(x_{t-5}-x_{t-3})\lambda(E'_{(t-5)(t-3)},\vec{x})-\delta^2\lambda(E'_{(t-5)(t-3)},\vec{x}). \qquad (54)
\end{aligned}$$

Let $\delta=\frac{x_{t-5}-x_{t-3}}{2}$. Clearly, $\vec{y}=(y_1,y_2,\cdots,y_t)$ is also a legal weighting. Also

$$\lambda(G',\vec{y})-\lambda(G',\vec{x})=\frac{(x_{t-5}-x_{t-3})^2}{4}\lambda(E'_{(t-5)(t-3)},\vec{x}). \qquad (55)$$

Let $\vec{z}=(z_1,z_2,\ldots,z_t)$ be given by $z_i=y_i$ for $i\neq t-1,\ \ i\neq t$ and $z_{t-1}=y_{t-1}+\eta,\ \ z_t=y_t-\eta$. Then

$$\begin{aligned}
&\lambda(G',\vec{z})-\lambda(G',\vec{y})\\
=&\eta[\lambda(E'_{t-1},\vec{y})-\lambda(E'_t,\vec{y})]-\eta^2\lambda(E'_{(t-1)t},\vec{y})\\
=&\eta[y_{t-3}y_{t-2}+y_{t-4}y_{t-2}+y_{t-5}y_{t-2}-(y_{t-1}-y_t)\lambda(E'_{(t-1)t},\vec{y})]-\eta^2\lambda(E'_{(t-1)t},\vec{y})\\
=&\eta[x_{t-3}x_{t-2}+x_{t-4}x_{t-2}+x_{t-5}x_{t-2}-(x_{t-1}-x_t)\lambda(E_{(t-1)t},\vec{x})]-\eta^2\lambda(E_{(t-1)t},\vec{x})\\
=&\eta[x_{t-3}x_{t-2}+x_{t-4}x_{t-2}+x_{t-5}x_{t-2}-\frac{\lambda(E_{(t-1)\setminus t},\vec{x})}{\lambda(E_{(t-1)t},\vec{x})}\lambda(E_{(t-1)t},\vec{x})]-\eta^2\lambda(E_{(t-1)t},\vec{x})\\
=&\eta(x_{t-3}x_{t-2}+x_{t-5}x_{t-2})-\eta^2\lambda(E_{(t-1)t},\vec{x}). \qquad (56)
\end{aligned}$$

Let $\eta=\frac{x_{t-3}x_{t-2}+x_{t-5}x_{t-2}}{2\lambda(E_{(t-1)t},\vec{x})}$. By the condition of $|E\triangle E(C_{3,m})|=6$ we have $\{1,2,3\}\subseteq E_{(t-1)t}$, so

$$\eta\le\frac{x_{t-2}}{3}.$$

Applying Remark 2.2 (b), we have

$$\begin{aligned}x_{t-2}=&x_t+\frac{\lambda(E_{(t-2)\backslash t},\vec{x})}{\lambda(E_{(t-2)t},\vec{x})}\le x_t+\frac{(x_{t-4}+\cdots+x_4)x_{t-1}}{1-x_t-x_{t-1}-x_{t-2}-x_{t-3}-x_{t-4}}\\ \le&x_t+x_{t-1}=x_t+x_t+\frac{\lambda(E_{(t-1)\backslash t},\vec{x})}{\lambda(E_{(t-1)t},\vec{x})}\\ \le&2x_t+\frac{x_{t-4}x_{t-2}}{x_1+x_2+x_3}\le 2x_t+\frac{x_{t-2}}{3}.\end{aligned}\tag{57}$$

So $\frac{x_{t-2}}{3}\le x_t$. Recall that $\eta\le\frac{x_{t-2}}{3}$. Therefore, $\eta\le x_t$. Hence $\vec{z}=(z_1,z_2,\cdots,z_t)$ is also a legal weighting, and

$$\lambda(G',\vec{z})-\lambda(G',\vec{y})=\frac{(x_{t-5}+x_{t-3})^2x_{t-2}^2}{4\lambda(E_{(t-1)t},\vec{x})}.\tag{58}$$

By Remark 2.2 (b), we have

$$x_{t-5}=x_{t-3}+\frac{(x_{t-1}+x_t)x_{t-2}}{\lambda(E_{(t-5)(t-3)},\vec{x})}.\tag{59}$$

Note that

$$\begin{aligned}&\lambda(E_{(t-5)(t-3)},\vec{x})-\lambda(E_{(t-1)t},\vec{x})\\ \ge&(1-x_{t-5}-x_{t-3})-(1-x_{t-5}-x_{t-4}-x_{t-3}-x_{t-2}-x_{t-1}-x_t)\ge 0.\end{aligned}\tag{60}$$

Combing (53), (55), (58), (59) and (60), we have

$$\begin{aligned}\lambda(G',\vec{z})-\lambda(G,\vec{x})\ge&\frac{(x_{t-5}-x_{t-3})^2\lambda(E'_{(t-5)(t-3)},\vec{x})}{4}+\frac{(x_{t-5}+x_{t-3})^2x_{t-2}^2}{4\lambda(E_{(t-1)t},\vec{x})}\\ &-\frac{x_{t-2}^2(x_{t-1}+x_t)x_t}{\lambda(E_{(t-5)(t-3)},\vec{x})}\\ =&\frac{x_{t-2}^2(x_{t-1}+x_t)^2}{4\lambda(E_{(t-5)(t-3)},\vec{x})}+\frac{(x_{t-5}+x_{t-3})^2x_{t-2}^2}{4\lambda(E_{(t-1)t},\vec{x})}-\frac{x_{t-2}^2(x_{t-1}+x_t)x_t}{\lambda(E_{(t-5)(t-3)},\vec{x})}\\ \ge&0.\end{aligned}\tag{61}$$

Hence $\lambda(G)=\lambda(G,\vec{x})\le\lambda(G',\vec{z})=\lambda(G')$. □

Lemma 3.7. *Let $G=([t],E)$ be the left-compressed 3-graphs with $m=\binom{t}{3}-6$ edges and $G^c=\{(t-2)(t-1)t,(t-3)(t-1)t,(t-4)(t-1)t,(t-3)(t-2)t,(t-4)(t-2)t,(t-3)(t-2)(t-1)\}$. Let $G'=([t],E')$, $E'=E\cup\{(t-4)(t-2)t\}\backslash\{(t-5)(t-1)t\}$. Then $\lambda(G)\le\lambda(G')\le\lambda(C_{3,m})$.*

Proof. By Lemma 3.4, $\lambda(G')\le\lambda(C_{3,m})$. So it is sufficient to show that $\lambda(G)\le\lambda(G')$. Let $\vec{x}=(x_1,x_2,\ldots,x_t)$ be an optimal weighting for G satisfying $x_1\ge x_2\ge\ldots\ge x_t\ge 0$. Then

$$\lambda(G',\vec{x})-\lambda(G,\vec{x})=x_{t-4}x_{t-2}x_t-x_{t-5}x_{t-1}x_t.\tag{62}$$

By Remark 2.2 (b), $x_{t-2}=x_{t-1}$. Consider a new weighting $\vec{y}=(y_1,y_2,\cdots,y_t)$ given by $y_i=x_i$ for $i\ne t-5,\ i\ne t-4$ and $y_{t-5}=x_{t-5}-\delta,\ y_{t-4}=x_{t-4}+\delta$. Then

$$\begin{aligned}\lambda(G',\vec{y})-\lambda(G',\vec{x})=&\delta[\lambda(E'_{t-4},\vec{x})-\lambda(E'_{t-5},\vec{x})]-\delta^2\lambda(E'_{(t-5)(t-4)},\vec{x})\\ =&\delta(x_{t-5}-x_{t-4})\lambda(E'_{(t-5)(t-4)},\vec{x})-\delta^2\lambda(E'_{(t-5)(t-4)},\vec{x}).\end{aligned}$$

Let $\delta = \frac{x_{t-5}-x_{t-4}}{2}$. Clearly, $\vec{y} = (y_1, y_2, \ldots, y_t)$ is also a legal weighting, and

$$\lambda(G', \vec{y}) - \lambda(G', \vec{x}) = \frac{(x_{t-5} - x_{t-4})^2}{4}\lambda(E'_{(t-5)(t-4)}, \vec{x}) = \frac{(x_{t-5} - x_{t-4})^2}{4}\lambda(E'_{(t-5)(t-4)}, \vec{x}). \quad (63)$$

Let $\vec{z} = (z_1, z_2, \cdots, z_t)$ given by $z_m = y_m$ for $m \neq t-2,\ \ m \neq t-1$ and $z_{t-2} = y_{t-2} + \eta$, $z_{t-1} = y_{t-1} - \eta$. Then

$$\begin{aligned}\lambda(G', \vec{z}) - \lambda(G', \vec{y}) =& \eta[\lambda(E'_{t-2}, \vec{y}) - \lambda(E'_{t-1}, \vec{y})] - \eta^2\lambda(E'_{(t-2)(t-1)}, \vec{y}) \\ =& \eta(y_{t-4}y_t + y_{t-5}y_t) - \eta^2\lambda(E'_{(t-2)(t-1)}, \vec{y}). \end{aligned} \quad (64)$$

Let $\eta = \frac{y_{t-5}y_t + y_{t-4}y_t}{2\lambda(E'_{(t-2)(t-1)}, \vec{y})}$. Clearly, $\eta < y_t$. Hence, $\vec{z} = (z_1, z_2, \cdots, z_t)$ is also a legal weighting, and

$$\lambda(G', \vec{z}) - \lambda(G', \vec{y}) = \frac{(y_{t-5} + y_{t-4})^2 y_t^2}{4\lambda(E'_{(t-2)(t-1)}, \vec{y})} = \frac{(x_{t-5} + x_{t-4})^2 x_t^2}{4\lambda(E_{(t-2)(t-1)}, \vec{x})}. \quad (65)$$

By Remark 2.2 (b), we have

$$x_{t-5} = x_{t-4} + \frac{2x_{t-1}x_t}{\lambda(E_{(t-5)(t-4)}, \vec{x})} \leq x_{t-4} + x_t, \quad (66)$$

$$x_{t-4} = x_{t-1} + \frac{\lambda(E_{(t-4)\backslash(t-1)}, \vec{x})}{\lambda(E_{(t-4)(t-1)}, \vec{x})} = x_{t-1} + \frac{x_{t-3}x_t + x_{t-3}x_{t-2}}{1 - x_{t-4} - x_{t-1} - x_t}, \quad (67)$$

$$x_{t-5} = x_{t-1} + \frac{\lambda(E_{(t-5)\backslash(t-1)}, \vec{x})}{\lambda(E_{(t-5)(t-1)}, \vec{x})} = x_{t-1} + \frac{(x_{t-4} + x_{t-3} + x_{t-2})x_t + x_{t-3}x_{t-2}}{1 - x_{t-5} - x_{t-1}}. \quad (68)$$

Combing (66)–(68) we have

$$\begin{aligned} x_{t-4} + x_{t-5} \leq & 2x_{t-1} + \frac{(x_{t-4} + x_{t-3} + x_{t-2} + x_{t-3})x_t + 2x_{t-3}x_{t-2}}{1 - x_{t-4} - x_{t-1} - x_t} \\ \leq & 2x_{t-1} + x_t + x_{t-3}. \end{aligned}$$

Hence,

$$\lambda(E_{(t-5)(t-4)}, \vec{x}) - \lambda(E_{(t-2)(t-1)}, \vec{x}) = x_{t-3} + 2x_{t-1} + x_t - x_{t-4} - x_{t-5} \geq 0. \quad (69)$$

Combing (62), (63), (65), (66) and (69), we have

$$\lambda(G', \vec{z}) - \lambda(G, \vec{x}) = \frac{x_{t-1}^2 x_t^2}{\lambda(E_{(t-5)(t-4)}, \vec{x})} + \frac{(x_{t-5} + x_{t-4})^2 x_t^2}{4\lambda(E_{(t-2)(t-1)}, \vec{x})} - \frac{2x_{t-1}^2 x_t^2}{\lambda(E_{(t-5)(t-4)}, \vec{x})} \geq 0.$$

Hence $\lambda(G') \geq \lambda(G', \vec{z}) \geq \lambda(G, \vec{x}) = \lambda(G)$. □

Now we are ready to show Theorem 12.

Proof of Theorem 1.12. Let G be a left-compressed extremal 3-graph with m edges satisfying $|E(G)\Delta E(C_{3,m})| \leq 6$. Let t be a positive integer such that $\binom{t-1}{3} \leq m < \binom{t}{3}$. By Remark 1.9, we can assume that $G = ([t], E)$. If $m < \binom{t-1}{3} + \binom{t-2}{2}$, let G' be obtained by adding the first $\binom{t-1}{3} + \binom{t-2}{2} - m$ triples in colex ordering in $E(G^c)$ to $E(G)$, then $\lambda(E(G')) \geq \lambda(E(G))$ and $|E(G')\Delta E(C_{3,m'})| \leq 6$ for $m' = \binom{t-1}{3} + \binom{t-2}{2}$. In view of Lemma 2.3, we may assume that $m \geq \binom{t-1}{3} + \binom{t-2}{2}$. Then $C_{3,m} \supseteq [t-1]^{(3)} \cup \{1, 2, \cdots, t-2, t\}^{(3)}$. Since $|E\Delta E(C_{3,m})| \leq 6$, there are at most 3 triples in $\{1, 2, \cdots, t-2, t\}^{(3)} \cup [t-1]^{(3)} - E$. Since G is left-compressed, in view of Figure 1, there are only the following possible cases for $\{1, 2, \cdots, t-2, t\}^{(3)} \cup [t-1]^{(3)} - E$:

Case 1. $\{1,2,\cdots,t-2,t\}^{(3)} \cup [t-1]^{(3)} - E = \{(t-3)(t-2)t\}$;

Case 2. $\{1,2,\cdots,t-2,t\}^{(3)} \cup [t-1]^{(3)} - E = \{(t-3)(t-2)t, (t-4)(t-2)t\}$;

Case 3. $\{1,2,\cdots,t-2,t\}^{(3)} \cup [t-1]^{(3)} - E = \{(t-3)(t-2)t, (t-3)(t-2)(t-1)\}$;

Case 4. $\{1,2,\cdots,t-2,t\}^{(3)} \cup [t-1]^{(3)} - E = \{(t-3)(t-2)t, (t-4)(t-2)t, (t-5)(t-2)t\}$;

Case 5. $\{1,2,\cdots,t-2,t\}^{(3)} \cup [t-1]^{(3)} - E = \{(t-3)(t-2)t, (t-4)(t-2)t, (t-4)(t-3)t\}$;

Case 6. $\{1,2,\cdots,t-2,t\}^{(3)} \cup [t-1]^{(3)} - E = \{(t-3)(t-2)t, (t-4)(t-2)t, (t-3)(t-2)(t-1)\}$.

If Cases 1, 2, 4, 5 happen, then by Lemma 3.3, $\lambda(G) \le \lambda(C_{3,m})$. If Case 3 and $a \ge 5$ happen, then by Lemma 3.4, $\lambda(G) \le \lambda(C_{3,m})$. If Case 3 and $a < 5$ happen, then $a = 4$. By Lemma 3.5, $\lambda(G) \le \lambda(C_{3,m})$. If Case 6 and $a \ge 7$ happen, then by Lemma 3.6, $\lambda(G) \le \lambda(C_{3,m})$. If Case 6 and $a < 7$ happen, then $a = 6$. By Lemma 3.7, $\lambda(G) \le \lambda(C_{3,m})$. The proof of Theorem 1.12 is completed. □

Proof of Corollary 1.13. We can assume G is a left-compressed 3-graph on $[t]$ by Remark 1.9. The range of m gurantees that $|E(G)\Delta E(C_{3,m})| \le 6$. Therefore $\lambda(G) \le \lambda(C_{3,m})$ by Theorem 1.12. □

The following result is also implied by Theorem 1.12.

Corollary 3.8. *Let m, t and b be positive integers satisfying*

$$\binom{t-1}{3} + \binom{t-2}{2} + b < m \le \binom{t}{3}.$$

Let $G = (V, E)$ be a left-compressed 3-graph on the vertex set $[t]$ with m edges satisfying $|E_{(t-1)t}| \le b+3$. Then $\lambda(G) \le \lambda(C_{3,m})$.

Proof. Since $|E_{(t-1)t}| \le b+3$, we have $|E\Delta E(C_{3,m})| \le 6$. So $\lambda(G) \le \lambda(C_{3,m})$ by Theorem 1.12. □

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