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Analytical Solution for Time-dependent Flow of a Third Grade Fluid Induced Due to Impulsive Motion of a Flat Porous Plate

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Abstract The aim of the present communication is to discuss the analytical solution for the unsteady flow of a third grade fluid which occupies the space $y > 0$ over an infinite porous plate. The flow is generated due to the motion of the plate in its own plane with an impulsive velocity $V(t)$. Translational symmetries in variables t and y are utilized to reduce the governing non-linear partial differential equation into an ordinary differential equation. The reduced problem is then solved using homotopy analysis method (HAM). Graphs representing the solution are plotted and discussed and proper conclusions are drawn.

Keywords third grade fluid; Porous plate; translational symmetries; Homotopy analysis method (HAM)
2000 MR Subject Classification 35Q51; 35B35; 35A07

1 Introduction

Due to increasing importance of non-Newtonian fluids in innovative technology and modern industries, the interest of the investigators to study the problems dealing with the flow of non-Newtonian fluids have enormously increased. The motivation of study the flow problems of non-Newtonian fluids stems because of several important engineering and industrial applications of non-Newtonian fluids, particularly in the extraction of crude oil from petroleum products, slurry transporting, synthetic fibers, processing of food, drilling of petroleum products and other suchlike activities. The non-Newtonian fluids are mainly classified into three types namely differential, rate and integral. The equations which govern the flow problem of non-Newtonian fluids are of higher order, nonlinear and much more subtle in nature as compared with that of Newtonian fluids. Because of this fact several models of non-Newtonian fluids have been proposed in the recent years to study the complex physical structure of these models. One special class of non-Newtonian fluid which has acquired special focus in the past few years is differential type non-Newtonian fluid of second grade[1,2]. A second grade fluid model is the simplest subclass of differential type fluids for which one can reasonably hope to establish an analytic result. In most of the flow aspects, the governing equations for a second grade fluid

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are linear. The fluid of second grade has a remarkable property in common with the classical linearly viscous fluid in that viscometric measurements sufficient to determine the form of the constitutive relation for all flows. A second grade fluid model for steady flows is used to predict the normal stress differences, it does not correspond to shear thinning or thickening if the shear viscosity is assumed to be constant. Therefore some experiments may be well described by third grade fluid [3]. The mathematical model of a third grade fluid represents a more realistic description of the behavior of non-Newtonian fluids. A third grade fluid model represents a further attempt towards the study of the flow properties of non-Newtonian fluids. Therefore, a third grade fluid model has considered in this study. Some useful studies regarding the flow of third grade fluid are given in [4–9].

In this study, we shall focus on Hayat et al.[10] and Fakhar [11], they both have discussed the unsteady flow of a third grade fluid over the porous plate. In Hayat et al.[10], the governing model has been solved by employing the successive symmetry reductions. The solution obtained in this article is not a meaningful solution. The solution does not satisfy any boundary condition, also it does not show any effects of the material parameters and the novel phenomena of suction/injection on the flow. In Fakhar[11], the same model has been considered once again. In this article translational type symmetries have been utilized to perform the travelling wave reduction on the governing model and the reduced model has been solved using the power series method. The method developed in this article also does not give any substantial solution of the physical model considered, the solution obtained is not valid for any particular boundary conditions and does not show any effects of the different emerging parameters on the flow.

In this note, we use an alternative approach to find an analytical solution of the problem. We employed translational symmetries to reduce the governing non-linear partial differential equation and then solve the reduced ordinary differential equation along with the necessary boundary conditions using homotopy analysis method (HAM). In 1992, Liao[12,13] developed the basic ideas of the homotopy in topology to purpose the HAM for highly nonlinear problems. The HAM is an analytic approach to get convergent series solutions of various types of strong nonlinear equations, including algebraic equations, ordinary differential equations, partial differential equations, difference-differential equations and coupled equations. Unlike perturbation techniques, the HAM is independent of small/large physical parameters, and thus is valid whether a nonlinear model contains small/large physical parameters or not. The HAM provides us with a simple way to ensure the convergence of series solution, and therefore, the HAM is valid even for strongly nonlinear equations. Apart from this fact, the HAM provides us with great freedom to choose proper base functions to approximate a nonlinear problem. From the past few years, more and more researchers have been successfully applying this method to various nonlinear problems in science and engineering, such as the viscous flows of non-Newtonian fluids [14−20], the KdV-type equation [21,22], nonlinear heat transfer [23,24], Rayleigh equation governing the radial dynamics of a multielectron bubble [25], delay differential equations [26], nonlinear water waves [27], time-dependent Emden-Fowler type equation [28], boundary layer flows of non-Newtonian fluids[29−31]. This shows the great potential of the HAM for strongly nonlinear problems in science and engineering.

2 Geometry of the Flow Model

Consider a Cartesian coordinate system $OXYZ$ with the y-axis pointing in the vertically upward direction. The third grade fluid occupies the half space $y > 0$ and is in contact with an infinite porous plate at $y = 0$. The flow is generated due to the motion of the porous plate in its own plane with an impulsive velocity $V(t)$. Since the plate is infinite in the XZ-plane and therefore all the physical quantities except the pressure depend on y only. Far away from the plate the fluid will be considered to be at rest. The geometry of the physical model is shown in Figure 1.

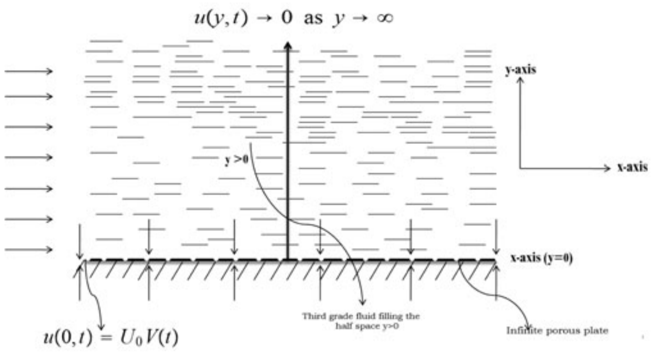

Figure 1. Geometry of the Physical Model and Coordinate System

3 Governing Equations

By following [10], the equation of motion for the unsteady flow of third grade fluid over the porous plate, with slight change of notation, is given by:

$$\rho\Big[\frac{\partial u}{\partial t} - W_0\frac{\partial u}{\partial y}\Big] = \mu\frac{\partial^2 u}{\partial y^2} + \alpha_1\frac{\partial^3 u}{\partial y^2\partial t} - \alpha_1 W_0\frac{\partial^3 u}{\partial y^3} + 6\beta_3\Big(\frac{\partial u}{\partial y}\Big)^2\frac{\partial^2 u}{\partial y^2}. \tag{1}$$

Here u is the velocity component, ρ is the density, μ is the coefficient of viscosity, α_1 and β_3 are the material constants (for details on these material constants and the conditions that are satisfied by these constants, the reader is referred to [3]). W_0 (constant) $\gtrless 0$ corresponds to suction (injection).

In order to solve the above (1), the required boundary and initial conditions are specified as follows:

$$u(0,t) = U_0V(t), \qquad t > 0, \tag{2}$$
$$u(\infty,t) = 0, \qquad t > 0, \tag{3}$$
$$u(y,0) = g(y), \qquad y > 0, \tag{4}$$

where U_0 is the reference velocity and $V(t)$ and $g(y)$ are as yet unspecified functions. The first boundary Condition (2) is the no-slip condition and the second boundary Condition (3) says that the main stream velocity is zero. This is not a restrictive assumption since we can always measure velocity relative to the main stream. The initial Condition (4) indicates that initially the fluid is moving with some non-uniform velocity $g(y)$. These functions are constrained in the next section when we seek analytical solution using the HAM.

On introducing the non-dimensional quantities

$$\widehat{u} = \frac{u}{U_0}, \qquad \widehat{y} = \frac{U_0 y}{\nu}, \qquad \widehat{t} = \frac{U_0^2 t}{\nu}, \qquad \widehat{\alpha} = \frac{\alpha_1 U_0^2}{\rho\nu^2}, \qquad \widehat{\beta} = \frac{6\beta_3 U_0^4}{\rho\nu^3}, \qquad \widehat{W}_0 = \frac{W_0}{U_0}. \tag{5}$$

So (1) and the corresponding initial and the boundary conditions take the form

$$\Big[\frac{\partial \widehat{u}}{\partial \widehat{t}} - W_0\frac{\partial \widehat{u}}{\partial \widehat{y}}\Big] = \frac{\partial^2 \widehat{u}}{\partial \widehat{y}^2} + \widehat{\alpha}\frac{\partial^3 \widehat{u}}{\partial \widehat{y}^2 \partial \widehat{t}} - \widehat{\alpha}\widehat{W}_0\frac{\partial^3 \widehat{u}}{\partial \widehat{y}^3} + \widehat{\beta}\Big(\frac{\partial \widehat{u}}{\partial \widehat{y}}\Big)^2\frac{\partial^2 \widehat{u}}{\partial \widehat{y}^2}, \tag{6}$$

$$\widehat{u}(0,t) = V(\widehat{t}), \qquad \widehat{t} > 0, \tag{7}$$

$$\widehat{u}(\widehat{y},\widehat{t}) \longrightarrow 0 \qquad \text{as} \;\; \widehat{y} \longrightarrow \infty, \;\; \widehat{t} > 0, \tag{8}$$

$$\widehat{u}(\widehat{y},0) = g(\widehat{y}), \qquad \widehat{y} > 0. \tag{9}$$

4 Reduction of the Problem

We know that from the principal of Lie symmetry methods that if a differential equation is explicitly independent of any dependent or independent variable, then this particular differential equation remains invariant under the translation symmetry corresponding to that particular variable. We noticed that (6) admits Lie point symmetry generators, $\partial/\partial\widehat{t}$ (time-translation) and $\partial/\partial\widehat{y}$ (space-translation in y). For a detail analysis the readers are referred to [32].

Let X_1 and X_2 be time-translation and space-translation symmetry generators respectively. Then the solution corresponding to the generator

$$X = X_1 + cX_2 = \frac{\partial}{\partial \widehat{t}} + c\frac{\partial}{\partial \widehat{y}}, \qquad c > 0, \tag{10}$$

would represent travelling wave solution with constant wave speed c. The Langrangian system corresponding to Equation (10) is

$$\frac{d\widehat{y}}{c} = \frac{d\widehat{t}}{1} = \frac{d\widehat{u}}{0}, \tag{11}$$

Solving Equation (11), invariant solutions are given by

$$\widehat{u}(\widehat{y},\widehat{t}) = f(\eta), \qquad \text{where} \;\; \eta = \widehat{y} - c\widehat{t}. \tag{12}$$

With these change of variables, we have the following reduced equation for (6),

$$-c\frac{df}{d\eta} = \frac{d^2 f}{d\eta^2} - c\alpha\frac{d^3 f}{d\eta^3} - \alpha W_0\frac{d^3 f}{d\eta^3} + \beta\Big(\frac{df}{d\eta}\Big)^2\frac{d^2 f}{d\eta^2} + W_0\frac{df}{d\eta}, \tag{13}$$

with the transformed boundary conditions are

$$\begin{aligned} &f(0) = l_1, \qquad \text{where} \;\; l_1 \;\; \text{is a constant,} \\ &f(\eta) = 0, \qquad \text{as} \;\; \eta \longrightarrow \infty. \end{aligned} \tag{14}$$

For simplicity we have omitted the caps from the non-dimensional parameters α, β and W_0.

5 The HAM Solution

Now we employ the HAM to obtain the solution of the nonlinear reduced Equation (13) subject to the boundary Conditions (14).

5.1 Rule of Solution Expression

When the HAM is applied to solve a nonlinear differential equation, a set of base functions are selected to represent the required solution. This is the so-called “rule of solution expression”.

There are no precise method to direct us to choose the basis functions. From the practical point of view, the role of solution expression play crucial role within the homotopy analysis method. The rule of solution implies an assumption that we should have some knowledge about a given nonlinear problem. So theoretically, this assumption impairs the HAM, although we can always use the same base functions even if a given nonlinear problem is completely new. As mentioned by Liao[13], a solution may be expressed with different base functions, among which some converge to the exact solution of the problem faster than others. Such base functions are better suited for the expression of the final solution. Thus according to (6) and boundaries Conditions (7)–(9), we have decided to express $f(\eta)$ by the set of base functions of the form:

$$f(\eta)=\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}c_{q,r}\eta^{q}\exp(-q\eta), \tag{15}$$

where $c_{q,r}$ are coefficients.

5.2 Choosing Initial Guess and Auxiliary Linear Operator

It is under the rule of solution expression that the initial guess and the auxiliary linear operator are selected. Here we choose initial guess $f_0(\eta)$ and auxiliary linear operator L_f in the following forms

$$f_0(\eta)=l_1\exp(-\eta), \tag{16}$$

and

$$L_f[\widehat{f}(\eta;p)]=\frac{\partial^2\widehat{f}(\eta;p)}{\partial\eta^2}+\frac{\partial\widehat{f}(\eta;p)}{\partial\eta}, \tag{17}$$

with the property

$$L_f[c_1+c_2\exp(-\eta)]=0. \tag{18}$$

5.3 The Zeroth-order Deformation Problem

Making use of the above definitions, we construct the zero-order deformation problems as follows:

$$(1-p)L_f[\widehat{f}(\eta;p)-f_0(\eta)]=p\hbar_f N_f[\widehat{f}(\eta;p)], \tag{19}$$

$$\widehat{f}(0;p)=l_1, \tag{20}$$

$$\widehat{f}(\eta;p)\longrightarrow 0,\qquad \text{as }\ \eta\longrightarrow\infty, \tag{21}$$

$$\begin{aligned}N_f[\widehat{f}(\eta;p)]=&\frac{\partial^2\widehat{f}(\eta;p)}{\partial\eta^2}+c\frac{\partial\widehat{f}(\eta;p)}{\partial\eta}-c\alpha\frac{\partial\widehat{f}(\eta;p)}{\partial\eta}-\alpha W_0\frac{\partial^3\widehat{f}(\eta;p)}{\partial\eta^3}\\&+\beta\Big(\frac{\partial\widehat{f}(\eta;p)}{\partial\eta}\Big)^2\frac{\partial^2\widehat{f}(\eta;p)}{\partial\eta^2}+W_0\frac{\partial\widehat{f}(\eta;p)}{\partial\eta},\end{aligned} \tag{22}$$

where $p(\in[0,1])$ is the embedding parameter and $\hbar_f$ is the non-zero convergence control parameter. For $p=0$ and $p=1$

$$\widehat{f}(\eta;0)=f_0(\eta),\qquad \widehat{f}(\eta;1)=f(\eta). \tag{23}$$

As p varies from 0 to 1, $\widehat{f}(\eta;p)$ vary from the initial guess $f_0(\eta)$ to the solution $f(\eta)$. By expanding $\widehat{f}(\eta;p)$ using Taylor's series with respect to p we have:

$$\widehat{f}(\eta;\,p)=\widehat{f}(\eta;0)+\sum_{m=1}^{\infty}f_m(\eta)p^m, \tag{24}$$

where

$$f_m(\eta) = \frac{1}{m!}\frac{\partial^m \widehat{f}(\eta;p)}{\partial p^m}\Big|_{p=0}, \tag{25}$$

The zero-order deformation problems contain the auxiliary parameter $\hbar_f$, In principle, $\hbar_f$ is chosen so that the above series remain convergent at $p = 1$, then by using (24) one obtains

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta). \tag{26}$$

5.4 The mth-order Deformation Problem

Differentiating the zero-order deformation Problems (19)–(21) m-times with respect to p and then dividing by $m!$ and setting $p = 0$, we obtain the following $mth-$order deformation problems

$$L_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \hbar_f R_m(\eta), \tag{27}$$

$$f_m(0) = 0, \tag{28}$$

$$f_m(\eta) = 0, \qquad \text{as} \quad \eta \longrightarrow \infty, \tag{29}$$

where

$$\begin{aligned} R_m(\eta) =& \frac{\partial^2 f_{m-1}}{\partial \eta^2} + c\frac{\partial f_{m-1}}{\partial \eta} - c\alpha\frac{\partial f_{m-1}}{\partial \eta} - \alpha W_0\frac{\partial^3 f_{m-1}}{\partial \eta^3} \\ &+ \beta\sum_{k=0}^{m-1}\frac{\partial f_{m-1-k}}{\partial \eta}\sum_{l=0}^{k}\frac{\partial f_{k-l}}{\partial \eta}\frac{\partial f_l}{\partial \eta} + W_0\frac{\partial f_{m-1}}{\partial \eta} \end{aligned} \tag{30}$$

and

$$\chi_m = \begin{cases} 0; & m \leq 1, \\ 1; & m > 1. \end{cases} \tag{31}$$

The linear non-homogeneous Problems (25)–(28) can be solved by using a symbolic computation software Mathematica in the order $m = 1, 2, 3, \cdots$.

6 Convergence of the Homotopy Solution

As mentioned earlier, HAM provides us with a easy way to control and adjust the convergence region and gives us great freedom to use different base functions to express solutions of nonlinear problem so that one can find approximate solution of a nonlinear problem more effeciently by a better choice of base functions. This has a great effect on the convergence region because the convergence region and rate of convergence of the homotopy series solution is chiefly determined by the base functions use to express the solution. The optimal values of the convergence control parameters $\hbar_f$ are calculated by minimizing the discrete square residual error[33]

$$E_{f,m} = \frac{1}{N+1}\sum_{j=0}^{N}\Big[\mathcal{N}\Big(\sum_{i=0}^{m} F_j(i\triangle\eta)\Big)\Big]. \tag{32}$$

By choosing the best values of $\hbar_f$, we can attain the required accuracy more quickly as the order of approximation tends to infinity.

7 Pade-approximation

The Pade technique can be employed with in the frame of homotopy analysis method. Such as for the problem we considered, it is very important to ensure that the series solution is convergent $p = 1$. A Pade approximation is very useful in improving the accuracy of an approximate series solution available in the form of a polynomial. Also the Pade approximation can greatly enlarge the convergence region and rate of solution series. Thus, for this problem the convergence is also achieved through Pade-approximations. The result using the Pade-approximations are shown in Table 1. From the Table it is seen that the Pade-approximations accelerate the convergence of the series solution.

Table 1. The Homotopy-Pade Approximation of $f'(0)$ with $\beta = 1.5$, $\alpha = 0.5$, $c = 1.4$, $W_0 = 1$

Homotopy-Pade approximation	$f'(0)$
[2/2]	−0.809524
[3/3]	−0.772300
[4/4]	−0.763734
[5/5]	−0.761261
[7/7]	−0.760020
[8/8]	−0.759925
[9/9]	−0.759882
[11/11]	−0.759851
[12/12]	−0.759849
[14/14]	−0.759846
[15/15]	−0.759846

8 Results and Discussion

The reduced (11) subject to the boundary Conditions (12) has been solved analytically via HAM. The graphical results are given to carry out a parametric study showing influences of the non-dimensional parameters on the velocity.

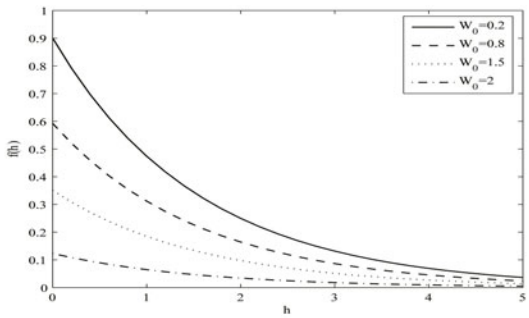

Figure 2. The Influence of the Suction Parameter $W_0 > 0$ on the Velocity Field f when $c = 1$, $\alpha = 1.5$, $\beta = 0.5$ are Fixed.

8.1 Suction/Injection Characteristics

Figure 2 and Figure 3 shows the influence of suction and injection occurring at the plate surface (which is the main focus of the present study). For $W_0 > 0$ suction occurs and $W_0 < 0$ corresponds to injection or blowing. It is observed that with the increase in the suction parameter the velocity decreases so does the boundary layer thickness. This is quite in accordance with the fact that suction causes a reduction in the boundary layer thickness, whereas the effect of increasing injection are quite opposite to that of suction. So from this figure it is obvious that our HAM solution is valid for suction and as well as for the injection.

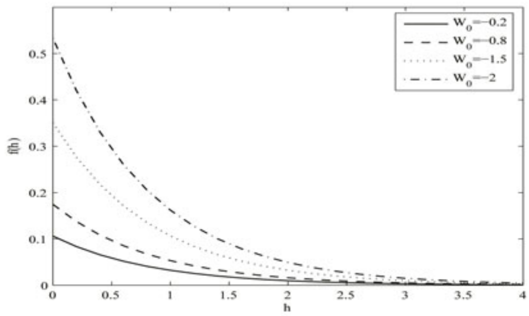

Figure 3. The Influence of the Injection Parameter $W_0 < 0$ on the Velocity Field f when $c = 1$, $\alpha = 1.5$, $\beta = 0.5$ are Fixed.

8.2 Material Parameters Characteristics

In Figures 4 and 5, we see the effects of the second grade fluid parameter α and the grade fluid parameters β on the velocity field f. These figures reveal that α and β have opposite roles on the velocity field. From these figures, it is noticed that velocity increases for large values of α. On the other hand, with the increasing values of β the velocity field decreases which shows the shear thickening behavior of the fluid. So this figure reveals that third grade fluid either shows the shear thinning or shear thickening characteristic of the fluid.

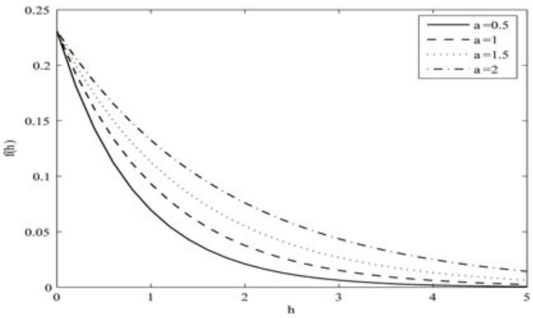

Figure 4. The Influence of the Second Grade Parameter α on the Velocity Field f when $c = 0.5$, $W_0 = 1$, $\beta = 1.5$ are Fixed.

8.3 Wave Speed Characteristics

Figure 6 shows the variation of the velocity field f with an increase in the wave speed c. It is observed that the velocity profile decreases as the value of the wave speed increases.

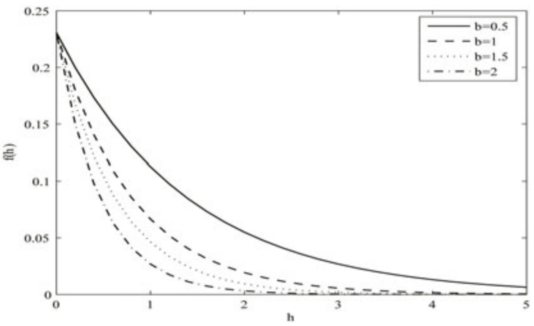

Figure 5. The Influence of the Third Grade Parameter β on the Velocity Field f when $c = 0.5$, $W_0 = 1$, $\alpha = 1.5$ are Fixed.

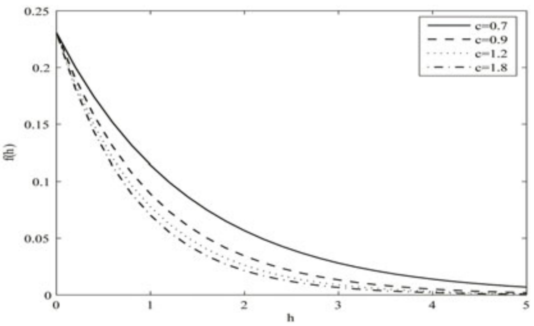

Figure 6. The Influence of the Wave Speed c on the Velocity Field f when $\alpha = 0.5$, $W_0 = 1$, $\beta = 1.5$ are Fixed.

9 Closing Remarks

In this study, we have revisited the unsteady problem for the flow of third grade fluid over a flat porous plate which has been considered previously by the two different authors. In both the previous studies of the model none of them have been able to construct the physically meaningful solution of the problem. In the present communication we have constructed the analytical solution of the model with the combination of Lie symmetry and the homotopy analysis method. The results obtained here satisfy the relevant physical boundary conditions and also take into account the interesting flow behavior of the model. The HAM solution clearly justify how various physical parameters play their role in determining the properties of the flow. The model considered here is a theoretical in nature and a prototype one but the method developed in this article is helpful for tacking a wide range of non-Newtonian fluid flow problems.

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