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Rotating-Symmetric Solutions for Nonlinear Systems with Symmetry

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Abstract It is proved that if a nonlinear system possesses some group-symmetry, then under certain transversality it admits solutions with the corresponding symmetry. The method is due to Mawhin's guiding function one.

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1 Introduction

The Lyapunov method is a fundamental one in studying the stability and invariance of differential equations. Actually it also plays a basic role in proving the existence of periodic solutions. For this we can go back to the Mawhin's guiding function method[18].

Consider the system

$$x' = f(t,x), \tag{1}$$

where $f : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. When $f(t+T,x) \equiv f(t,x)$, namely, f is T-periodic in t, Mawhin established the following well known result on the existence of periodic solutions.

Theorem 1[18]**.** *Assume that there exist $\mathbb{C}^1$ functions $V_i : \mathbb{R}^n \to \mathbb{R}^1$, $i = 0, 1, \cdots, m$, such that*

i) for M_i large enough,

$$\big\langle \nabla V_i(x), f(t,x) \big\rangle \neq 0, \qquad \forall\, |x| \geq M_i;$$

ii) $\sum_{i=0}^{m} |V_i(x)| \to \infty$, as $|x| \to \infty$;

iii)

$$\deg(\nabla V_0, B_{M_0}, 0) \neq 0.$$

Then (1) has T-periodic solutions.

The functions "$V_i(x)$" mentioned above are called the "guiding functions".

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Our question is whether (1) has a rotating-symmetric solution $x(t)$, i.e., $x(t+T) = Qx(t)$, $\forall\, t$, if we assume that for some $Q \in \mathrm{SO}(n)$,

$$f(t+T, x) = Qf(t, Q^{-1}x), \qquad \forall\,(t, x). \tag{2}$$

In the present paper, we investigate the question mentioned above and give an affirmative answer. Our main result is the following:

Theorem 2. *Assume that (2) is true and there exist* $\mathbb{C}^1$ *functions* $V_i(x)$, $i = 0, 1, \cdots, m$, *such that*

i) for M_i *large enough,*

$$\big\langle \nabla V_i(x), f(t, x) \big\rangle \neq 0, \qquad \forall\, |x| \geq M_i;$$

ii) $\sum\limits_{i=0}^{m} |V_i(x)| \to \infty$, *as* $|x| \to \infty$;

iii)

$$\deg(\nabla V_0, B_{M_0} \cap \mathrm{Ker}(\mathrm{id} - Q), 0|_{\mathrm{Ker}(\mathrm{id}-Q)}) \neq 0, \qquad \textit{if } \mathrm{Ker}\,(\mathrm{id} - Q) \neq \{0\},$$

where $\deg(\nabla V_0, B_{M_0}, 0)$ *denotes the Brouwer degree, and* $B_M = \{p \in \mathbb{R}^n : |p| < M_0\}$.

Then (1) has Q*-rotating symmetric solutions* $x(t)$*, i.e.,*

$$x(t+T) = Qx(t), \qquad \forall\, t.$$

Let us make the following comments:

a) The cases of $Q = \mathrm{id}$ or $-\mathrm{id}$ correspond to the ones of T-periodic or T-anti-periodic solutions, respectively. The former is Mawhin's theorem on the existence of T-periodic solutions, and the latter corresponds to T-anti-periodic solutions. If for some positive integer m_0 such that $Q^{m_0} = \mathrm{id}$, then the Q-symmetric solution is just the harmonic solution, i.e., $x(t+m_0T) \equiv x(t)$. For some recent achievements (see [1–3, 6, 8–25]).

b) The general $Q \in \mathrm{SO}(n)$ is thus correspondent to the solutions with Q-rotating symmetry, in particular to some special quasi-periodic solutions. Theorem 2 presents thus a guiding function method to solutions with the rotating symmetry. Some related results can be found in, for example, [4,5,7]. They assumed $f(t,\cdot)$ is periodic in t, and discussed the existence of periodic solutions with space symmetry such as antisymmetry. In our result, we do not assume $f(t,\cdot)$ is periodic in t. Moreover when $Q = \mathrm{diag}(e^{i\theta_1}, \cdots, e^{i\theta_n})$, and $\theta_1, \cdots, \theta_n$ are rationally independent, our solutions are actually the usual quasiperiodic ones with the frequency $(\theta_1, \cdots, \theta_n)$.

The paper is organized as follows. In Section 2, we give the proof of Theorem 2 via the guiding function method. In Section 3, we illustrate some applications of Theorem 2.

2 Proof of the Main Result

Let us start the proof of Theorem 2. Consider the auxiliary system

$$x' = \lambda f(t, x), \tag{3}$$

where $\lambda \in [0, 1]$. Without loss of generality, we assume that the solutions of (3) with respect to initial values are unique. Let $x(t, x_0, \lambda)$ denote the solution of (3) at $x(0) = x_0$.

In the following, we will consider two cases:

Case 1. $(\mathrm{id} - Q)^{-1}$ exists. In order to prove that (3) has a Q-rotating symmetric solution $x(t)$ it suffices to prove

$$Q^{-1}x(T) = x(0).$$

Equivalently, we need to prove that there is $x_0 \in \mathbb{R}^n$ such that

$$H(x_0, \lambda) \equiv x_0 + \lambda(\mathrm{id} - Q)^{-1} \int_0^T f(s, x(s, x_0, \lambda))ds = 0. \tag{4}$$

We claim that there is $K > 0$ such that

$$H(\partial B_k \times [0, 1]) \neq 0. \tag{5}$$

Set

$$K_i = \sup\{|V_i(x)| : |x| \leq M_i\}, \qquad K_* = \sum_{i=0}^{m} K_i,$$

$$D = \Big\{x \in \mathbb{R}^n : \sum_{i=0}^{m} |V_i(x)| < K_* + 1\Big\}.$$

By assumption ii), D is bounded. Hence there is a $K > (m+1)K_*$ such that

$$D \subset B_K.$$

Note that

$$\min_{M_i \leq |x| \leq K} \big|\big\langle f(t, x), \nabla V_i(x)\big\rangle\big| \geq \alpha > 0, \qquad i = 0, 1, \cdots, m. \tag{6}$$

Put

$$V(x) = \sum_{i=0}^{m} |V_i(x)|.$$

Let $y : \mathbb{R}^1 \to \mathbb{R}^n$ such that $y(t + T) = Qy(t),\ \forall\, t$. Then

$$y(t + mT) = Q^m y(t),$$
$$\Longrightarrow |y(t + mT)| = |y(t)|.$$

Hence for any Q-symmetric solution $x(t)$ of (3), if (5) fails, then there is $\{t_k^i\} \subset \mathbb{R}^1$ such that

$$|V_i(x(t_k^i))| \to \sup_{\mathbb{R}^1} |V_i(x(t))|,$$
$$\Longrightarrow \big\langle \nabla V_i(x(t_k^i)), f(t_k^i, x(t_k^i))\big\rangle \longrightarrow 0, \qquad \text{as } \ k \longrightarrow \infty,$$

which together with (6) implies that

$$|x(t_k^i)| < M_i, \qquad i = 0, 1, \cdots, m.$$

This shows that $x(t) \in D,\ \forall\, t$, a contradiction. By the homotopy invariance of the topological degree, we have

$$1 = \deg(\mathrm{id}, B_K, 0) = \deg(H(\cdot, 0), B_K, 0) = \deg(H(\cdot, 1), B_K, 0).$$

Thus there is $x_0^* \in B_K$ such that

$$x_0^* + (\mathrm{id} - Q)^{-1} \int_0^T f(s, x(s, x_0^*, 1))ds = 0,$$

and hence $x(t, x_0^*, 1)$ is the Q-symmetric solution of (1).

Case 2. $\operatorname{Ker}(\mathrm{id}-Q) \neq \{0\}$. We set

$$\mathbb{R}^n = \operatorname{Ker}(\mathrm{id}-Q) \oplus \operatorname{Im}(\mathrm{id}-Q).$$

Thus any $x_0 \in \mathbb{R}^n$ can be rewritten as

$$x_0 = \widehat{x} + \overline{x}, \qquad \widehat{x} \in \operatorname{Ker}(\mathrm{id}-Q), \qquad \overline{x} \in \operatorname{Im}(\mathrm{id}-Q).$$

Let $\widehat{P}: \mathbb{R}^n \to \operatorname{Ker}(\mathrm{id}-Q)$ be a projection. We can rewrite (5) as

$$H((\widehat{x},\overline{x}),\lambda) = \begin{cases} \displaystyle\int_0^T \widehat{P} f(s, x(s, \widehat{x}+\overline{x}, \lambda))ds = 0, \\ \displaystyle\overline{x} + \lambda \int_0^T (\mathrm{id}-\widehat{P}) f(s, x(s, \widehat{x}+\overline{x}, \lambda))ds = 0. \end{cases}$$

By the previous arguments, we have $H(\partial(B_K^{n_1} \times B_K^{n_2}) \times [0,1]) \neq 0$, where $n_1 = \dim(\operatorname{Ker}(\mathrm{id}-Q))$, $n_2 = \dim(\operatorname{Im}(\mathrm{id}-Q))$. Then the homotopy invariance of the topological degree implies

$$\begin{aligned} &\deg\Big(\Big(\int_0^T \widehat{P} f(s,\cdot)ds, \mathrm{id}\Big), B_K^{n_1} \times B_K^{n_2}, 0\Big) \\ =& \deg(H(\cdot,0), B_K^{n_1} \times B_K^{n_2}, 0) = \deg(H(\cdot,1), B_K^{n_1} \times B_K^{n_2}, 0). \end{aligned}$$

Now we claim

$$\deg\Big(\Big(\int_0^T \widehat{P} f(s,\cdot)ds, \mathrm{id}\Big), B_K^{n_1} \times B_K^{n_2}, 0\Big) \neq 0.$$

Equivalently, it suffices to verify

$$\deg\Big(\int_0^T \widehat{P} f(s,\cdot)ds, B_K^{n_1}, 0\Big) \neq 0.$$

By i), we may assume without loss generality that

$$\big\langle \nabla V_0(x), f(t,x)\big\rangle > 0, \qquad \forall\, |x| \geq M_0.$$

Since for each $M \geq M_0$,

$$\deg(\nabla V_0, B_M \cap \operatorname{Ker}(\mathrm{id}-Q), 0|_{\operatorname{Ker}(\mathrm{id}-Q)}) \neq 0,$$

it follows that

$$\begin{aligned} &\big\langle \widehat{P}\nabla V_0(x), \widehat{P} f(t,x)\big\rangle > 0, \qquad \forall\, x \in \operatorname{Ker}(\mathrm{id}-Q) \text{ and } |x| \geq M_0, \\ \Longrightarrow &\Big\langle \widehat{P}\nabla V_0(x), \int_0^T \widehat{P} f(t,x)\mathrm{dt}\Big\rangle > 0, \qquad \forall\, x \in \operatorname{Ker}(\mathrm{id}-Q) \text{ and } |x| \geq M_0. \end{aligned}$$

Consider the homotopy

$$H(x,\lambda) = \lambda \widehat{P}\nabla V_0(x) + (1-\lambda)\int_0^T \widehat{P} f(s,x)ds.$$

Then

$$\begin{aligned} &\big\langle \widehat{P}\nabla V_0(x), H(x,\lambda)\big\rangle \\ =&\lambda|\widehat{P}\nabla V_0(x)|^2 + (1-\lambda)\Big\langle \widehat{P}\nabla V_0(x), \int_0^T \widehat{P} f(s,x)ds\Big\rangle \\ >&0, \qquad \forall\, x \in \operatorname{Ker}(\mathrm{id}-Q) \text{ and } |x| \geq M_0. \end{aligned}$$

The homotopy invariance implies thus

$$\begin{aligned}&\deg\Big(\int_0^T \widehat{P}f(s,\cdot)ds, B_K^{n_1}, 0|_{\mathrm{Ker(id}-Q)}\Big)\\ =&\deg(H(\cdot,1), B_K^{n_1}, 0|_{\mathrm{Ker(id}-Q)}) = \deg(H(\cdot,0), B_K^{n_1}, 0|_{\mathrm{Ker(id}-Q)})\\ =&\deg(\nabla V_0, B_{M_0}\cap \mathrm{Ker(id}-Q), 0|_{\mathrm{Ker(id}-Q)}) \neq 0.\end{aligned}$$

The proof is complete. □

3 Applications

In this section, we describe some applications of Theorem 2.

Let us consider the gradient system

$$x' = -\nabla V(x) + f(t), \tag{7}$$

where $V:\mathbb{R}^n \to \mathbb{R}^1$ is a $\mathbb{C}^1$ even function, $f(t+T) = -f(t)$; moreover, as $|x|\to\infty$,

$$|V(x)| \longrightarrow \infty, \qquad |\nabla V(x)| \longrightarrow \infty.$$

We have

Theorem 3. *Under the above assumptions, (7) has an anti-symmetric solution.*

Proof. Put $V_0(x) = V(x)$. Then

$$\begin{aligned}&\langle \nabla V(x), -\nabla V(x) + f(t)\rangle\\ =&-|\nabla V(x)|^2 + \langle \nabla V(x), f(t)\rangle\\ \le&-|\nabla V(x)|^2 + \frac{1}{2}|\nabla V(x)|^2 + \frac{1}{2}|f(t)|^2\\ =&-\frac{1}{2}|\nabla V(x)|^2 + \frac{1}{2}|f(t)|^2\\ <&0, \qquad \text{as } |x|\gg 1.\end{aligned}$$

Since $V(x)$ is even, $\nabla V(x)$ is odd. By Borsuk's Theorem, $\deg(\nabla V, B_M, 0) \neq 0$ for M large. The conclusion follows from Theorem 2. This completes the proof. □

In applications, the following result seems more convenient.

Theorem 4. *Assume that there is an $M>0$ such that*

$$\langle Bx, Bf(t,x)\rangle \le -\alpha < 0, \qquad \forall\, x\in\mathbb{R}^n \text{ and } |x|\ge M,$$

where B is a nonsingular matrix of order n. Then (1) has a Q-symmetric solution.

Proof. Put

$$V(x) = \frac{1}{2}|Bx|^2.$$

Then

$$\langle \nabla V, f(t,x)\rangle = \langle Bx, Bf(t,x)\rangle \le -\alpha < 0, \qquad \forall\, x\in\mathbb{R}^n \text{ and } |x|\ge M.$$

Clearly, $\deg(\nabla V, B_M, 0) = (-1)^\beta$, where β is the sum of the multiplicity of all negative eigenvalues for the matrix B^*B. Hence $\beta = 0$ and $\deg(\nabla V, B_M, 0) = 1$. The conclusion of the theorem follows. □

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