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Uniformity in Double Designs

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Abstract In constructing two-level fractional factorial designs, the so-called doubling method has been employed. In this paper, we study the problem of uniformity in double designs. The centered L_2-discrepancy is employed as a measure of uniformity. We derive results connecting the centered L_2-discrepancy value of $D(X)$ and generalized wordlength pattern of X, which show the uniformity relationship between $D(X)$ and X. In addition, we also obtain lower bounds of centered L_2-discrepancy value of $D(X)$, which can be used to assess uniformity of $D(X)$.

Keywords double design; generalized wordlength pattern; uniform design; uniformity pattern
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1 Introduction

Two-level fractional factorial designs are widely used in industrial and agriculture experiments and many scientific investigations. The issue of construction and justifiable interpretation for fractional factorial designs with two-level has received a great deal of attention in the design literature. Recently, the so-called doubling method, which is employed to construct two-level fractional factorial designs, is a simple but powerful method, particularly in constructing two-level fractional factorial designs of resolution IV (Chen and Cheng[1]). Suppose X is an $n \times k$ matrix with two distinct entries, $+1$ and -1. We call the $2n \times 2k$ matrix $\begin{pmatrix} X & X \\ X & -X \end{pmatrix}$ the double of X, denoted by $D(X)$. Suppose X defines an $n-$run design with k two-level factors, where the two levels are denoted by $+1$ and -1, each column of X corresponds to a factor and each row defines a factor-level combination. Then $D(X)$ defines a design which has $2n$ factor-level combinations and $2k$ two-level factors. We call $D(X)$ a double design of X, and X the original design of $D(X)$.

Doubling method was firstly used by Plackett and Burman[11] to construct orthogonal main-effect plans. Recently, Chen and Cheng[1] attempted to construct a doubling design $D(X)$ of resolution IV via a regular fractional factorial design X of resolution IV, and proved that there exists a projection design of $D(X)$ with resolution IV or higher. Xu and Cheng[14] developed a general complementary design theory for doubling designs. We say that a regular design of resolution IV or higher is maximal if its resolution reduces to three whenever an extra factor is added. Chen and Cheng[1] and Xu and Cheng[14] respectively discussed the function of doubling method in constructing maximal designs with two-level. Xu and Cheng[14] also showed that one can choose some minimum aberration (Fries and Hunter[6]) projection designs from some maximal designs. In this paper, our aim is to discuss the issue of double designs in terms of uniformity.

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In this paper, we shall employ the centered L_2-discrepancy as a measure of uniformity, see Hickernell[7] and Fang and Qin[5] for details. If one fractional factorial design has smaller centered L_2-discrepancy value, then this design possesses better uniformity. The uniformity criterion favors designs with the best uniformity. Fang and Mukerjee[4] firstly found the analytic connection between uniformity and aberration (Fries and Hunter[6]) in an arbitrary regular two-level factorials, and indicated that the uniformity criterion is almost equivalent to the aberration criterion. Fang, Ma and Mukerjee[3] further showed that uniformity, orthogonality and aberration criteria agree quite well. Fang and Qin[5] investigated the projection properties of two-level factorials onto different dimensions in view of uniformity, and proposed the so-called projection uniformity pattern to assess and compare two-level factorials. Zhang and Qin[17] and Song and Qin[13] developed the theory of projection uniformity pattern. In this paper, we will discuss some analytic links between the centered L_2-discrepancy value of $D(X)$ and generalized wordlength pattern /uniformity pattern of X, and derive some lower bounds of centered L_2-discrepancy value of $D(X)$, which can be used to discuss the issue of uniformity of $D(X)$.

The paper is organized as follows. In Section 2, the centered L_2-discrepancy, generalized minimum aberration and minimum projection uniformity are introduced. In Section 3, the connection between the centered L_2-discrepancy value of $D(X)$ and generalized wordlength pattern/uniformity pattern is established. Section 4 provides two new lower bounds of the centered L_2-discrepancy value of $D(X)$. We close in Remark section with some notes and comments.

2 Basic Concepts

Consider a set, denoted by $\mathcal{D}(n, 2^k)$, of all two-level fractional factorial designs with n runs and k factors, where n runs are not necessarily distinct. For any design $X = (x_{il}) \in \mathcal{D}(n, 2^k)$, its centered L_2-discrepancy value, denoted by $CD(X)$, can be calculated as follows (Hickernell[7]):

$$\begin{aligned}
&[CD(X)]^2\\
=&\Big(\frac{13}{12}\Big)^k - \frac{2}{n}\sum_{i=1}^{n}\prod_{l=1}^{k}\Big(1+\frac{1}{2}\Big|d_{il}-\frac{1}{2}\Big|-\frac{1}{2}\Big|d_{il}-\frac{1}{2}\Big|^2\Big)\\
&+\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\prod_{l=1}^{k}\Big(1+\frac{1}{2}\Big|d_{il}-\frac{1}{2}\Big|+\frac{1}{2}\Big|d_{jl}-\frac{1}{2}\Big|-\frac{1}{2}|d_{il}-d_{jl}|\Big),
\end{aligned} \tag{1}$$

where $d_{il} = (x_{il} + 2)/4$.

Following (1), the centered L_2-discrepancy value of $D(X)$, denoted by $CD(D(X))$, can be easily computed by the following formula

$$\begin{aligned}
&\big[CD(D(X))\big]^2\\
=&\Big(\frac{13}{12}\Big)^{2k} - \frac{2}{n}\sum_{i=1}^{n}\prod_{l=1}^{k}\Big(1+\frac{1}{2}\Big|d_{il}-\frac{1}{2}\Big|-\frac{1}{2}\Big|d_{il}-\frac{1}{2}\Big|^2\Big)^2\\
&+\frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\prod_{l=1}^{k}\Big(1+\frac{1}{2}\Big|d_{il}-\frac{1}{2}\Big|+\frac{1}{2}\Big|d_{jl}-\frac{1}{2}\Big|-\frac{1}{2}|d_{il}-d_{jl}|\Big)^2\\
&+\frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\prod_{l=1}^{k}\Big[\Big(1+\frac{1}{2}\Big|d_{il}-\frac{1}{2}\Big|+\frac{1}{2}\Big|d_{jl}-\frac{1}{2}\Big|-\frac{1}{2}|d_{il}-d_{jl}|\Big)
\end{aligned}$$

$$\cdot\Big(1+\frac{1}{2}\Big|d_{il}-\frac{1}{2}\Big|+\frac{1}{2}\Big|d_{jl}-\frac{1}{2}\Big|-\frac{1}{2}|d_{il}+d_{jl}-1|\Big)\Big], \tag{2}$$

where d_{il} was defined in (1).

To make our work self contained, below is a brief description of generalized minimum aberration proposed by Xu and Wu [15] and minimum projection uniformity proposed by Fang and Qin[5].

For any design $X=(x_{il})\in\mathcal{D}(n,2^k)$ and j $(0\le j\le k)$, define

$$E_j(X)=\frac{1}{n}\big|\{(x_1,x_2):d_H(x_1,x_2)=j\}\big|,$$

where x_1 and x_2 are two runs of X, and $d_H(x_1,x_2)$ is the Hamming distance between x_1 and x_2, namely, the number of places where they differ. $|\Omega|$ denotes the cardinality of the set Ω. Define

$$A_j(X)\equiv\frac{1}{n}\sum_{j=0}^{k}P_i(j;k)E_j(X), \tag{3}$$

where $P_i(j;k)=\sum\limits_{r=0}^{i}(-1)^r\binom{j}{r}\binom{k-j}{i-r}$ is the Krawtchouk polynomial. The vector $(A_1(X),\cdots,A_k(X))$ is called generalized wordlength pattern of X by Xu and Wu[15]. The generalized minimum aberration criterion is to sequentially minimize $A_j(X)$ for $j=1,\cdots,k$.

Based on the centered L_2-discrepancy, Fang and Qin[5] used a quantity, $I_i(X)$, to measure the overall projection uniformity of X on i-subdimension, where

$$I_i(X)=\frac{1}{8^i}\sum_{j=1}^{i}\binom{k-j}{k-i}A_j(X), \tag{4}$$

$1\le i\le k$. The smaller $I_i(X)$-value, the better the uniformity of X on i-subdimension. The vector $(I_1(X),\cdots,I_k(X))$ is referred to as uniformity pattern of X in Fang and Qin[5]. The minimum projection uniformity criterion is to sequentially minimize $I_j(X)$ for $j=1,\cdots,k$.

3 Relationship Between $CD(D(X))$ and Generalized Wordlength Pattern of X

Fang and Mukerjee[4] firstly gave an analytic link between $CD(X)$ and the wordlength pattern of X. According to this result, we can easily connect $CD(D(X))$ to the generalized wordlength pattern of $D(X)$. However, in this section, we will give an analytic relationship between $CD(D(X))$ and the generalized wordlength pattern of X.

To begin with, we give another expression of $CD(D(X))$ in the following lemma, whose proof is due to Fang, Lu and Winker[2].

Lemma 1. *$X\in\mathcal{D}(n,2^k)$, we have*

$$[CD(D(X))]^2=C_0+\frac{1}{n^2}\sum_{1\le i<j\le n}\Big(\frac{25}{16}\Big)^{\lambda_{ij}}, \tag{5}$$

where λ_{ij} is the coincidence number between the ith and jth runs of X,

$$C_0=\Big(\frac{13}{12}\Big)^{2k}-2\Big(\frac{35}{32}\Big)^{2k}+\frac{1}{2}\Big(\frac{5}{4}\Big)^{k}+\frac{1}{2n}\Big(\frac{25}{16}\Big)^{k}.$$

The following theorem gives an analytic relationship between $CD(D(X))$ and the generalized wordlength pattern of X.

Theorem 1. *Let $X \in \mathcal{D}(n, 2^k)$, then*

$$[CD(D(X))]^2 = C_1 + \frac{1}{2}\Big(\frac{41}{32}\Big)^k \sum_{j=1}^{k} \Big(\frac{9}{41}\Big)^j A_j(X), \tag{6}$$

where

$$C_1 = \Big(\frac{13}{12}\Big)^{2k} - 2\Big(\frac{35}{32}\Big)^{2k} + \frac{1}{2}\Big(\frac{5}{4}\Big)^k + \frac{1}{2}\Big(\frac{41}{32}\Big)^k.$$

Proof. Suppose $x_i = (x_{i1}, \cdots, x_{ik})$ is the ith run of X, $1 \le i \le k$ and

$$C_{00} = \Big(\frac{13}{12}\Big)^{2k} - 2\Big(\frac{35}{32}\Big)^{2k} + \frac{1}{2}\Big(\frac{5}{4}\Big)^k.$$

Note that $d_H(x_i, x_j) = k - \lambda_{ij}$.

By (5), we have

$$\begin{aligned}[CD(D(X))]^2 =& C_{00} + \frac{1}{2n^2}\Big(\frac{25}{16}\Big)^k \sum_{i=1}^{n}\sum_{j=1}^{n} \Big(\frac{16}{25}\Big)^{d_H(x_i,x_j)} \\ =& C_{00} + \frac{1}{2n}\Big(\frac{25}{16}\Big)^k \sum_{i=0}^{k} E_i(X)\Big(\frac{16}{25}\Big)^i \\ =& C_{00} + \frac{1}{2^{k+1}}\Big(\frac{25}{16}\Big)^k \sum_{i=0}^{k}\sum_{j=0}^{k} A_j(X)P_i(j;k)\Big(\frac{16}{25}\Big)^i \\ =& C_{00} + \frac{1}{2}\Big(\frac{41}{32}\Big)^k + \frac{1}{2}\Big(\frac{41}{32}\Big)^k \sum_{j=1}^{k} \Big(\frac{9}{41}\Big)^j A_j(X),\end{aligned}$$

where, the second last equality holds due to the equation

$$E_i(X) = \frac{n}{2^k}\sum_{j=0}^{k} A_j(X)P_i(j;k),$$

the last equality holds since $\sum\limits_{i=0}^{k} P_i(j;k)a^i = (1+a)^{k-j}(1-a)^j$. This completes the proof. □

Theorem 1 indicates a link between the double design $D(X)$ and its original design X, namely, the centered L_2-discrepancy value of $D(X)$ only depends on the generalized wordlength pattern of X. Note that the leading factor, $\frac{1}{2}\big(\frac{41}{32}\big)^k\big(\frac{9}{41}\big)^j$, of $A_j(X)$ in (6) is a positive fraction, and decreases exponentially with j. Hence, the double design $D(X)$ should have smaller centered L_2-discrepancy value, that is, $D(X)$ possesses better uniformity when its original design X has less aberration. Fang and Mukerjee[4] and Fang, Ma and Mukerjee[3] showed that if $D(X)$ has better uniformity, then $D(X)$ also has less aberration. Thus, in order to construct one double design $D(X)$ with less aberration, we should choose a design X with best uniformity as an original design.

As an application of Theorem 1, we can obtain the following theorem, which gives the link between $CD(D(X))$ and the uniformity pattern of X.

Theorem 2. *Let $X \in \mathcal{D}(n, 2^k)$. Then*

$$[CD(D(X))]^2 = C_1 + \frac{1}{2}\sum_{i=1}^{k}\Big(\frac{9}{4}\Big)^i I_i(X), \tag{7}$$

where C_1 was defined in Theorem 1.

Proof. From (4), it is not hard to see that

$$A_j(X) = \sum_{i=1}^{j}(-1)^{j-i}\binom{k-i}{j-i}8^i I_i(X). \tag{8}$$

If we substitute (8) into (6), we have

$$\begin{aligned}[CD(D(X))]^2 =& C_1 + \frac{1}{2}\Big(\frac{41}{32}\Big)^k \sum_{i=1}^{k} 8^i I_i(X)\sum_{j=i}^{k}(-1)^{j-i}\binom{k-i}{j-i}\Big(\frac{9}{41}\Big)^j \\ =& C_1 + \frac{1}{2}\Big(\frac{41}{32}\Big)^k \sum_{i=1}^{k}\Big(\frac{72}{41}\Big)^i I_i(X)\sum_{t=0}^{k-i}\binom{k-i}{t}\Big(-\frac{9}{41}\Big)^t \\ =& C_1 + \frac{1}{2}\sum_{i=1}^{k}\Big(\frac{9}{4}\Big)^i I_i(X),\end{aligned}$$

and the proof is completed. □

4 Lower Bounds of $CD(D(X))$

For any design X in $\mathcal{D}(n, 2^k)$, the lower bound of $CD(X)$ had been obtained by Fang and Mukerjee[4], Fang, Ma and Mukerjee[3] and Fang, Lu and Winker[2], respectively. The corresponding lower bound of $CD(D(X))$ can be directly obtained in those references above literature. We state this result in the following lemma.

Lemma 2. *Let $X \in \mathcal{D}(n, 2^k)$, we have*

$$\begin{aligned}[CD(D(X))]^2 \geq& \Big(\frac{13}{12}\Big)^{2k} - 2\Big(\frac{35}{32}\Big)^{2k} + \Big(\frac{9}{8}\Big)^{2k} \\ &+ \frac{1}{4n^2}\sum_{i=1}^{2k}\Big(\frac{1}{8}\Big)^i\binom{2k}{i}R_{2n,i}(2^i - R_{2n,i}),\end{aligned} \tag{9}$$

where $R_{2n,i}$ is the residual of $2n$ (mod 2^i).

In this section, we will work out another lower bound of $CD(D(X))$ via the original design X. It is more valuable to use this lower bound to measure uniformity of given double design $D(X)$. This lower bound may also be used as a benchmark for searching uniform designs. If the centered L_2-discrepancy value of $D(X)$ achieves this lower bound, then $D(X)$ is called a uniform design.

Following Zhang and Qin[17], we know that for any i $(1 \leq i \leq k)$,

$$I_i(X) \geq \frac{1}{n^2 8^i}\binom{k}{i}R_{n,i}(2^i - R_{n,i}), \tag{10}$$

where $R_{n,i}$ was similarly defined in Lemma 2.

Substituting (10) into (7), we obtain the following theorem.

Theorem 3. *Let $X \in \mathcal{D}(n, 2^k)$, we have*

$$[CD(D(X))]^2 \geq C_1 + \frac{1}{2n^2}\sum_{i=1}^{k}\Big(\frac{9}{32}\Big)^i \binom{k}{i} R_{n,i}(2^i - R_{n,i}), \tag{11}$$

where C_1 was defined in Theorem 1.

Note that when all factor-level combinations of any i-subdimension projection design of X occur with the same frequency, $R_{n,i} = 0$. In this case, X is said to have strength i. Thus, Theorem 3 gives the lower bound of $CD(D(X))$ in view of orthogonality of the original design X. The lower bound in Theorem 3 is attainable. For example, the lower bound in Theorem 3 is attained if and only if X is a regular 2^{k-i} design of strength $k-i$.

Lemma 2 and Theorem 3 give two different lower bounds of $CD(D(X))$, respectively. Via numerical results, we find that the lower bound of $CD(D(X))$ in Theorem 3 is bigger than that in Lemma 2. Therefore, in assessing uniformity of the double design $D(X)$, we should use the lower bound of $CD(D(X))$ in Theorem 3 as a benchmark.

The following theorem gives another lower bound of $CD(D(X))$.

Theorem 4. *Let $X \in \mathcal{D}(n, 2^k)$ and the levels of each factor occur equally often in X. Then*

$$[CD(D(X))]^2 \geq C_0 + \frac{n-1}{2n}\Big(\frac{25}{16}\Big)^{\theta}\Big(1 + \frac{9}{16}f\Big), \tag{12}$$

where C_0 was defined in Lemma 1, and $\lambda = k(n-2)/[2(n-1)]$, θ is the largest integer contained in λ, $f = \lambda - \theta$.

Proof. Note that $|\{\lambda_{ij} : 1 \leq i < j \leq n\}| = n(n-1)/2 \equiv m$. The elements in $\{\lambda_{ij} : 1 \leq i < j \leq n\}$ are denoted by $\beta_r,\ r = 1, \cdots, m$. Define a function $\psi(\beta_r)$ by

$$\psi(\beta_r) = \Big(\frac{25}{16}\Big)^{\beta_r}.$$

Obviously, ψ is a Shchur-convex function on $\mathcal{R}^+ \to \mathcal{R}$. Furthermore, we define another function $\Psi(X; \psi)$ by

$$\Psi(X; \psi) = \sum_{r=1}^{m}\psi(\beta_r) = \sum_{r=1}^{m}\Big(\frac{25}{16}\Big)^{\beta_r}.$$

Following the definition of Schur-exponential criterion in [16], it is clear that $\Psi(X; \psi)$ is a Schur-exponential criterion. Applying Theorem 1 in [16], we may infer that for the Schur-exponential criterion $\Psi(X; \psi)$, its lower bound is $m(1-f)\psi(\theta) + mf\psi(\theta+1)$ and

$$\sum_{1 \leq i < j \leq n}\Big(\frac{25}{16}\Big)^{\lambda_{ij}} \geq \frac{n(n-1)}{2}\Big(\frac{25}{16}\Big)^{\theta}\Big(1 + \frac{9}{16}f\Big). \tag{13}$$

If we substitute (13) into (5), (12) follows. This completes the proof. □

Theorem 4 gives a lower bound of $CD(D(X))$, which is often attainable. If λ is an integer, and all Hamming distances between any distinct pair of runs of $X \in \mathcal{D}(n, 2^k)$ are equal, that is, $d_H(x_i, x_j) = k - \lambda$, $1 \leq i \neq j \leq n$, then the lower bound on the right-hand side of (12) can be attained. For example, when X is a two-level saturated orthogonal array (Mukerjee and Wu[10]) or a two-level supersaturated design obtained by Lin[9] from half-Hadamard designs, then the lower bound on the right-hand side of (12) is attained. In this case, the double design $D(X)$ is a uniform design.

If λ is not an integer, then the lower bound on the right-hand side of (12) may be achieved. When the difference among all Hamming distances between any distinct pair of runs of $X \in \mathcal{D}(n, 2^k)$ does not exceed one, that is, $2m(1-f)\, d_H(x_i, x_j)$'s have the value $k-\theta$, and other $2mf$ ones have the value $k-\theta-1$, the lower bound on the right-hand side of (12) is also attained. For example, when X is obtained by adding one balanced two-level factor to or removing one factor from a two-level saturated orthogonal array, the lower bound on the right-hand side of (12) is achieved.

Summarizing the above discussions, we are able to make the following conclusion.

Corollary 1. *For $X \in \mathcal{D}(n, 2^k)$, we have*

(i) if all Hamming distances between any distinct pair of runs of X are equal, then the double design $D(X)$ is a uniform design;

(ii) if X is obtained by adding one balanced two-level factor to or removing one factor from a two-level saturated orthogonal array, then the double design $D(X)$ is a uniform design.

Note that the lower bound of $CD(D(X))$ in Theorem 3 is based on the column balance of X. It is useful for assessing uniformity of $D(X)$ when X is an orthogonal array. Another lower bound of $CD(D(X))$ in Theorem 4 is based on the Hamming distance for rows of X. It is clear that the lower bound of $CD(D(X))$ in Theorem 3 is sharper than that in Theorem 4, since the latter lower bound may be achieved for some nearly saturated orthogonal arrays or supersaturated designs. The lower bound of $CD(D(X))$ in Theorem 4 is more useful for assessing uniformity of $D(X)$ when X is a nearly saturated orthogonal array or a supersaturated design.

5 Concluding Remarks

In this paper, we further study the justifiable interpretation of double designs in terms of uniformity. Some analysis relationships between the centered L_2-discrepancy value of $D(X)$ and generalized wordlength pattern/uniformity pattern of X are reported. These results show that in order to construct double designs with the best aberration or projection uniformity, the original designs with best uniformity should be chosen. Two lower bounds of centered L_2-discrepancy value of $D(X)$ are also obtained. They are not only used to assess uniformity of known double designs, but also used as a guided selection of searching optimal double designs.

Note that there are many other possible discrepancies to be employed as measures of uniformity, such as the symmetric L_2-discrepancy, the wrap-around L_2-discrepancy, the unanchored L_2-discrepancy proposed by Hickernell[8], the discrete discrepancy defined by Qin and Fang[12] and the Lee discrepancy proposed by Zhou, Ning and Song[18]. It is of theoretical interest to develop similar results for these discrepancies. It seems feasible to follow the current approach; however, the details are omitted.

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References

[1] Chen, H.G., Cheng, C.S. Doubling and projection: a method of constructing two-level designs of resolution Ⅳ. *Ann. Statist.*, 34: 546–558 (2006)

[2] Fang, K.T., Lu, X., Winker, P. Lower bounds for centered and wrap-around L_2-discrepancies and construction of uniform designs by threshold accepting. *J. Complexity*, 19: 692–711 (2003)

[3] Fang, K.T., Ma, C.X., Mukerjee, R. Uniformity in fractional factorials. In: Fang, K.T., Hickernell, F.J., Niederreiter H. (eds.), Monte Carlo and Quasi-Monte Carlo Methods, Springer-Verlag, Berlin, 2002

[4] Fang, K.T., Mukerjee, R. A connection between uniformity and aberration in regular fractions of two-level factorials. *Biometrika*, 87: 193–198 (2000)

[5] Fang, K.T, Qin, H. Uniformity pattern and related criteria for two-level factorials. *Science in China* (Series A), 48: 1–11 (2005)

[6] Fries, A., Hunter, W.G. Minimum aberration 2^{k-p} designs. *Technometrics*, 22: 601–608 (1980)

[7] Hickernell, F.J. A generalized discrepancy and quadrature error Bound. *Mathematics of Computation*, 67: 299–322 (1998)

[8] Hickernell, F.J. Lattice Rules: How Well do They Measure Up? In: Hellekalek P, Larcher G (eds.) Random and Quasi-Random Point Sets. Lecture Notes in Statistics, Vol.138, Springer-Verlag, New York, 1998, 109–166

[9] Lin, D.K.J. A new class of supersaturated designs. *Technometrics*, 35: 28–31 (1993)

[10] Mukerjee, R, Wu, C.F.J. On the existence of saturated and nearly saturated asymmetrical orthogonal arrays. *Ann. Statist.*, 23: 2102–2115 (1995)

[11] Plackett, R.L., Burman, J.P. The design of optimum multi-factorial experiments. *Biometrika*, 33: 305–325 (1946)

[12] Qin, H., Fang, K.T. Discrete discrepancy in factorial designs. *Metrika*, 60: 59–72 (2004)

[13] Song, S., Qin, H. Application of minimum projection uniformity in complementary designs. *Acta. Math. Sci.*, 30: 180–186 (2010)

[14] Xu, H.Q., Cheng, C.S. A complementary design theory for doubling. *Ann. Statist.*, 36: 445–457 (2006)

[15] Xu, H.Q., Wu, C.F.J. Generalized minimum aberration for asymmetrical fractional factorial designs. *Ann. Statist.*, 29: 549–560 (2001)

[16] Zhang, A.J., Fang, K.T., Li, R., Sudjianto, A. Majorization framework for balanced lattice designs. *Ann. Statist.*, 33: 2837–2853 (2005)

[17] Zhang, S.L., Qin, H. Minimum projection uniformity criterion and its application. *Statist. Probab. Letters*, 76: 634–640 (2006)

[18] Zhou, Y.D., Ning, J.H., Song, X.B. Lee discrepancy and its applications in experimental designs. *Statist. Probab. Letters*, 78: 1933–1942 (2008)