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Global Parametric Sufficient Efficiency Conditions for Semiinfinite Multiobjective Fractional Programming Problems Containing Generalized (α, η, ρ)-V-Invex Functions

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Abstract In this paper, we discuss numerous sets of global parametric sufficient efficiency conditions under various generalized (α, η, ρ)-V-invexity assumptions for a semiinfinite multiobjective fractional programming problem.

Keywords Semiinfinite programming, multiobjective fractional programming, generalized invex functions, infinitely many equality and inequality constraints, parametric sufficient efficiency conditions.

2000 MR Subject Classification 90C29; 90C30; 90C32; 90C34; 90C46

1 Introduction

Our aim in this paper is to state and prove a number of parametric sufficient efficiency results under various generalized (α, η, ρ)-V-invexity assumptions for the following semiinfinite multiobjective fractional programming problem:

$$\text{(P)} \qquad \text{Minimize } \varphi(x) = \big(\varphi_1(x), \cdots, \varphi_p(x)\big) = \Big(\frac{f_1(x)}{g_1(x)}, \dots, \frac{f_p(x)}{g_p(x)}\Big)$$

subject to

$$G_j(x,t) \leq 0, \qquad \text{for all} \quad t \in T_j, \quad j \in \underline{q},$$
$$H_k(x,s) = 0, \qquad \text{for all} \quad s \in S_k, \quad k \in \underline{r},$$
$$x \in \mathbb{R}^n,$$

where p, q and r are positive integers, $\mathbb{R}^n$ is the n-dimensional Euclidean space, for each $j \in \underline{q} \equiv \{1, 2, \cdots, q\}$ and $k \in \underline{r}$, T_j and S_k are compact subsets of complete metric spaces, for each $i \in \underline{p}$, f_i and g_i are real-valued functions defined on $\mathbb{R}^n$, for each $j \in \underline{q}$, $G_j(\cdot, t)$ is a real-valued function defined on $\mathbb{R}^n$ for all $t \in T_j$, for each $k \in \underline{r}$, $H_k(\cdot, s)$ is a real-valued function defined on $\mathbb{R}^n$ for all $s \in S_k$, for each $j \in \underline{q}$ and $k \in \underline{r}$, $G_j(x, \cdot)$ and $H_k(x, \cdot)$ are continuous real-valued functions defined, respectively, on T_j and S_k for all $x \in \mathbb{R}^n$, and for each $i \in \underline{p}$, $g_i(x) > 0$ for all x satisfying the constraints of (P).

Multiobjective programming problems like (P) but with a finite number of constraints, that is, when the functions G_j are independent of t, and the functions H_k are independent of s, have been the subject of numerous investigations in the past three decades. Several classes of static and dynamic optimization problems with multiple fractional objective functions have been studied and, consequently, a number of sufficient efficiency and duality results are

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currently available for these problems in the related literature. Fairly extensive lists of references pertaining to various aspects of multiobjective fractional programming are available in [42–44]. For more information about the vast general area of multiobjective programming, the reader may consult [25, 32, 39, 41].

A close examination of these and other related sources will readily reveal the fact that despite a phenomenal proliferation of publications in several areas of multiobjective programming, so far *semiinfinite nonlinear multiobjective fractional programming problems* have not been studied at all. In the present study, we shall formulate a number of parametric sufficient efficiency results for (P) under various generalized (α, η, ρ)-V-invexity assumptions. Their relevance to various parametric duality relations for (P) is discussed in [50].

A mathematical programming problem with a finite number of variables and infinitely many constraints is called a *semiinfinite programming problem*. Problems of this type have been utilized for the modeling and analysis of a staggering array of theoretical as well as concrete, real-world, practical problems. More specifically, semiinfinite programming concepts and techniques have found relevance and applications in approximation theory, statistics, game theory, engineering design (earthquake-resistant design of structures, design of control systems, digital filters, electronic circuits, etc.), boundary value problems, defect minimization for operator equations, geometry, random graphs, graphs related to Newton flows, wavelet analysis, reliability testing, environmental protection planning, decision making under uncertainty, semidefinite programming, geometric programming, disjunctive programming, optimal control problems, robotics, and continuum mechanics, among others. For a wealth of information pertaining to various aspects of semiinfinite programming, including areas of applications, optimality conditions, duality relations, and numerical algorithms, the reader is referred to [2, 5, 6, 9–13, 16–19, 23, 30, 33, 34]. Relatively more recent applications of generalized semiinfinite programming to the formulation and analysis of anticipatory systems and gene-environment networks are discussed in [4, 35–38], and to a very interesting gemstone cutting problem in [40].

From these and other related publications one can easily see that the two important trends, namely, the ubiquity of duality theories and generalized convexity concepts that have been playing significant roles in the evolution of optimization theory and methodology in general and in nonlinear programming in particular are conspicuously missing in the area of semiinfinite nonlinear programming. In fact, until very recently there were no publications dealing with nonlinear semiinfinite programming that made substantial use of any class of generalized convex functions in establishing sufficient optimality conditions or duality results. Some small steps toward bridging this gap have recently been taken by the authors in [45–49]. However, so far no sufficient efficiency results based on generalized convexity concepts have been published in the related literature for any kind of semiinfinite multiobjective fractional programming problems.

The rest of this paper is organized as follows. In Section 2, we present a number of definitions and auxiliary results which will be needed in the sequel. In Section 3, we begin our discussion of sufficient efficiency conditions where we formulate and prove numerous sets of sufficiency criteria under a variety of generalized (α, η, ρ)-V-invexity assumptions that are placed on the individual as well as certain combinations of the problem functions. Utilizing two partitioning schemes, in Section 4 we establish several sets of generalized parametric sufficient efficiency results each of which is in fact a family of such results whose members can easily be identified by appropriate choices of certain sets and functions. Finally, in Section 5 we summarize our main results and also point out some further research opportunities arising from certain modifications of the principal problem model considered in this paper.

Evidently, all the parametric sufficient efficiency results established in this paper can easily be modified and restated for each one of the following seven classes of nonlinear programming problems, which are special cases of (P):

$$\text{(P1)} \qquad \underset{x \in \mathbb{F}}{\text{Minimize}} \ \big(f_1(x), \cdots, f_p(x)\big);$$

$$\text{(P2)} \qquad \underset{x\in\mathbb{F}}{\text{Minimize}} \ \frac{f_1(x)}{g_1(x)};$$

$$\text{(P3)} \qquad \underset{x\in\mathbb{F}}{\text{Minimize}} \ f_1(x),$$

where $\mathbb{F}$ (assumed to be nonempty) is the feasible set of (P), that is,

$$\mathbb{F} = \big\{x \in \mathbb{R}^n : G_j(x,t) \leqq 0 \ \text{ for all } t \in T_j, \ j \in \underline{q}, H_k(x,s) = 0 \ \text{ for all } s \in S_k, \ \ k \in \underline{r}\big\};$$

$$\text{(P4)} \qquad \text{Minimize} \ \Big(\frac{f_1(x)}{g_1(x)}, \cdots, \frac{f_p(x)}{g_p(x)}\Big)$$

subject to

$$\widetilde{G}_j(x) \leqq 0, \quad j \in \underline{q}; \qquad \widetilde{H}_k(x) = 0, \quad k \in \underline{r}; \qquad x \in \mathbb{R}^n,$$

where f_i and g_i, $i \in \underline{p}$, are as defined in the description of (P), $\widetilde{G_j}$, $j \in \underline{q}$ and $\widetilde{H}_k$, $k \in \underline{r}$, are real-valued functions defined on $\mathbb{R}^n$, and for each $i \in \underline{p}$, the denominators of the objective functions of (P4) are positive for all feasible solutions;

$$\text{(P5)} \qquad \underset{x\in\mathbb{G}}{\text{Minimize}} \ \big(f_1(x), \cdots, f_p(x)\big);$$

$$\text{(P6)} \qquad \underset{x\in\mathbb{G}}{\text{Minimize}} \ \frac{f_1(x)}{g_1(x)};$$

$$\text{(P7)} \qquad \underset{x\in\mathbb{G}}{\text{Minimize}} \ f_1(x),$$

where $\mathbb{G}$ is the feasible set of (P4), that is,

$$\mathbb{G} = \big\{x \in \mathbb{R}^n : \ \widetilde{G}_j(x) \leqq 0, \ j \in \underline{q}, \ \widetilde{H}_k(x) = 0, \ k \in \underline{r}\big\}.$$

Since in most cases these results can easily be altered and rephrased for each one of the above seven problems, we shall not state them explicitly.

2 Preliminaries

In this section we recall, for convenience of reference, the definitions of certain classes of generalized convex functions which will be needed in the sequel. We begin by defining an invex function, which has been instrumental in creating a vast array of interesting and important classes of generalized convex functions.

Definition 2.1. *Let f be a real-valued differentiable function defined on $\mathbb{R}^n$. Then f is said to be η-*invex *at y if there exists a function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$,*

$$f(x) - f(y) \geqq \langle \nabla f(y), \eta(x,y)\rangle,$$

where $\nabla f(y) = (\partial f(y)/\partial y_1, \partial f(y)/\partial y_2, \cdots, \partial f(y)/\partial y_n)$ is the gradient of f at y, and $\langle a, b\rangle$ denotes the inner product of the vectors a and b; f is said to be η-invex on $\mathbb{R}^n$ if the above inequality holds for all $x, y \in \mathbb{R}^n$.

From this definition it is clear that every real-valued differentiable convex function is invex with $\eta(x,y) = x - y$. This generalization of the concept of convexity was originally proposed by Hanson[14] who showed that for a nonlinear programming problem of the form

$$\text{Minimize } f(x) \ \text{ subject to } \ g_i(x) \leqq 0, \qquad i \in \underline{m}, \ \ x \in \mathbb{R}^n,$$

where the differentiable functions $f, g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \underline{m}$, are invex with respect to the same function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, the Karush-Kuhn-Tucker necessary optimality conditions are also sufficient. The term *invex* (for *in*variant con*vex*) was coined by Craven[3] to signify the fact that the invexity property, unlike convexity, remains invariant under bijective coordinate transformations.

In a similar manner, one can readily define η-pseudoinvex and η-quasiinvex functions as generalizations of differentiable pseudoconvex and quasiconvex functions.

The notion of invexity has been generalized in several directions. For our present purposes, we shall need a simple extension of invexity, namely, ρ-invexity which was originally defined in [20].

Let h be a differentiable real-valued function defined on $\mathbb{R}^n$.

Definition 2.2. *The function h is said to be (strictly) (η,ρ)-invex at x^* if there exists a function $\eta:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ and $\rho\in\mathbb{R}$ such that for each $x\in\mathbb{R}^n$ $(x\neq x^*)$,*

$$h(x)-h(x^*)(>)\geqq\langle\nabla h(x^*),\eta(x,x^*)\rangle+\rho\|x-x^*\|^2.$$

Definition 2.3. *The function h is said to be* (strictly) (η,ρ)-pseudoinvex *at $x^*\in\mathbb{R}^n$ if there exists a function $\eta:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ and $\rho\in\mathbb{R}$ such that for each $x\in\mathbb{R}^n$ $(x\neq x^*)$,*

$$\langle\nabla h(x^*),\eta(x,x^*)\rangle\geqq-\rho\|x-x^*\|^2\implies h(x)(>)\geqq h(x^*).$$

Definition 2.4. *The function h is said to be* (prestrictly) (η,ρ)-quasiinvex *at $x^*\in\mathbb{R}^n$ if there exists a function $\eta:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ and $\rho\in\mathbb{R}$ such that for each $x\in\mathbb{R}^n$,*

$$h(x)(<)\leqq h(x^*)\implies\langle\nabla h(x^*),\eta(x,x^*)\rangle\leqq-\rho\|x-x^*\|^2.$$

From the above definitions it is clear that if h is (η,ρ)-invex at x^*, then it is both (η,ρ)-pseudoinvex and (η,ρ)-quasiinvex at x^*, if h is (η,ρ)-quasiinvex at x^*, then it is prestrictly (η,ρ)-quasiinvex at x^*, and if h is strictly (η,ρ)-pseudoinvex at x^*, then it is (η,ρ)-quasiinvex at x^*.

Let the function $F=(F_1,F_2,\cdots,F_N):\mathbb{R}^n\to\mathbb{R}^N$ be differentiable at x^*. The following generalizations of the notions of invexity, pseudoinvexity, and quasiinvexity for vector-valued functions were originally proposed in [21].

Definition 2.5. *The function F is said to be* (strictly) $(\alpha,\eta,\overline{\rho})$-V-invex at x^* *if there exist functions $\eta:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ and $\alpha_i:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}_+\backslash\{0\}\equiv(0,\infty)$, and $\overline{\rho}_i\in\mathbb{R}$, $i\in\underline{N}$, such that for each $x\in\mathbb{R}^n$ $(x\neq x^*)$ and $i\in\underline{N}$,*

$$F_i(x)-F_i(x^*)\,(>)\geqq\langle\alpha_i(x,x^*)\nabla F_i(x^*),\eta(x,x^*)\rangle+\overline{\rho}_i\|x-x^*\|^2.$$

Definition 2.6. *The function F is said to be* (strictly) $(\beta,\eta,\widetilde{\rho})$-V-pseudoinvex at x^* *if there exist functions $\eta:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ and $\beta_i:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}_+\backslash\{0\}$, $i\in\underline{N}$, and $\widetilde{\rho}\in\mathbb{R}$ such that for each $x\in\mathbb{R}^n$ $(x\neq x^*)$,*

$$\Big\langle\sum_{i=1}^N\nabla F_i(x^*),\eta(x,x^*)\Big\rangle\geqq-\widetilde{\rho}\|x-x^*\|^2$$
$$\Longrightarrow\sum_{i=1}^N\beta_i(x,x^*)F_i(x)(>)\geqq\sum_{i=1}^N\beta_i(x,x^*)F_i(x^*).$$

Definition 2.7. *The function F is said to be* (prestrictly) $(\gamma,\eta,\widehat{\rho})$-V-quasiinvex at x^* *if there exist functions $\eta:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ and $\gamma_i:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}_+\backslash\{0\}$, $i\in\underline{N}$, and $\widehat{\rho}\in\mathbb{R}$ such that for each $x\in\mathbb{R}^n$,*

$$\sum_{i=1}^N\gamma_i(x,x^*)F_i(x)(<)\leqq\sum_{i=1}^N\gamma_i(x,x^*)F_i(x^*)$$
$$\Longrightarrow\Big\langle\sum_{i=1}^N\nabla F_i(x^*),\eta(x,x^*)\Big\rangle\leqq-\widehat{\rho}\|x-x^*\|^2.$$

In contrast to the case of (η,ρ)-invex, (η,ρ)-pseudoinvex, and (η,ρ)-quasiinvex functions, the relationships among the three classes of functions specified in Definitions (2.5–2.7) are not immediately obvious. However, the underlying relationships can be determined by appropriate choices of the functions α_i,β_i, and $\gamma_i,\ i\in \underline{N}$, and the real numbers $\overline{\rho}_i,\ i\in\underline{N},\ \widetilde{\rho}$, and $\widehat{\rho}$. Indeed, it is easily seen that an $(\alpha,\eta,\overline{\rho})$-V-invex function is both $(\beta,\eta,\widetilde{\rho})$-V-pseudoinvex and $(\gamma,\eta,\widehat{\rho})$-V-quasiinvex if we choose $\gamma_i=\beta_i=1/\alpha_i,\ i\in\underline{N}$, and $\widehat{\rho}=\widetilde{\rho}=\sum_{i=1}^{N}\overline{\rho}_i/\alpha_i$.

In the proofs of the sufficiency theorems, sometimes it may be more convenient to use certain alternative but equivalent forms of the above definitions. These are obtained by considering the contrapositive statements. For example, $(\beta,\eta,\widetilde{\rho})$-V-pseudoinvexity can be defined in the following equivalent way:

The function F is said to be $(\beta,\eta,\widetilde{\rho})$-*V-pseudoinvex at* x^* if there exist functions $\eta:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ and $\beta_i:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}_+\backslash\{0\},\ i\in\underline{N}$, and $\widetilde{\rho}\in\mathbb{R}$ such that for each $x\in\mathbb{R}^n$,

$$\sum_{i=1}^{N}\beta_i(x,x^*)F_i(x)<\sum_{i=1}^{N}\beta_i(x,x^*)F_i(x^*)\Longrightarrow\Big\langle\sum_{i=1}^{N}\nabla F_i(x^*),\eta(x,x^*)\Big\rangle<-\widetilde{\rho}\|x-x^*\|^2.$$

The concept of ρ-invexity has been extended in many ways, and various types of generalized ρ-invex functions have been utilized for establishing a wide range of sufficient optimality criteria and duality relations for several classes of nonlinear programming problems. For more information about invex functions, the reader may consult [1, 3, 7, 8, 15, 24, 26, 28, 31], and for recent surveys of these and related functions, the reader is referred to [22, 29].

In the sequel, we shall also need a consistent notation for vector inequalities. For $a,b\in\mathbb{R}^m$, the following order notation will be used: $a\geqq b$ if and only if $a_i\geqq b_i$ for all $i\in\underline{m}$; $a\geqslant b$ if and only if $a_i\geqq b_i$ for all $i\in\underline{m}$, but $a\neq b$; $a>b$ if and only if $a_i>b_i$ for all $i\in\underline{m}$; and $a\ngeqslant b$ is the negation of $a\geqslant b$.

Consider the multiobjective problem

$$(\mathrm{P}^*)\qquad \underset{x\in\mathbb{F}}{\text{Minimize}}\ F(x)=(F_1(x),\cdots,F_p(x)),$$

where $F_i,\ i\in\underline{p}$, are real-valued functions defined on $\mathbb{R}^n$.

An element $x^\circ\in\mathbb{F}$ is said to be an *efficient (Pareto optimal, nondominated, noninferior)* solution of (P^*) if there exists no $x\in\mathbb{F}$ such that $F(x)\leqslant F(x^\circ)$. In the area of multiobjective programming, there exist several versions of the notion of efficiency most of which are discussed in [25, 32, 39, 41]. However, throughout this paper, we shall deal exclusively with the efficient solutions of (P) in the sense defined above.

For the purpose of comparison with the sufficient efficiency conditions that will be proposed and discussed in this paper, we next recall a set of necessary efficiency conditions established for (P) in [49].

Theorem 2.1[49]**.** *Let $x^*\in\mathbb{F}$, let $\lambda^*=\varphi(x^*)$, for each $i\in\underline{p}$, let f_i and g_i be continuously differentiable at x^*, for each $j\in\underline{q}$, let the function $G_j(\cdot,t)$ be continuously differentiable at x^* for all $t\in T_j$, and for each $k\in\underline{r}$, let the function $H_k(\cdot,s)$ be continuously differentiable at x^* for all $s\in S_k$. If x^* is an efficient solution of (P), if the generalized Guignard constraint qualification holds at x^*, and if for each $i_0\in\underline{p}$, the set cone $\big(\{\nabla G_j(x^*,t):t\in\widehat{T}_j(x^*),j\in\underline{q}\}\cup\{\nabla f_i(x^*)-\lambda_i^*\nabla g_i(x^*):i\in\underline{p},\ i\neq i_0\}\big)+span\big(\{\nabla H_k(x^*,s):s\in S_k,k\in\underline{r}\}\big)$ is closed, then there exist $u^*\in U$ and integers ν_0^* and ν^*, with $0\leqq\nu_0^*\leqq\nu^*\leqq n+1$, such that there exist ν_0^* indices j_m, with $1\leqq j_m\leqq q$, together with ν_0^* points $t^m\in\widehat{T}_{j_m}(x^*),\ m\in\underline{\nu_0^*},\ \nu^*-\nu_0^*$ indices k_m, with $1\leqq k_m\leqq r$, together with $\nu^*-\nu_0^*$ points $s^m\in S_{k_m}$ for $m\in\underline{\nu^*}\backslash\underline{\nu_0^*}$, and ν^**

real numbers v_m^*, with $v_m^* > 0$ for $m \in \underline{\nu_0^*}$, with the property that

$$\sum_{i=1}^{p} u_i^* [\nabla f_i(x^*) - \lambda_i^* \nabla g_i(x^*)] + \sum_{m=1}^{\nu_0^*} v_m^* \nabla G_{j_m}(x^*, t^m) + \sum_{m=\nu_0^*+1}^{\nu^*} v_m^* \nabla H_{k_m}(x^*, s^m) = 0,$$

where cone (V) is the conic hull of the set $V \subset \mathbb{R}^n$ (i.e., the smallest convex cone containing V), span(V) is the linear hull of V (i.e., the smallest subspace containing V), $\widehat{T}_j(x^*) = \{t \in T_j : G_j(x^*, t) = 0\}$, and $U = \{u \in \mathbb{R}^p : u > 0,\ \sum_{i=1}^{p} u_i = 1\}$.

3 Sufficient Efficiency Conditions

In this section, we present several sets of sufficiency results in which various generalized (α, η, ρ)-V-invexity assumptions are imposed on certain vector functions whose components are the individual as well as some combinations of the problem functions.

Theorem 3.1. *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*) \geqq 0$, let the functions $f_i, g_i,\ i \in \underline{p},\ G_j(\cdot, t)$, and $H_k(\cdot, s)$ be differentiable at x^* for all $t \in T_j$ and $s \in S_k,\ j \in \underline{q},\ k \in \underline{r}$, and assume that there exist $u^* \in U$ and integers ν_0 and ν, with $0 \leqq \nu_0 \leqq \nu \leqq n+1$, such that there exist ν_0 indices j_m, with $1 \leqq j_m \leqq q$, together with ν_0 points $t^m \in \widehat{T}_{j_m}(x^*),\ m \in \underline{\nu_0},\ \nu - \nu_0$ indices k_m, with $1 \leqq k_m \leqq r$, together with $\nu - \nu_0$ points $s^m \in S_{k_m},\ m \in \underline{\nu} \backslash \underline{\nu_0}$, and ν real numbers v_m^*, with $v_m^* > 0$ for $m \in \underline{\nu_0}$, with the property that*

$$\sum_{i=1}^{p} u_i^* \big[\nabla f_i(x^*) - \lambda_i^* \nabla g_i(x^*)\big] + \sum_{m=1}^{\nu_0} v_m^* \nabla G_{j_m}(x^*, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m^* \nabla H_{k_m}(x^*, s^m) = 0. \quad (1)$$

Assume, furthermore, that either one of the following two sets of conditions holds:

(a) *(i)* $(f_1, \cdots, f_p)$ *is* $(\theta, \eta, \overline{\rho})$*-V-invex at* x^**;*

(ii) $(-g_1, \cdots, -g_p)$ *is* $(\xi, \eta, \widetilde{\rho})$*-V-invex at* x^**;*

(iii) $\big(v_1^* G_{j_1}(\cdot, t^1), \cdots, v_{\nu_0}^* G_{j_{\nu_0}}(\cdot, t^{\nu_0})\big)$ *is* $(\pi, \eta, \widehat{\rho})$*-V-invex at* x^**;*

(iv) $\big(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot, s^{\nu_0+1}), \ldots, v_\nu^* H_{k_\nu}(\cdot, s^\nu)\big)$ *is* $(\delta, \eta, \breve{\rho})$*-V-invex at* x^**;*

(v) $\theta_i = \xi_j = \pi_k = \delta_l = \sigma$ *for all* $i,\ j \in \underline{p}, k \in \underline{\nu_0}$*, and* $l \in \underline{\nu} \backslash \underline{\nu_0}$*;*

(vi) $\sum_{i=1}^{p} u_i^* (\overline{\rho}_i + \lambda_i^* \widetilde{\rho}_i) + \sum_{m=1}^{\nu_0} \widehat{\rho}_m + \sum_{m=\nu_0+1}^{\nu} \breve{\rho}_m \geqq 0$*;*

(b) the function $\big(L_1(\cdot, u^*, v^*, \lambda^*, \overline{t}, \overline{s}), \cdots, L_p(\cdot, u^*, v^*, \lambda^*, \overline{t}, \overline{s})\big)$ *is* $(\theta, \eta, 0)$ *-V-pseudoinvex at* x^**, where*

$$L_i(z, u^*, v^*, \lambda^*, \overline{t}, \overline{s}) = u_i^* \Big[f_i(z) - \lambda_i^* g_i(z) + \sum_{m=1}^{\nu_0} v_m^* G_{j_m}(z, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m^* H_{k_m}(z, s^m)\Big], \qquad i \in \underline{p}.$$

Then x^* is an efficient solution of (P).

Proof. Let x be an arbitrary feasible solution of (P).

(a) Keeping in mind that $u^* > 0$ and $\lambda^* \geqq 0$, we have

$$
\begin{aligned}
&\sum_{i=1}^{p} u_i^*[f_i(x) - \lambda_i^* g_i(x)] \\
&= \sum_{i=1}^{p} u_i^*\{f_i(x) - f_i(x^*) - \lambda_i^*[g_i(x) - g_i(x^*)]\} \ \ (\text{since } \lambda^* = \varphi(x^*)) \\
&\geqq \sum_{i=1}^{p} u_i^*\big[\sigma(x,x^*)\langle \nabla f_i(x^*) - \lambda_i^* \nabla g_i(x^*), \eta(x,x^*)\rangle \\
&\quad + (\overline{\rho}_i + \lambda_i^* \widetilde{\rho}_i)\|x - x^*\|^2\big] \ \ (\text{by (i), (ii), and (v)}) \\
&= -\,\sigma(x,x^*)\Big\langle \sum_{m=1}^{\nu_0} v_m^* \nabla G_{j_m}(x^*, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m^* \nabla H_{k_m}(x^*, s^m), \eta(x,x^*)\Big\rangle \\
&\quad + \sum_{i=1}^{p} u_i^*(\overline{\rho}_i + \lambda_i^* \widetilde{\rho}_i)\|x - x^*\|^2 \ \ (\text{by (1)}) \\
&\geqq \sum_{m=1}^{\nu_0} v_m^*[G_{j_m}(x^*, t^m) - G_{j_m}(x, t^m)] + \sum_{m=\nu_0+1}^{\nu} v_m^* H_{k_m}(x^*, s^m) \\
&\quad + \Big[\sum_{i=1}^{p} u_i(\overline{\rho}_i + \lambda^* \widetilde{\rho}_i) + \sum_{m=1}^{\nu_0} \widehat{\rho}_m + \sum_{m=\nu_0+1}^{\nu} \breve{\rho}_m\Big]\|x - x^*\|^2 \\
&\qquad (\text{by (iii), (iv), (v), and the primal feasibility of } x) \\
&\geqq 0, \qquad\qquad (2)
\end{aligned}
$$

where the last inequality follows from (vi), the primal feasibility of x^*, and the fact that $t^m \in \widehat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$.

Since $u^* > 0$, the above inequality implies that

$$\big(f_1(x) - \lambda_1^* g_1(x), \cdots, f_p(x) - \lambda_p^* g_p(x)\big) \nleq (0, \cdots, 0),$$

which in turn implies that

$$\varphi(x) = \Big(\frac{f_1(x)}{g_1(x)}, \cdots, \frac{f_p(x)}{g_p(x)}\Big) \nleq (\lambda_1^*, \cdots, \lambda_p^*) = \varphi(x^*).$$

Since $x \in \mathbb{F}$ was arbitrary, we conclude from this inequality that x^* is an efficient solution of (P).

(b) By our $(\theta, \eta, 0)$-V-pseudoinvexity assumption, (1) implies that

$$
\begin{aligned}
&\sum_{i=1}^{p} \theta_i(x,x^*)\Big\{u_i^*[f_i(x) - \lambda_i^* g_i(x)] + \sum_{m=1}^{\nu_0} v_m^* G_{j_m}(x, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m^* H_{k_m}(x, s^m)\Big\} \\
\geqq &\sum_{i=1}^{p} \theta_i(x,x^*)\Big\{u_i^*[f_i(x^*) - \lambda_i^* g_i(x^*)] + \sum_{m=1}^{\nu_0} v_m^* G_{j_m}(x^*, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m^* H_{k_m}(x^*, s^m)\Big\}.
\end{aligned}
$$

Because $x^* \in \mathbb{F}$ and $t^m \in \widehat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$, the right-hand side of this inequality is equal to zero, and so we have that

$$\sum_{i=1}^{p} \theta_i(x,x^*)\Big\{u_i^*[f_i(x) - \lambda_i^* g_i(x)] + \sum_{m=1}^{\nu_0} v_m^* G_{j_m}(x, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m^* H_{k_m}(x, s^m)\Big\} \geqq 0.$$

But $x \in \mathbb{F}$ and $v_m^* > 0$ for each $m \in \underline{\nu_0}$, and hence the above inequality reduces to

$$\sum_{i=1}^{p} u_i^* \theta_i(x, x^*) \big[f_i(x) - \lambda_i^* g_i(x)\big] \geqq 0. \tag{3}$$

Since $u^* > 0$ and $\theta(x, x^*) > 0$, the above inequality implies that

$$\big(f_1(x) - \lambda_1^* g_1(x), \cdots, f_p(x) - \lambda_p^* g_p(x)\big) \nleq (0, \cdots, 0),$$

which in turn implies that

$$\varphi(x) = \Big(\frac{f_1(x)}{g_1(x)}, \cdots, \frac{f_p(x)}{g_p(x)}\Big) \nleq (\lambda_1^*, \cdots, \lambda_p^*) = \varphi(x^*).$$

Since $x \in \mathbb{F}$ was arbitrary, we conclude from this inequality that x^* is an efficient solution of (P).

In Theorem 3.1, separate (α, η, ρ)-V-invexity assumptions were imposed on the vector functions $(f_1, \cdots, f_p)$ and $(-g_1, \ldots, -g_p)$. In the remainder of this section, we shall formulate some sufficient efficiency conditions in which various generalized (α, η, ρ)-V-invexity requirements will be placed on the vector function $(\mathcal{E}_1(\cdot, \lambda, u), \ldots, \mathcal{E}_p(\cdot, \lambda, u))$, where for each $i \in \underline{p}$, the component function $\mathcal{E}_i(\cdot, \lambda, u)$ is defined, for fixed λ and u, on $\mathbb{R}^n$ by

$$\mathcal{E}_i(z, \lambda, u) = u_i\big[f_i(z) - \lambda_i g_i(z)\big].$$

Theorem 3.2. *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*)$, let the functions f_i, g_i, $i \in \underline{p}$, $G_j(\cdot, t)$, and $H_k(\cdot, s)$ be differentiable at x^* for all $t \in T_j$ and $s \in S_k$, $j \in \underline{q}$, $k \in \underline{r}$, and assume that there exist $u^* \in U$ and integers ν_0 and ν, with $0 \leqq \nu_0 \leqq \nu \leqq n+1$, such that there exist ν_0 indices j_m, with $1 \leqq j_m \leqq q$, together with ν_0 points $t^m \in \widehat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m, with $1 \leqq k_m \leqq r$, together with $\nu - \nu_0$ points $s^m \in S_{k_m}$, $m \in \underline{\nu}\backslash\underline{\nu_0}$, and ν real numbers v_m^*, with $v_m^* > 0$ for $m \in \underline{\nu_0}$, such that (1) holds. Assume, furthermore, that any one of the following four sets of hypotheses is satisfied:*

(a) *(i)* $\big(\mathcal{E}_1(\cdot, \lambda^*, u^*), \cdots, \mathcal{E}_p(\cdot, \lambda^*, u^*)\big)$ *is (θ, η, ρ)-V-pseudoinvex at x^*;*

(ii) $\big(v_1^* G_{j_1}(\cdot, t^1), \cdots, v_{\nu_0}^* G_{j_{\nu_0}}(\cdot, t^{\nu_0})\big)$ *is $(\pi, \eta, \widetilde{\rho})$-V-quasiinvex at x^*;*

(iii) $\big(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot, s^{\nu_0+1}), \cdots, v_\nu^* H_{k_\nu}(\cdot, s^\nu)\big)$ *is $(\delta, \eta, \widehat{\rho})$-V-quasiinvex at x^*;*

(iv) $\rho + \widetilde{\rho} + \widehat{\rho} \geqq 0$;

(b) *(i)* $\big(\mathcal{E}_1(\cdot, \lambda^*, u^*), \cdots, \mathcal{E}_p(\cdot, \lambda^*, u^*)\big)$ *is prestrictly (θ, η, ρ)-V-quasiinvex at x^*;*

(ii) $\big(v_1^* G_{j_1}(\cdot, t^1), \cdots, v_{\nu_0}^* G_{j_{\nu_0}}(\cdot, t^{\nu_0})\big)$ *is $(\pi, \eta, \widetilde{\rho})$-V-quasiinvex at x^*;*

(iii) $\big(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot, s^{\nu_0+1}), \cdots, v_\nu^* H_{k_\nu}(\cdot, s^\nu)\big)$ *is $(\delta, \eta, \widehat{\rho})$-V-quasiinvex at x^*;*

(iv) $\rho + \widetilde{\rho} + \widehat{\rho} > 0$;

(c) *(i)* $\big(\mathcal{E}_1(\cdot, \lambda^*, u^*), \cdots, \mathcal{E}_p(\cdot, \lambda^*, u^*)\big)$ *is prestrictly (θ, η, ρ)-V-quasiinvex at x^*;*

(ii) $\big(v_1^* G_{j_1}(\cdot, t^1), \cdots, v_{\nu_0}^* G_{j_{\nu_0}}(\cdot, t^{\nu_0})\big)$ *is strictly $(\pi, \eta, \widetilde{\rho})$-V-pseudoinvex at x^*;*

(iii) $\big(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot, s^{\nu_0+1}), \cdots, v_\nu^* H_{k_\nu}(\cdot, s^\nu)\big)$ *is $(\delta, \eta, \widehat{\rho})$-V-quasiinvex at x^*;*

(iv) $\rho + \widetilde{\rho} + \widehat{\rho} \geqq 0$;

(d) *(i)* $\big(\mathcal{E}_1(\cdot,\lambda^*,u^*),\cdots,\mathcal{E}_p(\cdot,\lambda^*,u^*)\big)$ *is prestrictly* (θ,η,ρ)*-V-quasiinvex at* x^**;*

(ii) $\big(v_1^*G_{j_1}(\cdot,t^1),\cdots,v_{\nu_0}^*G_{j_{\nu_0}}(\cdot,t^{\nu_0})\big)$ *is* $(\pi,\eta,\widetilde{\rho})$*-V-quasiinvex at* x^**;*

(iii) $\big(v_{\nu_0+1}^*H_{k_{\nu_0+1}}(\cdot,s^{\nu_0+1}),\cdots,v_\nu^*H_{k_\nu}(\cdot,s^\nu)\big)$ *is strictly* $(\delta,\eta,\widehat{\rho})$*-V-pseudoinvex at* x^**;*

(iv) $\rho+\widetilde{\rho}+\widehat{\rho}\geqq 0.$

Then x^* *is an efficient solution of (P).*

Proof. (a) Let x be an arbitrary feasible solution of (P). Since $x\in\mathbb{F}$ and $t^m\in\widehat{T}_{j_m}(x^*),\ m\in\underline{\nu_0}$, we have $G_{j_m}(x,t^m)\leqq 0=G_{j_m}(x^*,t^m)$, and hence

$$\sum_{m=1}^{\nu_0}v_m^*\pi_m(x,x^*)G_{j_m}(x,t^m)\leqq\sum_{m=1}^{\nu_0}v_m^*\pi_m(x,x^*)G_{j_m}(x^*,t^m),$$

which in view of (ii) implies that

$$\Big\langle\sum_{m=1}^{\nu_0}v_m^*\nabla G_{j_m}(x^*,t^m),\eta(x,x^*)\Big\rangle\leqq-\widetilde{\rho}\|x-x^*\|^2. \tag{4}$$

Similarly, we can show that our assumptions in (iii) combined with the feasibility of x and x^* lead to the following inequality:

$$\Big\langle\sum_{m=\nu_0+1}^{\nu}v_m^*\nabla H_{k_m}(x^*,s^m),\eta(x,x^*)\Big\rangle\leqq-\widehat{\rho}\|x-x^*\|^2. \tag{5}$$

Now because of (4), (5) and (iv), (1) reduces to

$$\Big\langle\sum_{i=1}^{p}u_i^*[\nabla f_i(x^*)-\lambda_i^*\nabla g_i(x^*)],\eta(x,x^*)\Big\rangle\geqq-\rho\|x-x^*\|^2,$$

which in view of (i) implies that

$$\sum_{i=1}^{p}u_i^*\theta_i(x,x^*)[f_i(x)-\lambda_i^*g_i(x)]\geqq\sum_{i=1}^{p}u_i^*\theta_i(x,x^*)[f_i(x^*)-\lambda_i^*g_i(x^*)]=0,$$

where the equality follows from the fact that $\lambda^*=\varphi(x^*)$. In the proof of part (b) of Theorem 3.1, it was shown that this inequality leads to the desired conclusion that x^* is an efficient solution of (P).

(b)–(d) The proofs are similar to that of part (a).

In the remainder of this section, we briefly discuss certain modifications of Theorems 3.1 and 3.2 obtained by replacing (1) with an inequality.

Theorem 3.3. *Let* $x^*\in\mathbb{F}$*, let* $\lambda^*=\varphi(x^*)\geqq 0$*, let the functions* $f_i,g_i,\ i\in\underline{p},G_j(\cdot,t)$*, and* $H_k(\cdot,s)$ *be differentiable at* x^* *for all* $t\in T_j$ *and* $s\in S_k,\ j\in\underline{q},\ k\in\underline{r}$*, and assume that there exist* $u^*\in U$ *and integers* ν_0 *and* ν*, with* $0\leqq\nu_0\leqq\nu\leqq n+1$*, such that there exist* ν_0 *indices* j_m*, with* $1\leqq j_m\leqq q$*, together with* ν_0 *points* $t^m\in\widehat{T}_{j_m}(x^*),\ m\in\underline{\nu_0},\ \nu-\nu_0$ *indices* k_m*, with* $1\leqq k_m\leqq r$*, together with* $\nu-\nu_0$ *points* $s^m\in S_{k_m},\ m\in\underline{\nu}\backslash\underline{\nu_0}$*, and* ν *real numbers* v_m^**, with* $v_m^*>0$ *for* $m\in\underline{\nu_0}$*, such that the following inequality holds:*

$$\Big\langle\sum_{i=1}^{p}u_i^*\big[\nabla f_i(x^*)-\lambda_i^*\nabla g_i(x^*)\big]+\sum_{m=1}^{\nu_0}v_m^*\nabla G_{j_m}(x^*,t^m)+\sum_{m=\nu_0+1}^{\nu}v_m^*\nabla H_{k_m}(x^*,s^m),\eta(x,x^*)\Big\rangle\geqq 0\qquad\text{for all }\ x\in\mathbb{F}, \tag{6}$$

where $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a given function. Furthermore, assume that either one of the two sets of conditions specified in Theorem 3.1 is satisfied. Then x^* is an efficient solution of (P).

Although the proofs of Theorems 3.1 and 3.3 are essentially the same, their contents are somewhat different. This can easily be seen by comparing (1) with (6). We observe that any solution of (1) is also a solution of (6), but the converse is not necessarily true. Moreover, (1) is a system of n equations, whereas (6) is a single inequality. Evidently, from a computational point of view, (1) is preferable to (6) because of the dependence of the latter on the feasible set of (P).

The modified version of Theorem 3.2 can be stated in a similar manner.

4 Generalized Sufficiency Criteria

In this section, we discuss several families of sufficient efficiency results under various generalized (α, η, ρ)-V-invexity hypotheses imposed on certain vector functions whose components are formed by considering different combinations of the problem functions. This is accomplished by employing a certain type of partitioning scheme which was originally proposed in [27] for the purpose of constructing generalized dual problems for nonlinear programming problems. For this we need some additional notation.

Let ν_0 and ν be integers, with $1 \leqq \nu_0 \leqq \nu \leqq n+1$, and let $\{J_0, J_1, \cdots, J_M\}$ and $\{K_0, K_1, \cdots, K_M\}$ be partitions of the sets $\underline{\nu_0}$ and $\underline{\nu} \backslash \underline{\nu_0}$, respectively; thus, $J_i \subseteq \underline{\nu_0}$ for each $i \in \underline{M} \cup \{0\}$, $J_i \cap J_j = \emptyset$ for each $i, j \in \underline{M} \cup \{0\}$ with $i \neq j$, and $\bigcup_{i=0}^{M} J_i = \underline{\nu_0}$. Obviously, similar properties hold for $\{K_0, K_1, \cdots, K_M\}$. Moreover, if m_1 and m_2 are the numbers of the partitioning sets of $\underline{\nu_0}$ and $\underline{\nu} \backslash \underline{\nu_0}$, respectively, then $M = \max\{m_1, m_2\}$ and $J_i = \emptyset$ or $K_i = \emptyset$ for $i > \min\{m_1, m_2\}$.

In addition, we use the real-valued functions $\Phi_i(\cdot, \lambda, u, v, \bar{t}, \bar{s})$, $i \in \underline{p}$, and $\Lambda_\tau(\cdot, v, \bar{t}, \bar{s})$, $\tau \in \underline{M}$, defined, for fixed $u, v, \lambda, \bar{t} \equiv (t^1, t^2, \cdots, t^{\nu_0})$, and $\bar{s} \equiv (s^{\nu_0+1}, s^{\nu_0+2}, \cdots, s^{\nu})$, on $\mathbb{R}^n$ as follows:

$$\begin{aligned}\Phi_i(z, u, v, \lambda, \bar{t}, \bar{s}) =& u_i \Big[f_i(z) - \lambda_i g_i(z) + \sum_{m \in J_0} v_m G_{j_m}(z, t^m) \\ &+ \sum_{m \in K_0} v_m H_{k_m}(z, s^m) \Big], \qquad i \in \underline{p}, \\ \Lambda_\tau(z, v, \bar{t}, \bar{s}) =& \sum_{m \in J_\tau} v_m G_{j_m}(z, t^m) + \sum_{m \in K_\tau} v_m H_{k_m}(z, s^m), \qquad \tau \in \underline{M}.\end{aligned}$$

Making use of the sets and functions defined above, we can now formulate our first collection of generalized sufficiency results for (P) as follows.

Theorem 4.1. *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*)$, let the functions f_i, g_i, $i \in \underline{p}$, $G_j(\cdot, t)$, and $H_k(\cdot, s)$ be differentiable at x^* for all $t \in T_j$ and $s \in S_k$, $j \in \underline{q}$, $k \in \underline{r}$, and assume that there exist $u^* \in U$ and integers ν_0 and ν, with $0 \leqq \nu_0 \leqq \nu \leqq n+1$, such that there exist ν_0 indices j_m, with $1 \leqq j_m \leqq q$, together with ν_0 points $t^m \in \widehat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m, with $1 \leqq k_m \leqq r$, together with $\nu - \nu_0$ points $s^m \in S_{k_m}$, $m \in \underline{\nu} \backslash \underline{\nu_0}$, and ν real numbers v_m^*, with $v_m^* > 0$ for $m \in \underline{\nu_0}$, such that (1) holds. Assume, furthermore, that any one of the following three sets of hypotheses is satisfied:*

(a) *(i)* $\big(\Phi_1(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s}), \cdots, \Phi_p(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s})\big)$ *is $(\theta, \eta, \bar{\rho})$-V-pseudoinvex at x^*;*

(ii) $\big(\Lambda_1(\cdot, v^*, \bar{t}, \bar{s}), \cdots, \Lambda_M(\cdot, v^*, \bar{t}, \bar{s})\big)$ *is $(\pi, \eta, \tilde{\rho})$-V-quasiinvex at x^*;*

(iii) $\bar{\rho} + \tilde{\rho} \geqq 0$*;*

(b) *(i)* $\big(\Phi_1(\cdot,u^*,v^*,\lambda^*,\overline{t},\overline{s}),\cdots,\Phi_p(\cdot,u^*,v^*,\lambda^*,\overline{t},\overline{s})\big)$ *is prestrictly* $(\theta,\eta,\overline{\rho})$*-V-quasiinvex at* x^**;*

(ii) $\big(\Lambda_1(\cdot,v^*,\overline{t},\overline{s}),\cdots,\Lambda_M(\cdot,v^*,\overline{t},\overline{s})\big)$ *is* $(\pi,\eta,\widetilde{\rho})$*-V-quasiinvex at* x^**;*

(iii) $\overline{\rho}+\widetilde{\rho}>0$*;*

(c) *(i)* $\big(\Phi_1(\cdot,u^*,v^*,\lambda^*,\overline{t},\overline{s}),\cdots,\Phi_p(\cdot,u^*,v^*,\lambda^*,\overline{t},\overline{s})\big)$ *is prestrictly* $(\theta,\eta,\overline{\rho})$*-V-quasiinvex at* x^**;*

(ii) $\big(\Lambda_1(\cdot,v^*,\overline{t},\overline{s}),\cdots,\Lambda_M(\cdot,v^*,\overline{t},\overline{s})\big)$ *is strictly* $(\pi,\eta,\widetilde{\rho})$*-V-pseudoinvex at* x^**;*

(iii) $\overline{\rho}+\widetilde{\rho}\geqq 0.$

Then x^* *is an efficient solution of (P).*

Proof. Let x be an arbitrary feasible solution of (P).

(a) It is clear that (1) can be expressed as follows:

$$\sum_{i=1}^{p}u_i^*\Big[\nabla f_i(x^*)-\lambda_i^*\nabla g_i(x^*)+\sum_{m\in J_0}v_m^*\nabla G_{j_m}(x^*,t^m)+\sum_{m\in K_0}v_m^*\nabla H_{k_m}(x^*,s^m)\Big]$$
$$+\sum_{\tau=1}^{M}\Big[\sum_{m\in J_\tau}v_m^*\nabla G_{j_m}(x^*,t^m)+\sum_{m\in K_\tau}v_m^*\nabla H_{k_m}(x^*,s^m)\Big]=0. \tag{7}$$

Since $x,x^*\in\mathbb{F}$, $v_m^*>0$, and $t^m\in\widehat{T}_{j_m}(x^*),\ m\in\underline{\nu_0}$, it follows that for each $\tau\in\underline{M}$,

$$\begin{aligned}\Lambda_\tau(x,v^*,\overline{t},\overline{s})&=\sum_{m\in J_\tau}v_m^*G_{j_m}(x,t^m)+\sum_{m\in K_\tau}v_m^*H_{k_m}(x,s^m)\leqq 0\\&=\sum_{m\in J_\tau}v_m^*G_{j_m}(x^*,t^m)+\sum_{m\in K_\tau}v_m^*H_{k_m}(x^*,s^m)\\&=\Lambda_\tau(x^*,v^*,\overline{t},\overline{s}),\end{aligned}$$

and hence

$$\sum_{\tau=1}^{M}\pi_\tau(x,x^*)\Lambda_\tau(x,v^*,\overline{t},\overline{s})\leqq\sum_{\tau=1}^{M}\pi_\tau(x,x^*)\Lambda_\tau(x^*,v^*,\overline{t},\overline{s}),$$

which because of (ii) implies that

$$\Big\langle\sum_{\tau=1}^{M}\Big[\sum_{m\in J_\tau}v_m^*\nabla G_{j_m}(x^*,t^m)+\sum_{m\in K_\tau}v_m^*\nabla H_{k_m}(x^*,s^m)\Big],\eta(x,x^*)\Big\rangle\leqq-\widetilde{\rho}\|x-x^*\|^2. \tag{8}$$

Combining (7) and (8), and using (iii) we get

$$\Big\langle\sum_{i=1}^{p}u_i^*\Big[\nabla f_i(x^*)-\lambda_i^*\nabla g_i(x^*)+\sum_{m\in J_0}v_m^*\nabla G_{j_m}(x^*,t^m)$$
$$+\sum_{m\in K_0}v_m^*\nabla H_{k_m}(x^*,s^m),\eta(x,x^*)\Big]\Big\rangle\geqq\widetilde{\rho}\|x-x^*\|^2\geqq-\overline{\rho}\|x-x^*\|^2, \tag{9}$$

which by virtue of (i) implies that

$$\sum_{i=1}^{p}\theta_i(x,x^*)\Phi_i(x,u^*,v^*,\lambda^*,\overline{t},\overline{s})\geqq\sum_{i=1}^{p}\theta_i(x,x^*)\Phi_i(x^*,u^*,v^*,\lambda^*,\overline{t},\overline{s})=0, \tag{10}$$

where the equality follows from the feasibility of x^*, the fact that $t^m \in \widehat{T}_{j_m}(x^*),\ m \in \underline{\nu_0}$, and $\lambda^* = \varphi(x^*)$. Because $x \in \mathbb{F}$ and $v_m^* > 0,\ m \in \underline{\nu_0}$, this inequality reduces to

$$\sum_{i=1}^{p} u_i^* \theta_i(x, x^*)[f_i(x) - \lambda_i^* g_i(x)] \geqq 0.$$

Now using this inequality, as in the proof of Theorem 3.1, we obtain $\varphi(x) \nleq \varphi(x^*)$. Since x was arbitrary, we conclude that x^* is an efficient solution of (P).

(b) Proceeding in exactly the same manner as in the proof of part (a), we obtain (9) in which the second inequality is strict. By (i), this implies that (10) holds and, therefore, the rest of the proof is identical to that of part (a).

(c) The proof is similar to those of parts (a) and (b).

Each one of the six sets of conditions given in Theorem 4.1 and its modified version obtained by replacing (1) with (6) can be viewed as a family of sufficient efficiency conditions whose members can easily be identified by appropriate choices of the partitioning sets J_μ and $K_\mu,\ \mu \in \underline{M} \cup \{0\}$.

In the remainder of this section we present another collection of sufficiency results which are somewhat different from those stated in Theorem 4.1. These results are formulated by utilizing a partition of $\underline{p}$ in addition to those of $\underline{\nu_0}$ and $\underline{\nu}\backslash\underline{\nu_0}$, and by placing appropriate generalized (α, η, ρ)-V-invexity requirements on certain vector functions involving $\mathcal{E}_i(\cdot, \lambda, u),\ i \in \underline{p},\ G_j,\ j \in \underline{q}$ and $H_k,\ k \in \underline{r}$.

Let $\{I_0, I_1, \cdots, I_d\},\ \{J_0, J_1, \cdots, J_e\}$ and $\{K_0, K_1, \cdots, K_e\}$ be partitions of $\underline{p},\ \underline{\nu_0}$ and $\underline{\nu}\backslash\underline{\nu_0}$, respectively, such that $D = \{0, 1, 2, \cdots, d\} \subset E = \{0, 1, \cdots, e\}$, and let the function $\Pi_\tau(\cdot, u, v, \lambda, \overline{t}, \overline{s}) : \mathbb{R}^n \to \mathbb{R}$ be defined, for fixed $u, v, \lambda, \overline{t}$, and $\overline{s}$, by

$$\Pi_\tau(z, u, v, \lambda, \overline{t}, \overline{s}) = \sum_{i \in I_\tau} u_i[f_i(z) - \lambda_i g_i(z)] + \sum_{m \in J_\tau} v_m G_{j_m}(z, t^m) + \sum_{m \in K_\tau} v_m H_{k_m}(z, s^m), \qquad \tau \in D.$$

Theorem 4.2. *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*)$, let the functions $f_i, g_i,\ i \in \underline{p},\ G_j(\cdot, t)$ and $H_k(\cdot, s)$ be differentiable at x^* for all $t \in T_j$ and $s \in S_k,\ j \in \underline{q},\ k \in \underline{r}$, and assume that there exist $u^* \in U$, and integers ν_0 and ν, with $0 \leqq \nu_0 \leqq \nu \leqq n+1$, such that there exist ν_0 indices j_m, with $1 \leqq j_m \leqq q$, together with ν_0 points $t^m \in \widehat{T}_{j_m}(x^*),\ m \in \underline{\nu_0},\ \nu - \nu_0$ indices k_m, with $1 \leqq k_m \leqq r$, together with $\nu - \nu_0$ points $s^m \in S_{k_m},\ m \in \underline{\nu}\backslash\underline{\nu_0}$, and ν real numbers v_m^*, with $v_m^* > 0$ for $m \in \underline{\nu_0}$, such that (1) holds. Assume, furthermore, that any one of the following three sets of hypotheses is satisfied:*

(a) *(i)* $\big(\Pi_0(\cdot, u^*, v^*, \lambda^*, \overline{t}, \overline{s}), \cdots, \Pi_d(\cdot, u^*, v^*, \lambda^*, \overline{t}, \overline{s})\big)$ *is $(\theta, \eta, \overline{\rho})$-V-pseudoinvex at x^*;*

(ii) $\big(\Lambda_{d+1}(\cdot, v^*, \overline{t}, \overline{s}), \cdots, \Lambda_e(\cdot, v^*, \overline{t}, \overline{s})\big)$ *is $(\pi, \eta, \widetilde{\rho})$-V-quasiinvex at x^*;*

(iii) $\overline{\rho} + \widetilde{\rho} \geqq 0$;

(b) *(i)* $\big(\Pi_0(\cdot, u^*, v^*, \lambda^*, \overline{t}, \overline{s}), \cdots, \Pi_d(\cdot, u^*, v^*, \lambda^*, \overline{t}, \overline{s})\big)$ *is prestrictly $(\theta, \eta, \overline{\rho})$-V-quasiinvex at x^*;*

(ii) $\big(\Lambda_{d+1}(\cdot, v^*, \overline{t}, \overline{s}), \cdots, \Lambda_e(\cdot, v^*, \overline{t}, \overline{s})\big)$ *is strictly $(\pi, \eta, \widetilde{\rho})$-V-pseudoinvex at x^*;*

(iii) $\overline{\rho} + \widetilde{\rho} \geqq 0$;

(c) *(i)* $\big(\Pi_0(\cdot, u^*, v^*, \lambda^*, \overline{t}, \overline{s}), \cdots, \Pi_d(\cdot, u^*, v^*, \lambda^*, \overline{t}, \overline{s})\big)$ *is prestrictly $(\theta, \eta, \overline{\rho})$-V-quasiinvex at x^*;*

(ii) $\big(\Lambda_{d+1}(\cdot, v^*, \bar{t}, \bar{s}), \cdots, \Lambda_e(\cdot, v^*, \bar{t}, \bar{s})\big)$ *is* $(\pi, \eta, \widetilde{\rho})$*-V-quasiinvex at* x^**;*

(iii) $\overline{\rho} + \widetilde{\rho} > 0$.

Then x^* *is an efficient solution of (P).*

Proof. (a) Suppose to the contrary that x^* is not an efficient solution of (P). Then there is $\overline{x} \in \mathbb{F}$ such that $\varphi(\overline{x}) \leqslant \varphi(x^*)$, and so it follows that

$$f_i(\overline{x}) - \lambda_i^* g_i(\overline{x}) \leqq 0, \qquad i \in \underline{p},$$

with strict inequality holding for at least one index $i \in \underline{p}$. Since $u^* > 0$, we see that for each $\tau \in D$,

$$\sum_{i \in I_\tau} u_i^*[f_i(\overline{x}) - \lambda_i^* g_i(\overline{x})] \leqq 0, \tag{11}$$

with strict inequality holding for at least one index $\tau \in D$.

Now using this inequality, we see that

$$\begin{aligned}
&\Pi_\tau(\overline{x}, u^*, v^*, \lambda^*, \bar{t}, \bar{s}) \\
&= \sum_{i \in I_\tau} u_i^*[f_i(\overline{x}) - \lambda_i^* g_i(\overline{x})] + \sum_{m \in J_\tau} v_m^* G_{j_m}(\overline{x}, t^m) + \sum_{m \in K_\tau} v_m^* H_{k_m}(\overline{x}, s^m) \\
&\leqq \sum_{i \in I_\tau} u_i^*[f_i(\overline{x}) - \lambda_i^* g_i(\overline{x})] \text{ (by the feasibility of } \overline{x} \text{ and positivity of } v_m^*,\ m \in \underline{\nu_0}) \\
&\leqq 0 \qquad \text{(by (11))} \\
&= \sum_{i \in I_\tau} u_i^*[f_i(x^*) - \lambda_i^* g_i(x^*)] + \sum_{m \in J_\tau} v_m^* G_{j_m}(x^*, t^m) + \sum_{m \in K_\tau} v_m^* H_{k_m}(x^*, s^m) \\
&\qquad \text{(since } \lambda^* = \varphi(x^*),\ x^* \in \mathbb{F}, \text{ and } t^m \in \widehat{T}_{j_m}(x^*),\ m \in \underline{\nu_0}) \\
&= \Pi_\tau(x^*, u^*, v^*, \lambda^*, \bar{t}, \bar{s}),
\end{aligned}$$

with strict inequality holding for at least one index $\tau \in D$, and hence

$$\sum_{\tau \in D} \theta_\tau(x, x^*) \Pi_\tau(\overline{x}, u^*, v^*, \lambda^*, \bar{t}, \bar{s}) < \sum_{\tau \in D} \theta_\tau(x, x^*) \Pi_\tau(x^*, u^*, v^*, \lambda^*, \bar{t}, \bar{s}),$$

which in view of (i) implies that

$$\begin{aligned}
&\Big\langle \sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda_i \nabla g_i(x^*)] + \sum_{\tau \in D} \Big[\sum_{m \in J_\tau} v_m^* \nabla G_{j_m}(x^*, t^m) \\
&\quad + \sum_{m \in K_\tau} v_m^* \nabla H_{k_m}(x^*, s^m) \Big], \eta(\overline{x}, x^*) \Big\rangle < -\overline{\rho} \|\overline{x} - x^*\|^2.
\end{aligned} \tag{12}$$

As shown in the proof of Theorem 4.1, for each $\tau \in E \backslash D$, $\Lambda_\tau(\overline{x}, v^*, \bar{t}, \bar{s}) \leqq \Lambda_\tau(x^*, v^*, \bar{t}, \bar{s})$, and hence

$$\sum_{\tau \in E \backslash D} \pi_\tau(x, x^*) \Lambda_\tau(\overline{x}, v^*, \bar{t}, \bar{s}) \leqq \sum_{\tau \in E \backslash D} \pi_\tau(x, x^*) \Lambda_\tau(x^*, v^*, \bar{t}, \bar{s}),$$

which in view of (ii) implies that

$$\begin{aligned}
&\Big\langle \sum_{\tau \in E \backslash D} \Big[\sum_{m \in J_\tau} v_m^* \nabla G_{j_m}(x^*, t^m) + \sum_{m \in K_\tau} v_m^* \nabla H_{k_m}(x^*, s^m) \Big], \eta(\overline{x}, x^*) \Big\rangle \\
&\leqq -\widetilde{\rho} \|\overline{x} - x^*\|^2.
\end{aligned} \tag{13}$$

Now combining (12) and (13) and using (iii), we see that

$$\Big\langle \sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda_i^* \nabla g_i(x^*)] + \sum_{m=1}^{\nu_0} v_m^* \nabla G_{j_m}(x^*, t^m)$$
$$+ \sum_{m=\nu_0+1}^{\nu} v_m^* \nabla H_{k_m}(x^*, s^m), \eta(\overline{x}, x^*) \Big\rangle < -(\overline{\rho} + \widetilde{\rho}) \|\overline{x} - x^*\|^2 \leqq 0,$$

which contradicts (1). Therefore, x^* is an efficient solution of (P).

(b) and (c) The proofs are similar to that of part (a). □

As we mentioned previously, one can readily identify numerous special cases of the six families of sufficiency results stated in Theorem 4.2 and its modified version obtained by replacing (1) with (6).

5 Concluding Remarks

In this study we have established a number of sets of global sufficient efficiency conditions under various generalized (α, η, ρ)-V-invexity hypotheses for a semiinfinite multiobjective fractional programming problem. It appears that all these results are new in the area of semiinfinite programming. Since all the results obtained here can be modified and restated in a straightforward manner for each one of the seven problems designated as (P1)–(P7) in Section 1, they collectively subsume a fairly large number of existing results in the areas of conventional nonlinear programming and semiinfinite nonlinear programming. Furthermore, the style and techniques employed in this paper can be utilized to establish similar results for some other classes of related optimization problems. For example, it seems reasonable to expect that a similar approach can be applied to investigate the optimality and duality aspects of the following closely related classes of semiinfinite minmax fractional programming problems:

$$\underset{x \in \mathbb{F}}{\text{Minimize}} \ \max_{1 \leqq i \leqq p} \frac{f_i(x)}{g_i(x)},$$

$$\underset{x \in \mathbb{F}}{\text{Minimize}} \ \max_{y \in Y} \frac{f(x,y)}{g(x,y)}.$$

We shall investigate these problems in subsequent papers.

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