Acta Mathematicae Applicatae Sinica, English Series
Vol. 27, No. 3 (2011) 463–470
DOI: 10.1007/s10255-011-0084-8
http://www.ApplMath.com.cn & www.SpringerLink.com

Acta Mathematicae Applicatae Sinica, English Series

Existence of Solutions to Nonlinear Neumann Boundary Value Problems with p-Laplacian Operator and Iterative Construction

Li Wei[1], **Hai-yun Zhou**[2], **Ravi P. Agarwal**[3]

[1]School of Mathematics and Statistics, Hebei University of Economics and Business, Shijiazhuang 050061, China (E-mail: diandianba@yahoo.com)

[2]Institute of Applied Mathematics, Ordnance Engineering College, Shijiazhuang 050003, China

[3]Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901, U.S.A.

Abstract By using some results of pseudo-monotone operator, we discuss the existence and uniqueness of the solution of one kind nonlinear Neumann boundary value problems involving the p-Laplacian operator. We also construct an iterative scheme converging strongly to this solution.

Keywords maximal monotone operator, pseudo-monotone operator, p-Laplacian operator, iterative scheme
2000 MR Subject Classification 47H05; 47H09

1 Introduction

Nonlinear boundary value problems involving p-Laplacian operator $-\Delta_p$ occur in a variety of physical phenomena, such as non-Newtonian fluids, reaction-diffusion problems, petroleum extraction, flow through porous in media, etc. Thus, the study of such problems and their far reaching generalizations have attracted several mathematicians in recent years.

Recall that we employed the perturbation results on sums of ranges of nonlinear maximal monotone operators of Calvert and Gupta[3], and showed in [7–12] that some kind of nonlinear Neumann boundary value problems with p-Laplacian operators had solutions in some Sobolev spaces. This paper can be considered as an extension of our previous ones.

Actually, we have two purposes in this paper. First, in Section 3, we shall show that under some conditions the following nonlinear Neumann boundary value problem with p-Laplacian operator has a unique solution in $W^{1,p}(\Omega)$:

$$\begin{aligned} &-\operatorname{div}(|\nabla u|^{p-2}\nabla u)+\varepsilon|u|^{q-2}u=f(x) \qquad &&\text{a.e. in } \Omega,\\ &-\langle\vartheta,|\nabla u|^{p-2}\nabla u\rangle=g(x) &&\text{a.e. on } \Gamma, \end{aligned} \tag{1.1}$$

where $f(x)\in L^{p'}(\Omega)$, $g(x)\in W^{-\frac{1}{p'},\,p'}(\Gamma)$, $\frac{1}{p}+\frac{1}{p'}=1$ and ϑ denotes the exterior normal derivative of Γ. Further details will be introduced in Section 3.1.

Later, in Section 4, we shall set up the relationship between the solution of Equation (1.1) and zero points of maximal monotone operators and construct an iterative sequence converging strongly to the solution of (1.1), which may help us to know the properties of the solution better.

Manuscript received October 16, 2008. Revised December 9, 2009.
Supported by the National Natural Science Foundation of China (No. 11071053), the Natural Science Foundation of Hebei Province (No.A2010001482), and the project of Science and Research of Hebei Education Department (the second round in 2010).

2 Preliminaries

Let X be a real reflexive Banach space with a dual space X^*. We shall use " $\to$ " and "$w-\lim$" to denote strong and weak convergences, respectively. A mapping $T : D(T) = X \to X^*$ is said to be hemi-continuous on X (c.f. [5]) if $w - \lim_{t\to 0} T(x + ty) = Tx$, for any $x, y \in X$.

Let J denote the *duality mapping* from X into 2^{X^*} defined by

$$J(x) = \{f \in X^* : (x, f) = \|x\| \cdot \|f\|,\ \|f\| = \|x\|\}, \qquad \forall x \in X,$$

where $(\cdot,\cdot)$ denotes the generalized duality pairing between X and X^*. Since X^* is strictly convex, J is a single-valued mapping (c.f. [5, 16]).

A multi-valued mapping $B : X \to 2^{X^*}$ is said to be monotone (c.f. [5, 16]) if the inequality

$$(u_1 - u_2, w_1 - w_2) \ge 0 \tag{2.1}$$

holds for any $u_i \in D(B)$ and $w_i \in Bu_i$, $i = 1, 2$. The mapping B is said to be strictly monotone if the equality in (2.1) implies that $u_1 = u_2$. The monotone operator B is said to be maximal monotone if $R(J + rB) = X^*$, for $\forall r > 0$. The mapping B is said to be coercive (c.f. [5, 16]) if $\lim_{n\to+\infty} (x_n, x_n^*)/\|x_n\| = +\infty$ for all $x_n \in D(B)$, $x_n^* \in Bx_n$ such that $\lim_{n\to+\infty} \|x_n\| = +\infty$. A point $x \in D(B)$ is said to be a zero point of B if $0 \in Bx$, and we denote by $B^{-1}(0) = \{x \in X : 0 \in Bx\}$ the set of zero points of B.

Let C be a closed convex subset of X and let $A : C \to 2^{X^*}$be a multi-valued mapping. Then A is said to be a pseudo-monotone operator (c.f. [5]) provided that

(i) For each $x \in C$, the image Ax is a non-empty closed and convex subset of X^*;

(ii) If $\{x_n\}$ is a sequence in C converging weakly to $x \in C$ and if $f_n \in Ax_n$ is such that

$$\limsup_{n\to\infty}(x_n - x, f_n) \le 0,$$

then to each element $y \in C$, there corresponds an $f(y) \in Ax$ with the property that

$$(x - y, f(y)) \le \liminf_{n\to\infty}(x_n - x, f_n);$$

(iii) For each finite-dimensional subspace F of X, the operator A is continuous from $C \cap F$ to X^* in the weak topology.

Lemma 2.1[5]**.** *If $B : X \to 2^{X^*}$ is a maximal monotone operator such that $D(B) = X$, then B is pseudo-monotone.*

Remark 2.1. For $u \in W^{1,p}(\Omega)$, we use $\|u\|_{1,p,\Omega}$ to denote the norm of u in $W^{1,p}(\Omega)$. For $v \in W^{\frac{1}{p'},p}(\Gamma)$, which is the trace space of $W^{1,p}(\Omega)$, we use $\|v\|_{W^{\frac{1}{p'},p}(\Gamma)}$ to denote the norm of it.

Lemma 2.2[1]**.** *Let Ω be a domain of R^N with its boundary $\Gamma \in C^1$, then we have the following results:*

(i) If $u \in W^{1,p}(\Omega)$, then the trace $\gamma u \in W^{\frac{1}{p'},p}(\Gamma)$ and $\|\gamma u\|_{W^{\frac{1}{p'},p}(\Gamma)} \le K_1\|u\|_{1,p,\Omega}$;

(ii) If $v \in W^{\frac{1}{p'},p}(\Gamma)$, then there exists $u \in W^{1,p}(\Omega)$ such that $v = \gamma u$ and $\|u\|_{1,p,\Omega} \le K_2\|v\|_{W^{\frac{1}{p'},p}(\Gamma)}$, where $\gamma : W^{1,p}(\Omega) \to W^{\frac{1}{p'},p}(\Omega)$ denotes the trace operator and $\frac{1}{p} + \frac{1}{p'} = 1$.

Lemma 2.3[5]**.** *If $B : X \to 2^{X^*}$ is a maximal monotone and coercive operator, then B is surjective.*

Theorem 2.1[2]**.** *Let $T : X \to X^*$ be a bounded and pseudo-monotone operator, K be a closed and convex subset of X. Suppose that Φ is a lower-semi-continuous and convex function defined on K which is not always $+\infty$ such that for any $v \in K$, $\Phi(v) \in (-\infty, +\infty]$. Suppose there exists $v_0 \in K$ such that $\Phi(v_0) < +\infty$, and satisfies the following:*

$$\frac{(v - v_0, Tv) + \Phi(v)}{\|v\|} \to \infty \qquad as \ \ \|v\| \to \infty,\ v \in K,$$

then there exists $u \in K$ such that

$$(u - v, Tu) \leq \Phi(v) - \Phi(u) \qquad for \ \ \forall v \in K.$$

3 Solution of Equation (1.1)

3.1 Explanation of Equation (1.1)

In this paper, unless otherwise stated, we shall assume that $\frac{2N}{N+1} < p < +\infty$, $1 \leq q < +\infty$ if $p \geq N$, and $1 \leq q \leq \frac{Np}{N-p}$ if $p < N$, where $N \geq 1$. We use $\langle \cdot, \cdot \rangle$ to denote the inner product in R^N.

In Equation (1.1), Ω is a bounded domain of a Euclidean space R^N, where $N \geq 1$. Γ is the boundary of Ω such that $\Gamma \in C^1$ (see [10]). We shall assume that Green's Formula is available. $f(x) \in L^{p'}(\Omega)$ is a given function. ε is a non-negative constant, $g(x) \in W^{-\frac{1}{p'},\, p'}(\Gamma) = (W^{\frac{1}{p'},\, p}(\Gamma))^*$ and ϑ denotes the exterior normal derivative of Γ. Moreover, $\frac{1}{p} + \frac{1}{p'} = 1$.

3.2 Discussion of Equation (1.1)

Lemma 3.1. *Define the mapping $B_{p,q} : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ by*

$$(v, B_{p,q}u) = \int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla v \rangle dx + \varepsilon \int_\Omega |u(x)|^{q-2} u(x) v(x) dx$$

for any $u, v \in W^{1,p}(\Omega)$. Then, $B_{p,q}$ is everywhere defined, strictly monotone, hemi-continuous and coercive.

Proof. We only need to show that $B_{p,q}$ is strictly monotone since we have proved in [9] that $B_{p,q}$ is everywhere defined, monotone, hemi-continuous and coercive.

In fact, for any $u, v \in W^{1,p}(\Omega)$,

$$\begin{aligned}|(u - v, B_{p,q}u - B_{p,q}v)| \geq &\int_\Omega (|\nabla u|^{p-1} - |\nabla v|^{p-1})(|\nabla u| - |\nabla v|) dx \\ &+ \varepsilon \int_\Omega (|u|^{q-2}u - |v|^{q-2}v)(|u| - |v|) dx \geq 0. \qquad (3.1)\end{aligned}$$

Now we can see from (3.1) that if $(u - v, B_{p,q}u - B_{p,q}v) = 0$, then $u(x) = v(x)$ and $\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial x_i}$ $(i = 1, 2, \cdots, N)$, a.e. in Ω, which implies that $u(x) = v(x)$ in $W^{1,p}(\Omega)$. Therefore, $B_{p,q}$ is strictly monotone. □

From Lemmas 3.1 and 2.3, we have the following result:

Lemma 3.2. $B_{p,q} : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ *is maximal monotone and* $R(B_{p,q}) = (W^{1,p}(\Omega))^*$.

From Lemmas 3.1 , 3.2 and Lemma 2.1, we can easily get the following result:

Lemma 3.3. *$B_{p,q} : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is pseudo-monotone and for $v_0 \in W^{1,p}(\Omega)$, we have*

$$\lim_{\|v\|_{1,p,\Omega}\to+\infty} \frac{(v-v_0, B_{p,q}v)}{\|v\|_{1,p,\Omega}} = +\infty.$$

Theorem 3.1. *Equation* (1.1) *has a unique solution in $W^{1,p}(\Omega)$, for $f \in L^{p'}(\Omega)$ and $g(x) \in W^{-\frac{1}{p'},p'}(\Gamma)$.*

Proof. Let $\gamma : W^{1,p}(\Omega) \to W^{\frac{1}{p'},p}(\Gamma)$ be the trace operator. Define $T : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ as follows:

$$(v, Tu) = (v, B_{p,q}u) - \int_\Omega fv dx + (\gamma v, g)_\Gamma \qquad \text{for } \ \forall u, v \in W^{1,p}(\Omega),$$

then $T : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is everywhere defined, monotone, hemi-continuous and bounded. Therefore, it is pseudo-monotone.

Define $\Phi_1 : W^{\frac{1}{p'},p}(\Gamma) \to (-\infty, +\infty]$ by $\Phi_1(u) \equiv 0$, for $\forall u \in W^{\frac{1}{p'},p}(\Gamma)$. Denote by $\partial\Phi_1 : W^{\frac{1}{p'},p}(\Omega) \to W^{-\frac{1}{p'},p'}(\Omega)$ the subdifferential of Φ_1.

Define $\Phi_2 : W^{1,p}(\Omega) \to (-\infty, +\infty]$ by $\Phi_2(v) \equiv \Phi_1(\gamma v)$, for $\forall v \in W^{1,p}(\Omega)$. It is obviously that Φ_2 is a lower-semi-continuous and convex function defined on $W^{1,p}(\Omega)$. Moreover, for any given $w_0 \in W^{1,p}(\Omega), \forall w \in W^{1,p}(\Omega)$, we have

$$\frac{(w-w_0, Tw) + \Phi_2(w)}{\|w\|_{1,p,\Omega}} \to \infty \qquad \text{as } \ \|w\|_{1,p,\Omega} \to \infty.$$

Then from Theorem 2.1, we know that there exists $u(x) \in W^{1,p}(\Omega)$ such that for $\forall v \in W^{1,p}(\Omega)$, there holds the following inequality:

$$(u - w, Tu) \le \Phi_2(w) - \Phi_2(u) = 0. \tag{3.2}$$

Putting $w = u \pm \varphi \in K$, where $\varphi \in C_0^\infty(\Omega)$, into inequality (3.2), then $(\varphi, Tu) = 0$. That is,

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle dx + \int_\Omega \varepsilon |u|^{q-2}\varphi dx - \int_\Omega f\varphi dx = 0.$$

From the properties of generalized function, we have :

$$-\text{div}(|\nabla u|^{p-2}\nabla u) + \varepsilon|u|^{q-2}u = f(x) \qquad \text{a.e. in } \ \Omega. \tag{3.3}$$

Notice Green's formula, we have for $\forall w \in W^{1,p}(\Omega)$,

$$\int_\Omega [\text{div}(|\nabla u|^{p-2}\nabla u)] w dx + \int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla w\rangle dx = \int_\Gamma \langle \vartheta, |\nabla u|^{p-2}\nabla u\rangle \gamma w d\Gamma(x), \tag{3.4}$$

where $u(x)$ is the function which satisfies (3.3).

Then we have from (3.2) that

$$\int_\Gamma \langle \vartheta, |\nabla u|^{p-2}\nabla u\rangle \gamma(u-w) d\Gamma(x) + (\gamma(u-w), g)_\Gamma \le 0 = \Phi_1(\gamma w) - \Phi_1(\gamma u),$$

which implies that

$$-\langle \vartheta, |\nabla u|^{p-2}\nabla u\rangle - g(x) \in \partial\Phi_1(\gamma u) \equiv 0, \quad \forall w \in W^{1,p}(\Omega).$$

That is, $-\langle \vartheta, |\nabla u|^{p-2}\nabla u\rangle = g(x)$.

Therefore, $u(x) \in W^{1,p}(\Omega)$ is the solution of Equation (1.1).

Next, we shall prove the uniqueness. In fact, it suffices to show that if $u, v \in W^{1,p}(\Omega)$ satisfy the Equation (1.1), then $u(x) = v(x)$ in $W^{1,p}(\Omega)$.

For this, notice that

$$\begin{aligned}
&(u - v, B_{p,q}u - B_{p,q}v) \\
=&\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla u\rangle dx + \varepsilon \int_\Omega |u(x)|^{q-2}u(x)u(x)dx \\
&- \int_\Omega \langle |\nabla v|^{p-2}\nabla v, \nabla u\rangle dx - \varepsilon \int_\Omega |v(x)|^{q-2}v(x)u(x)dx \\
&- \int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla v\rangle dx - \varepsilon \int_\Omega |u(x)|^{q-2}u(x)v(x)dx \\
&+ \int_\Omega \langle |\nabla v|^{p-2}\nabla v, \nabla v\rangle dx + \varepsilon \int_\Omega |v(x)|^{q-2}v(x)v(x)dx \\
=&\int_\Omega [-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \varepsilon|u(x)|^{q-2}u(x)]u(x)dx \\
&- \int_\Omega [-\mathrm{div}(|\nabla v|^{p-2}\nabla v) + \varepsilon|v(x)|^{q-2}v(x)]u(x)dx \\
&- \int_\Omega [-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \varepsilon|u(x)|^{q-2}u(x)]v(x)dx \\
&+ \int_\Omega [-\mathrm{div}(|\nabla v|^{p-2}\nabla v) + \varepsilon|v(x)|^{q-2}v(x)]v(x)dx = 0.
\end{aligned}$$

Since $B_{p,q}$ is strictly monotone, then $u(x) = v(x)$. □

Remark 3.1. Compared to the related work in [3, 7–12], we may notice that the boundary condition in (1.1) is more general. That is, the function $g(x)$ in (1.1) is no longer required to be the subdifferential of a proper, convex and lower-semi-continuous function as ever before. This makes difference in proving the existence of solution of (1.1). Specifically, if in (1.1), Ω is reduced to be a bounded interval of $R_+ = (0, +\infty)$, $p = q = 2$ and $g(x) = 0$, we can get the following special example of (1.1), which can be found in [5]:

$$\begin{aligned}
&- u'' + \varepsilon u = f \in L^2(\Omega), \\
&- u' = 0.
\end{aligned} \tag{3.5}$$

From the discussions of both Theorem 3.1 in our paper and that in Chapter 3 in [5], we know that (3.5) has a solution in $H^1(\Omega)$. However, the methods used are different since ours based on the Theorem 2.1 and theirs not. In a word, our method is a complement and continuation of the previous work.

4 Iterative Construction of the Solution of Equation(1.1)

Definition 4.1[4]**.** *Let X be a real smooth Banach space. Then the Lyapunov functional $\varphi : X \times X \to R^+$ is defined as follows:*

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \qquad \forall x, y \in X.$$

Lemma 4.1[4]. *Let X be a real reflexive, strictly convex and smooth Banach space, C be a nonempty closed and convex subset of X and let $x \in X$. Then there exists a unique element $x_0 \in C$ such that*

$$\varphi(x_0, x) = \min\{\varphi(z, x) : z \in C\}.$$

Define a mapping π_C from X onto C by $\pi_C x = x_0$ for all $x \in X$. π_C is called the *generalized projection mapping* from X onto C. It is easy to see that π_C is coincide with the metric projection P_C in a Hilbert space.

Theorem 4.1. *Suppose that X is a real smooth and uniformly convex Banach space and $A : X \to 2^{X^*}$ is a maximal monotone operator with $A^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following iterative scheme:*

$$\begin{aligned} &x_1 \in X, \qquad r_1 > 0, \\ &x_{n+1} = J^{-1}[\beta_n J x_1 + (1-\beta_n) J (J + r_n A)^{-1} J x_n], \qquad \forall n \geq 1. \end{aligned} \tag{4.1}$$

If $\{r_n\} \subset (0, +\infty)$ with $\lim\limits_{n\to\infty} r_n = +\infty$, $\{\beta_n\} \subset [0,1]$ with $\sum\limits_{n=1}^{\infty} \beta_n = +\infty$ and $\beta_n \to 0$ as $n \to \infty$, then $\{x_n\}$ converges strongly to $\pi_{A^{-1}(0)}(x_1)$, where $\pi_{A^{-1}(0)}$ is the generalized projection operator from X onto $A^{-1}(0)$.

Theorem 4.1 is a special case of Theorem1 in [13], therefore, the proof is omitted.

Definition 4.2. *Define the mapping $C_{p,q} : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ by*

$$(v, C_{p,q} u) = \int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla v \rangle dx + \varepsilon \int_\Omega |u(x)|^{q-2} u(x) v(x) dx - \int_\Omega f(x) v(x) dx + (\gamma v, g)_\Gamma$$

for any $u, v \in W^{1,p}(\Omega)$, where $f \in L^{p'}(\Omega)$ and $g(x) \in W^{-\frac{1}{p'}, p'}(\Gamma)$ are the same as those in Equation (1.1).

Similarly to Lemma 3.1 or theorem 3.1, we know that $C_{p,q}$ is also a maximal monotone operator. Moreover, we can easily get the following result:

Lemma 4.2. *If $u \in W^{1,p}(\Omega)$ is the solution of Equation (1.1), then $u \in W^{1,p}(\Omega)$ is the zero point of $C_{p,q}$.*

Lemma 4.3[6]. *Let X be a Banach space and J be a duality mapping defined on X. If J is single-valued, then X is smooth.*

Based on the facts of Lemmas 4.2 , 4.3 and Theorem 4.1, we can construct an iterative sequence to approximate the solution of Equation (1.1).

Theorem 4.2. *Let $\{u_n(x)\}$ be a sequence generated by the following iterative scheme:*

$$\begin{aligned} &u_1(x) \in W^{1,p}(\Omega), \qquad r_1 > 0, \quad \textit{chosen arbitrarily}, \\ &u_{n+1}(x) = J^{-1}[\beta_n J u_1(x) + (1-\beta_n) J (J + r_n C_{p,q})^{-1} J u_n(x)], \qquad \forall n \geq 1. \end{aligned} \tag{4.2}$$

If $\{r_n\} \subset (0, +\infty)$ with $\lim_{n\to\infty} r_n = +\infty$, $\{\beta_n\} \subset [0,1]$ with $\sum\limits_{n=1}^{\infty} \beta_n = +\infty$ and $\beta_n \to 0$ as $n \to \infty$, then $\{u_n(x)\}$ converges strongly to $\pi_{C_{p,q}^{-1}(0)}(u_1(x))$.

Remark 4.1. Theorem 4.2 not only tells us that the sequence $\{u_n(x)\}$ generated by (4.2) converges strongly to the solution of Equation (1.1), but also that the unique solution of Equation (1.1) is the generalized projection of the initial function $u_1(x)$ onto $C_{p,q}^{-1}(0)$.

Remark 4.2. If $p \equiv 2$, then Equation (1.1) reduces to the case of Laplacian operator Equation. We can get the following Corollaries:

Corollary A. *The following Equation* $(1.1)'$ *has a unique solution in* $H^1(\Omega)$, *for* $f \in L^2(\Omega)$ *and* $g \in (H^{\frac{1}{2}}(\Gamma))^*$:

$$\begin{aligned} &-\triangle u + \varepsilon|u|^{q-2}u = f(x), && a.e.\ in\ \ \Omega, \\ &-\frac{\partial u}{\partial \vartheta} = g(x), && a.e.\ on\ \ \Gamma. \end{aligned} \tag{1.1$'$}$$

Corollary B. *Let* $\{u_n(x)\}$ *be a sequence generated by* $(4.2)'$:

$$\begin{aligned} &u_1(x) \in H^1(\Omega), \qquad r_1 > 0, \ \ \textit{chosen arbitrarily}, \\ &u_{n+1}(x) = \beta_n u_1(x) + (1-\beta_n)(I + r_n C_{2,q})^{-1} u_n(x), \qquad \forall n \geq 1, \end{aligned} \tag{4.2$'$}$$

where $C_{2,q}$ *is a special case of* $C_{p,q}$ *defined in Definition 4.2 if* $p \equiv 2$.

If $\{r_n\} \subset (0,+\infty)$ with $\lim\limits_{n\to\infty} r_n = +\infty$, $\{\beta_n\} \subset [0,1]$ with $\sum\limits_{n=1}^{\infty} \beta_n = +\infty$ and $\beta_n \to 0$ as $n \to \infty$, then $\{u_n(x)\}$ converges strongly to $P_{C_{2,q}^{-1}(0)}(u_1(x))$, where $P_{C_{2,q}^{-1}(0)}(u_1(x))$ denotes the metric projection from $H^1(\Omega)$ onto $C_{2,q}^{-1}(0)$.

Remark 4.3. Notice our recent work in [14], the whole discussion can be applied to the following nonlinear Neumann boundary value problem with generalized p-Laplacian operator:

$$\begin{aligned} &-\operatorname{div}[(C(x) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u] + \varepsilon|u|^{q-2}u = f(x), && \text{a.e. in } \ \Omega, \\ &-\langle \vartheta, (C(x) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u\rangle = g(x), && \text{a.e. on } \ \Gamma, \end{aligned} \tag{1.1$''$}$$

where $0 \leq C(x) \in L^p(\Omega)$, the other conditions are the same as those in Equation (1.1).

Remark 4.4. Notice our recent work in [15], the whole discussion can also be applied to the following nonlinear Neumann boundary value problem with so-called generalized p-Laplacian operator which is different from that in Remark 4.3:

$$\begin{aligned} &-\operatorname{div}(\alpha(\text{grad u})) + \varepsilon|u|^{q-2}u = f(x), && \text{a.e. in } \ \Omega, \\ &-\langle \vartheta, \alpha(\text{grad u})\rangle = g(x), && \text{a.e. on } \ \Gamma, \end{aligned} \tag{1.1$'''$}$$

where $\alpha : R^N \to R^N$ is a given monotone and continuous function, and there exist positive constants k_1, k_2 and k_3 such that for $\forall \xi, \xi' \in R^N$, the following are satisfied:
(i) $|\alpha(\xi)| \leq k_1|\xi|^{p-1}$;
(ii) $|\alpha(\xi) - \alpha(\xi')| \leq k_2||\xi|^{p-2}\xi - |\xi'|^{p-2}\xi'|$;
(iii) $< \xi, \alpha(\xi) > \geq k_3|\xi|^p$.

The other conditions are the same as those in Equation (1.1).

References

[1] Adams, R.A. The Sobolev Space (Version of Chinese Translation). People's Education Press, Beijing, 1981
[2] Brezis, H. Equations et inequations nonlinearies dans les espaces vectoriels en dualite. *Ann. Inst.Fourier Grenoble*, 18(1): 115–157 (1968)
[3] Calvert,B.D., Gupta, C.P. Nonlinear elliptic boundary value problems in L^p-spaces and sums of ranges of accretive operators. *Nonlinear Anal.*, 2: 1–26 (1978)
[4] Kamimura, S., Takahashi,W. Strong convergence of a proximal-type algorithm in a Banach space. *SIAM J Optim.*, 13(3): 938–945 (2003)
[5] Pascali,D., Sburlan, S. Nonlinear Mappings of Monotone Type. Sijthoff and Noordhoff, Romania, 1978
[6] Takahashi, W. Nonlinear functional analysis-fixed point theory and its applications. Yokohama Publishers, Yokohama, 2000
[7] Wei, L. The existence of a solution of nonlinear elliptic boundary value problems involving the p-Laplacian operator. *Acta Analy. Funct. Appl.*, 4: 46–54 (2002) (in Chinese)

[8] Wei, L. On the existence of solutions for Neumann boundary value problems with p-Laplacian operator. *Acta Math. Appli. Sinica*, 30(3): 517–526 (2007) (in Chinese)
[9] Wei, L., Agarwal, R.P. On the existence of solutions for nonlinear Neumann boundary value problems with p-Laplacian operator. *Advances in Math. Sci. Appli.*, 18(1): 69–81 (2008)
[10] Wei, L., He, Z. The applications of theories of accretive operators to nonlinear elliptic boundary value problems in L^p-spaces. *Nonlinear Anal.*, 46: 199–211 (2001)
[11] Wei, L., Zhou, H.Y. The existence of solutions of nonlinear boundary value problem involving the p-Laplacian operator in L^s-spaces. *J. Syst. Sci. Complexity*, 18: 511–521 (2005)
[12] Wei, L., Zhou, H.Y. Research on the existence of solution of equation involving the p-Laplacian operator. *Appl. Math. J. Chinese Univ.*, Ser B, 21(2): 191–202 (2006)
[13] Wei, L., Zhou, H.Y. The iterative convergence theorem of zero point for maximal monotone operator in Banach space and its application. *Math.in Prac. Theory*, 36(5): 235–242 (2006) (in Chinese)
[14] Wei, L., Agarwal, R.P. Existence of solutions to nonlinear Neumann boundary value problems with generalized p-Laplacian operator. *Comput. Math. Appli.*, 56 : 530–541 (2008)
[15] Wei, L., Zhou, H.Y., Agarwal, R.P. Existence of solutions for nonlinear Neumann boundary value problems. *J. Math. Research Exposi.*, 30(1): 99–109 (2010)
[16] Zeidler, E. Nonlinear functional analysis and its applications, III variational methods and optimization. Springer-Verlag, New York, Berlin, 1992